SEVERAL COMPLEX VARIABLES AND THE ORDER OF GROWTH OF THE RESONANCE COUNTING FUNCTION IN EUCLIDEAN SCATTERING

T. CHRISTIANSEN

Abstract. We study four classes of compactly supported perturbations of the Laplacian on $\mathbb{R}^d$, $d \geq 3$ odd. They are a fairly general class of black box perturbations, a class of second order, self-adjoint elliptic differential operators, Laplacians associated to metric perturbations, and the Dirichlet Laplacian on the exterior of a star-shaped obstacle. In each case, we show that generically the resonance counting function has maximal order of growth.

1. Introduction

This paper studies the resonance counting function for several types of compactly supported perturbations of the Laplacian on $\mathbb{R}^d$, $d \geq 3$ odd. The resonances, or scattering poles, for such an operator $P$ are defined as the poles of the meromorphic continuation of the resolvent $(P - \lambda^2)^{-1}$. They may be regarded as replacements for discrete spectral data. For the types of operators we consider in this paper, upper bounds on the number of resonances are now well-understood. They are of the type $n_P(r) \leq c_P(1 + r^d)$, where $n_P(r)$ is the resonance counting function; that is, the number of poles, counted with multiplicity, of the meromorphic continuation of $R_P(\lambda) = (P - \lambda^2)^{-1}$ which have norm at most $r$. In the generality we use here this was proved in [22] and [26], although there were many earlier results for more specific classes of operators. Lower bounds have proved more elusive. There are relatively few cases in which asymptotics, or even lower bounds of the “right” order, are known, and they are mostly in some sense one-dimensional or degenerate. For an introduction to resonances and further results on their distribution, see the surveys [27, 30] or the shorter introduction [33]. The recent paper [24] also has a fairly extensive bibliography, with many references to previous results.

One may ask, then, what is the order of the resonance counting function for a “generic” operator of this type. The paper [4] answers this question for Schrödinger operators on $\mathbb{R}^d$, $d \geq 3$ odd, using results of [29] and [2]. More precisely, for $K \subset \mathbb{R}^d$ a compact set with non-empty interior, the set of potentials $V \in L^\infty(K)$ for which
the resonance counting function for $\Delta + V$ has order of growth $d$ is Baire typical ("generic") in $L^\infty(K)$. In particular, it is dense in $L^\infty(K)$. This is true even for complex potentials, though it is known that there are compactly supported $L^\infty$ complex-valued potentials with no resonances [3]. It is natural to ask if similar density and genericity results hold for other classes of compactly supported perturbations of the Laplacian on $\mathbb{R}^d$. This paper addresses this question.

In this paper we prove density and genericity results for a restricted class of “black box” perturbations of the Laplacian of the type studied by Sjöstrand-Zworski [22]. We also study three smaller classes of compactly supported perturbations of the Laplacian. We show that black box operators with resonance counting function having the maximum order of growth is dense in, and generic in, a set of restricted black box operators. We do the same for the set of elliptic, self-adjoint second order differential operators. Next we consider compactly supported perturbations of the Euclidean metric and their associated Laplacians. Again, we prove density and genericity results for the set of such metrics so that the associated resonance counting function has maximal order of growth. Finally, we obtain similar results for the Dirichlet Laplacian in the exterior of star-shaped obstacles.

Although the results of this paper resemble the results of [2] and [4], the complex-analytic techniques required here are more sophisticated. Here we give a heuristic explanation of one reason why the techniques of [2] do not work for the operators we study. One of the primary results of [2] is that the set of potentials in $L^\infty_{\text{comp}}(\mathbb{R}^d)$ with corresponding resonance counting function having maximal order of growth is dense in $L^\infty_{\text{comp}}(\mathbb{R}^d)$. The proof involves introducing a family of operators depending holomorphically on a complex parameter $z$, and using some results from the theory of several complex variables. For the Schrödinger operator, this means studying $\Delta_x + V(z,x)$. The paper [2] uses extensively the fact that even when $V$ is complex, there is a half plane on which $R_{\Delta + V}(\lambda) = (\Delta + V - \lambda^2)^{-1}$ has no poles. It is then possible to study the order of growth of the resonance counting function by studying the order of growth of a function (the determinant of the scattering matrix) holomorphic in a half-plane. In this paper we consider more general classes of operators. When we introduce a complex parameter, it may appear, for example, in the coefficients of the second order part of a differential operator. For such operators $P$, there may well be no half-plane on which $R_P(\lambda) = (P - \lambda^2)^{-1}$ has no poles. To compensate for this, here we use the Nevanlinna characteristic function and the notion of the order of a meromorphic function (e.g. Section 2 or [11, 12]) in place of the order of a function holomorphic on a half-plane. Our scattering-theoretic results then require Theorem 3.4, a result in several complex variables which may be considered the central technical result of this paper. We are unaware of this result in the literature of Nevanlinna theory or several complex variables, though our proof uses many techniques from [8].

We now give our scattering-theoretic results. Let $\mathcal{BB}(\mathcal{H})$ denote the set of all black box perturbations of the non-negative Laplacian on $\mathbb{R}^d$ acting on the Hilbert space $\mathcal{H}$ as in (7) – see Section 4.1 for a precise definition, following [22]
but including the restriction (9). For $P \in \mathfrak{B}(\mathcal{H})$, let $n_P(r)$ denote the numbers of poles of $R_P(\lambda) = (P - \lambda^2)^{-1}$ with norm at most $r$, and set

$$\mathfrak{MB}(\mathcal{H}) = \{ P \in \mathfrak{B}(\mathcal{H}) : \limsup_{r \to \infty} \frac{\log n_P(r)}{\log r} = d \}.$$  

Using the restriction (9) and the results of [22] or [26], $d$ is the maximum value the limit in (1) can obtain. Our first result is then

**Theorem 1.1.** For $d \geq 3$ odd, $\mathfrak{MB}(\mathcal{H})$ is dense in $\mathfrak{B}(\mathcal{H})$ in the topology compatible with norm resolvent convergence.

Since the operators we consider are self-adjoint and bounded below, a topology compatible with norm resolvent convergence is given, for example by the metric $\|P - Q\| = \|(P + i)^{-1} - (Q + i)^{-1}\|_{\mathcal{H}}$. This theorem follows from the somewhat stronger Theorem 4.1. Theorem 4.5 is a genericity result for a more restricted class of black box operators.

We also consider three smaller classes of operators: second order, elliptic, self-adjoint differential operators on $L^2(\mathbb{R}^d)$ which are the Laplacian outside a compact set; Laplacians associated with perturbations of the standard Euclidean metric; and the Dirichlet Laplacian on the exterior of star-shaped obstacles. We note that although these all are black box operators, the results below do not follow from the black box results.

For $R > 0$, let $\mathfrak{P}_R$ denote the set of second order elliptic, self-adjoint differential operators on $L^2(\mathbb{R}^d)$ which have smooth coefficients and which are equal to the non-negative Laplacian for $|x| > R$. For $P \in \mathfrak{P}_R$, let $n_P(r)$ be the number of poles of $R_P(\lambda)$ with norm less than or equal to $r$, and set

$$\mathfrak{MP}_R = \{ P \in \mathfrak{P}_R : \limsup_{r \to \infty} \frac{\log n_P(r)}{\log r} = d \}.$$  

For $i = 1, 2$, and $P_i \in \mathfrak{P}_R$, let

$$P_i = \sum_{j \leq k} a_{i,jk}(x)D_{x_j}D_{x_k} + \sum_{j=1}^d b_{i,j}(x)D_j + V_i(x)$$

where $D_{x_j} = i^{-1} \frac{\partial}{\partial x_j}$. Then define

$$\|P_1 - P_2\|_{\mathfrak{P}} = \sum_{j \leq k} \|a_{1,jk} - a_{2,jk}\|_{C^\infty} + \sum_j \|b_{1,j} - b_{2,j}\|_{C^\infty} + \|V_1 - V_2\|_{C^\infty}$$

where $\| \cdot \|_{C^\infty}$ is a norm on bounded smooth functions on $\mathbb{R}^d$. We will call the topology induced by $\| \cdot \|_{\mathfrak{P}}$ the $C^\infty$ topology on $\mathfrak{P}$. Note that if $P_j, P \in \mathfrak{P}$ and $P_j \to P$ in the $C^\infty$ topology, then $P_j \to P$ in the norm resolvent sense.

**Theorem 1.2.** Let $d \geq 3$ be odd. Then the set $\mathfrak{MP}_R$ is dense in $\mathfrak{P}_R$ in the $C^\infty$ topology.

Using notation of B. Simon [21], for a metric space $X$, we call a dense $G_\delta$ set $S \subset X$ Baire typical. “Baire typical” sets are often called “generic.”
Theorem 1.3. Let $d \geq 3$ be odd. The set $\mathcal{MP}_R$ is Baire typical in $\mathcal{P}_R$ under the $C^\infty$ metric.

The proof of this theorem resembles the proof of a related result for Schrödinger operators [4].

There are similar results for at least some smaller classes of operators, and we consider as an example the class of Laplacians associated with metric perturbations. Upper bounds in this case are originally due to [25]. Let $\mathcal{G}_R$ denote the set of all smooth metrics $g$ on $\mathbb{R}^d$ such that $g(x)$ is the Euclidean metric for $|x| \geq R$, and set $\mathcal{G} = \cup_R \mathcal{G}_R$. We use the $C^\infty$ topology on $\mathcal{G}$.

Theorem 1.4. For $d \geq 3$ odd and $R > 0$, the set
\[
\mathcal{MG}_R = \{ g \in \mathcal{G}_R : \limsup_{r \to \infty} \frac{\log n_{\Delta_g}(r)}{\log r} = d \}
\]
is dense in $\mathcal{G}_R$.

This theorem has the following corollary, which really follows from Theorems 1.4 and 1.3.

Theorem 1.5. Let $d \geq 3$ be odd. The set $\mathcal{MG}_R$ is Baire typical in $\mathcal{G}_R$ under the $C^\infty$ metric.

Next we turn to obstacle scattering. The upper bound of order $d$ for the resonance counting function was proved in [9]. A compact obstacle $\mathcal{O} \subset \mathbb{R}^d$ which is star-shaped with respect to the origin is given by
\[
\mathcal{O} = \left\{ x \in \mathbb{R}^d : |x| \leq b \left( \frac{x}{|x|} \right) \right\}
\]
for a function $b > 0$ defined on $\mathbb{S}^{d-1}$. For $R \in \mathbb{R}_+$, let $B(R)$ be the open ball of radius $R$ centered at the origin. For $R > 1$, set
\[
\mathcal{O}_{ss,R} = \{ \text{smooth obstacles } \mathcal{O} \subset \mathbb{R}^d \text{ with } B(1/R) \subset \mathcal{O} \subset \overline{B(R)} , \text{ and } \mathcal{O} \text{ is star-shaped with respect to the origin} \}.
\]
The set $\mathcal{D}_{ss} = \cup_R \mathcal{O}_{ss,R}$. On $\mathcal{D}_{ss}$ we will use the topology generated by the metric
\[
\text{dist}_{ss}(\mathcal{O}_1, \mathcal{O}_2) = \| b_1(\omega) - b_2(\omega) \|_{C^\infty(\mathbb{S}^{d-1})}
\]
where for $i = 1, 2$, $\mathcal{O}_i \in \mathcal{D}_{ss}$ and, as in (2), is given by $b_i$. In analogy with the previous cases, for odd $d$ we set
\[
\mathcal{MD}_{ss,R} = \left\{ \mathcal{O} \in \mathcal{D}_{ss,R} : \limsup_{r \to \infty} \frac{\log n_{\mathcal{O}}(r)}{\log r} = d \right\}
\]
where $n_{\mathcal{O}}(r)$ is the resonance counting function for the Laplacian on $\mathbb{R}^d \setminus \mathcal{O}$ with Dirichlet boundary conditions.

Theorem 1.6. Let $d \geq 3$ be odd and $R > 1$. Then $\mathcal{MD}_{ss,R}$ is dense in $\mathcal{D}_{ss,R}$.
Of course, we could equally well consider obstacles star-shaped with respect to any other point in the plane. A result similar to Theorem 1.3 is

**Theorem 1.7.** Let \( d \geq 3 \) be odd and \( R > 1 \). Then \( \mathcal{M}\Omega_{ss,R} \) is Baire typical in \( \Omega_{ss,R} \).

**Notation.** Throughout this paper we use \( \Delta \) to denote the non-negative Laplacian. For \( \operatorname{Im} \lambda > 0 \) we write \( R_P(\lambda) = (P - \lambda^2)^{-1} \), and continue to denote by \( R_P(\lambda) \) its meromorphic continuation to the complex plane.

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**2. Review of some complex analysis**

In this section we review some complex analysis that we will use in this paper. Definitions and results for meromorphic functions of one complex variable can be found in, for example, [11] or [12].

We begin with a function \( f(\lambda) \) which is meromorphic on \( \mathbb{C} \). For \( r \geq 0 \), let \( n(r, f, \infty) \) denote the number of poles \( b_k \) of \( f \) (counted with multiplicity) with \( |b_k| \leq r \). Let

\[
N(r, f, \infty) = \int_0^r \frac{n(r, f, \infty) - n(0, f, \infty)}{t} dt + n(0, f, \infty) \log r
\]

and let

\[
m(r, f, \infty) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\varphi})| d\varphi,
\]

where \( \log^+ (a) = \max(\log a, 0) \). We also define, for \( a \in \mathbb{C} \),

\[
N(r, f, a) = N(r, \frac{1}{f-a}, \infty).
\]

**Definition 2.1.** If \( f \) is meromorphic for \( |\lambda| < R \leq \infty \), for \( r < R \) the Nevanlinna characteristic function of \( f \) is

\[
T(r, f) = T(r, f, \infty) = N(r, f, \infty) + m(r, f, \infty).
\]

**Definition 2.2.** For \( r > 0 \), let \( s(r) > 0 \) be a monotone increasing function of \( r \). If

\[
\limsup_{r \to \infty} \frac{\log s(r)}{\log r} = \mu < \infty,
\]

then \( s(r) \) is said to be of order \( \mu \). The order of a meromorphic function \( f \) on \( \mathbb{C} \) is the same as the order of its characteristic function \( T(r, f) \).
The order of a holomorphic function \( f \) is the same as its order as a meromorphic function [11, Theorem 2.8]. We shall later use the following lemma, which is closely related to Lemma 4.2 of [2]. We include the proof for the convenience of the reader.

**Lemma 2.1.** Suppose \( f(\lambda) \) is a meromorphic function on \( \mathbb{C} \) with the property that \( \lambda_0 \) is a pole of \( f \) if and only if \(-\lambda_0\) is a zero of \( f \), and the multiplicities coincide. In addition, suppose no zeros of \( f \) lie on the real axis. Moreover, suppose

\[
\int_0^r \frac{d}{dt} \log f(t) dt = O(r^m)
\]

and \( p > 1 \). Then \( f \) is of order \( p > m \) if and only if \( n(r, f, \infty) \) is of order \( p \).

**Proof.** From the definition, the order of \( f \) must be at least the order of \( n(r, f, \infty) \). We will show that in this case it cannot be greater.

Suppose \( f \) is of order \( p > m \) but \( n(r, f, \infty) \) is of order \( q < p \). For integers \( l \geq 1 \), let \( G(u; l) \) be the canonical factor

\[
G(u; l) = (1 - u)e^{u + u^2/2 + \ldots + u^l/l}.
\]

We may write

\[
f(\lambda) = \alpha e^{ig(\lambda)} \frac{P(-\lambda)}{P(\lambda)}
\]

where \( \alpha \) is a constant,

\[
P(\lambda) = \prod_{\lambda_j \text{ pole of } f} G(\lambda/\lambda_j; [q]),
\]

\([q] \) is the greatest integer less than or equal to \( q \) and \( g(\lambda) \) is a polynomial of order at most \( p \). Then \( P(-\lambda)/P(\lambda) \) is of order \( q \), so that \( g(\lambda) \) must be of order \( p \), but not of order \( p - 1 \). On the other hand, from a slight strengthening of Lemma 4.1 of [2],

\[
\left| \int_0^r \frac{d}{dt} (\log P(-t) - \log P(t)) dt \right| \leq C_\epsilon (1 + r^{q+\epsilon})
\]

for any \( \epsilon > 0 \). But our assumption

\[
\int_0^r \frac{d}{dt} \log f(t) dt = O(r^m)
\]

means then that \( g \) cannot have order greater than \( \max(m, q) \), a contradiction. □

Now we turn to functions of several complex variables, particularly plurisubharmonic functions. For further information about plurisubharmonic functions, see, for example, [7, 8].

**Definition 2.3.** Let \( \Omega \subset \mathbb{C}^m \) be a domain; that is, an open, connected set. A function \( \varphi(z) \) which takes its values in \([ -\infty, \infty \) is plurisubharmonic in \( \Omega \) if

- \( \varphi(z) \) is upper semi-continuous and \( \varphi \not\equiv -\infty \).
• For every $z \in \Omega$ and every $w \in \mathbb{C}^m$, $r > 0$ such that $\{z + uw : |u| \leq r, u \in \mathbb{C}\} \subset \Omega$,

$$\varphi(z) \leq (2\pi)^{-1} \int_0^{2\pi} \varphi(z + re^{i\theta}w) d\theta.$$ 

An important example of a plurisubharmonic function is $\log|f(z)|$, where $f(z)$ is holomorphic. We shall write $\varphi \in \text{PSH}(\Omega)$ if $\varphi$ is plurisubharmonic on $\Omega$. Plurisubharmonicity can be checked locally. Let $\Omega \subset \mathbb{C}^m$ be a domain. If $\varphi$ is upper semi-continuous on $\Omega$, $\varphi \not\equiv -\infty$, and for every $z \in \Omega$ there is a $b(z)$ such that

$$\varphi(z) \leq (2\pi)^{-1} \int_0^{2\pi} \varphi(z + we^{i\theta}) d\theta$$

for all $w \in \mathbb{C}^m$, $|w| < b(z)$, then we say that $\varphi$ is locally plurisubharmonic on $\Omega$. But if $\varphi$ is locally plurisubharmonic on $\Omega$, it is plurisubharmonic on $\Omega$ (e.g. [8, Proposition I.19]).

**Definition 2.4.** A set $E \subset \mathbb{C}^m$ is pluripolar if for each $a \in E$ there is a neighborhood $V$ of $a$ and $\varphi \in \text{PSH}(V)$ such that $E \cap V \subset \{z \in V : \varphi(z) = -\infty\}$.

This is equivalent to the definition given in [8] via the Josefson Theorem e.g. [7, Theorem 4.7.4]. We note that if $E \subset \mathbb{C}$ (i.e. $m = 1$) is pluripolar, then $E$ is a polar set. We shall use “pluripolar,” however, to be consistent with results from [8] which we shall use.

### 3. Proof of main technical result

The principal results of this section are Proposition 3.3 and Theorem 3.4, which give some results about the order, in one variable, of a meromorphic function of two variables.

**Lemma 3.1.** Let $\Omega \subset \mathbb{C}$ be a connected open set, and suppose $g(z, \lambda)$ and $h(z, \lambda)$ are holomorphic on $\Omega \times \mathbb{C}$. Let

$$\psi(z, r) = (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} \log |g(z, re^{i\varphi}) - e^{i\theta}h(z, re^{i\varphi})| d\theta d\varphi.$$ 

Then either $\psi(z, |v|)$ is a plurisubharmonic function of $(z, v) \in \Omega \times \mathbb{C}$ or $\psi$ is identically $-\infty$.

**Proof.** Note that

$$\psi(z, |v|) = (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} \log |g(z, |v|e^{i\varphi}) - e^{i\theta}h(z, |v|e^{i\varphi})| d\theta d\varphi$$

$$= (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} \log |g(z, ve^{i\varphi}) - e^{i\theta}h(z, ve^{i\varphi})| d\theta d\varphi$$

by a change of variable of integration. Since $g(z, ve^{i\varphi}) - e^{i\theta}h(z, ve^{i\varphi})$ is a holomorphic function of $(z, v)$, $(z, v) \mapsto \log|g(z, ve^{i\varphi}) - e^{i\theta}h(z, ve^{i\varphi})|$ is plurisubharmonic
Moreover, the next lemma shows that $\tilde{\pi}$ restricted to a compact set, $(2\pi)^{-2} \log |g(z, ve^{i\varphi}) - e^{i\theta} h(z, ve^{i\varphi})|$ is bounded above. If $(z, u)$ is restricted to a compact set, $(2\pi)^{-2} \log |g(z, ve^{i\varphi}) - e^{i\theta} h(z, ve^{i\varphi})|$ is bounded above. Moreover,

$$(\varphi, \theta, \omega) \mapsto (2\pi)^{-2} \log |g(z+w_1e^{i\omega}, (v+w_2e^{i\omega})e^{i\varphi}) - e^{i\theta} h(z+w_1e^{i\omega}, (v+w_2e^{i\omega})e^{i\varphi})|$$

is $d\theta d\varphi d\omega$ measurable. Hence, by [8, Proposition I.14], $\psi(z, |v|)$ is plurisubharmonic or identically $-\infty$.  

□

Let $\Omega \subset \mathbb{C}$ be an open set and suppose $f$ is meromorphic on $\Omega \times \mathbb{C}$. For $z_0 \in \Omega$ let $\Omega_0 \subset \Omega$ be an open ball containing $z_0$. Then there are holomorphic, relatively prime functions $g_{\Omega_0}$, $h_{\Omega_0}$ on $\Omega_0 \times \mathbb{C}$, such that

$$(3) \quad f(z, \lambda) = g_{\Omega_0}(z, \lambda)/h_{\Omega_0}(z, \lambda) \text{ for } (z, \lambda) \in \Omega_0 \times \mathbb{C}$$

[6, Chapter VIII B, Proposition 13]. Suppose, in addition, that near $\lambda = 0$, $h_{\Omega_0}(z, \lambda) = \lambda^j \tilde{h}_{\Omega_0}(z, \lambda)$, with $\tilde{h}_{\Omega_0}(z, 0)$ holomorphic on $\Omega_0$ and $\tilde{h}_{\Omega_0}(z, 0) \neq 0$. Let

$K_{f, \Omega_0} = \{z_1 \in \Omega_0 : \tilde{h}_{\Omega_0}(z_1, 0) = 0 \text{ or } h_{\Omega_0}(z_1, \lambda) \equiv 0\}.$

Note that $K_{f, \Omega_0}$ is independent of the choice of $g_{\Omega_0}$, $h_{\Omega_0}$ holomorphic and relatively prime such that (3) holds. Moreover, if $\Omega_1 \subset \Omega_0$ is a ball, then $K_{f, \Omega_1} = K_{f, \Omega_0} \cap \Omega_1$. Let

$$K_f = \bigcup_{\Omega_0 \subset \Omega} K_{f, \Omega_0}.$$  

The intersection of $K_f$ with any compact set in $\Omega$ is a finite number of points.

Let $z_0 \in \Omega$ and let $\Omega_0$, $g_{\Omega_0}$ and $h_{\Omega_0}$ be as before. For $z \in \Omega_0 \setminus K_{f, \Omega_0}$, $r \geq 0$, set

$$\tilde{T}(z, r, f, \infty) = T(z, r, f) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log |g_{\Omega_0}(z, re^{i\varphi}) - e^{i\theta} h_{\Omega_0}(z, re^{i\varphi})| d\theta d\varphi - \log |\tilde{h}_{\Omega_0}(z, 0)|.$$  

Since $\Omega$ can be covered by such sets, $\tilde{T}(z, r, f, \infty)$ is thus defined for $z \in \Omega \setminus K_f$.  

The next lemma shows that $\tilde{T}$ is indeed well-defined.

**Lemma 3.2.** The function $\tilde{T}(z, r, f, \infty)$ is well-defined independent of choice of $g_{\Omega_0}$ and $h_{\Omega_0}$ which are holomorphic and relatively prime and satisfy (3). Moreover, $\tilde{T}(z, |v|, f, \infty)$ is plurisubharmonic on $(\Omega \setminus K_f) \times \mathbb{C}$.

**Proof.** The first part of the lemma follows from Jensen’s Theorem and the requirement that $g_{\Omega_0}$ and $h_{\Omega_0}$ are relatively prime. Recall that to show a function is plurisubharmonic we need only show it is locally plurisubharmonic. The function $\log |\tilde{h}_{\Omega_0}(z, 0)|$ is harmonic away from the zeros of $\tilde{h}_{\Omega_0}(z, 0)$. This, together with Lemma 3.1, proves the second part of the lemma.  

□

Let $\Omega \subset \mathbb{C}$ be an open connected set and let $f$ be a function meromorphic on $\Omega \times \mathbb{C}$. Let $f_z(\lambda) = f(z, \lambda)$. In analogy with Definition 2.1, for $z \in \Omega$ such that $f(z, \lambda)$ is a meromorphic function of $\lambda$, we define

$$T(z, r, f, \infty) = T(r, f_z, \infty).$$
Proof. Using intermediate steps in [12, VI.3.5], for \( T \in \Omega \setminus K_f \),

\[
T(z, r, f, \infty) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log |f(z, re^{i\varphi})| \, d\varphi d\theta + N(z, r, f, \infty).
\]

Let \( z_0 \in \Omega \) and let \( \Omega_0 \subset \Omega \) be an open ball containing \( z_0 \). As before, let \( g_{\Omega_0}, h_{\Omega_0} \) be holomorphic, relatively prime functions on \( \Omega_0 \times \mathbb{C} \), such that \( f(z, \lambda) = g_{\Omega_0}(z, \lambda)/h_{\Omega_0}(z, \lambda) \) on \( \Omega_0 \times \mathbb{C} \) [6, Chapter VIII B, Proposition 13]. For \( z \in \Omega_0 \setminus K_f, \Omega_0 \),

\[
(4) \quad T(z, r, f, \infty) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log |g_{\Omega_0}(z, re^{i\varphi})| \, d\varphi d\theta - \frac{1}{(2\pi)} \int_0^{2\pi} \log |h_{\Omega_0}(z, re^{i\varphi})| \, d\varphi + N(z, r, g_{\Omega_0}/h_{\Omega_0}, \infty).
\]

Let \( \tilde{h}_{\Omega_0} \) be as in the definition of \( \tilde{T} \). By Jensen’s Theorem,

\[
(5) \quad -\frac{1}{(2\pi)} \int_0^{2\pi} \log |h_{\Omega_0}(z, re^{i\varphi})| \, d\varphi + N(z, r, h_{\Omega_0}, 0) = -\log \tilde{h}_{\Omega_0}(z, 0).
\]

Since \( N(z, r, g_{\Omega_0}/h_{\Omega_0}, \infty) \leq N(z, r, h_{\Omega_0}, 0) \), we obtain from (4), (5), and the definition of \( \tilde{T} \) that \( T(z, r, f, \infty) \leq \tilde{T}(z, r, f, \infty) \) for \( z \in \Omega \setminus K_f,\Omega_0 \). Moreover, if \( z_1 \in \Omega_0 \setminus K_f,\Omega_0 \) and \( g_{\Omega_0}(z_1, \lambda), h_{\Omega_0}(z_1, \lambda) \) have no common zeros, then by (4) and (5) \( T(z_1, r, f, \infty) = \tilde{T}(z_1, r, f, \infty) \). Thus \( T(z, r, f, \infty) = \tilde{T}(z, r, f, \infty) \) on \( \Omega \setminus (K_f \cup E_1) \), where

\[
E_1 = \bigcup_{j \in \mathbb{N}} \Omega_j \subset \Omega \{z \in \Omega_0 : \exists \lambda_j \in \mathbb{C} : g_{\Omega_0}(z, \lambda_j) = 0 = h_{\Omega_0}(z, \lambda_j)\}.
\]

It remains to show that the set \( E_1 \) is pluripolar. Recalling Definition 2.4, it is enough to show, for any \( \Omega_0 \in \Omega \) an open, bounded ball, that \( E_1 \cap \Omega_0 \) is a pluripolar set. Let \( g_{\Omega_0}, h_{\Omega_0} \) be as before, and set, for \( j \in \mathbb{N} \),

\[
(6) \quad E_{1,\Omega_0,j} = \{z \in \Omega_0 : \text{dist}(z, \partial \Omega_0) < 1/j \text{ and } \exists \lambda \in \mathbb{C} \text{ with } |\lambda| < j \text{ and } g_{\Omega_0}(z, \lambda) = 0 = h_{\Omega_0}(z, \lambda)\}.
\]

Since \( g_{\Omega_0} \) and \( h_{\Omega_0} \) are relatively prime, the set of their common zeros has dimension zero, and the set \( E_{1,\Omega_0,j} \) consists of a finite number of points, and is thus pluripolar. But

\[
E_1 \cap \Omega_0 = \bigcup_{j \in \mathbb{N}} E_{1,\Omega_0,j}
\]
is the countable union of pluripolar sets, and is thus pluripolar [8, Proposition 1.37]. □

The following theorem is related to Theorem 1.41 and Corollary 1.42 of [8].

**Theorem 3.4.** Let $\Omega \subset \mathbb{C}$ be a domain, and let $f$ be meromorphic on $\Omega \times \mathbb{C}$. If the order $\rho(z)$ of the function $r \mapsto T(z,r,f,\infty)$ is at most $\rho_0$ for $z \in \Omega \setminus K_f$ and there is a $z_0 \in \Omega \setminus K_f$ such that $\rho(z_0) = \rho_0$, then there is a pluripolar set $E \subset \Omega \setminus K_f$ such that $\rho(z) = \rho_0$ for $z \in \Omega \setminus (K_f \cup E)$.

**Proof.** Consider $\tilde{T}(z,r,f,\infty)$ on $\Omega \setminus K_f$, and let $\tilde{\rho}(z)$ be the order of the function $r \mapsto \tilde{T}(z,r,f,\infty)$.

Let $\Omega' \subset \Omega \setminus K_f$ be a domain with $z_0 \in \Omega'$. By [8, Proposition 1.40], there is a sequence of negative plurisubharmonic functions $\{\psi_q\}$ on $\Omega'$ such that

$$(\rho(z))^{-1} = \limsup_{q \to \infty} \psi_q(z) \text{ for } z \in \Omega'.$$

Let $(-1/\tilde{\rho})^*(z) = \limsup_{q \to z}(1/\tilde{\rho})(z')$. By [8, Theorem I.27] and Remark 1 following it, $(-1/\tilde{\rho})^* \in \mathrm{PSH}(\Omega')$ and $\rho(z) \leq -1/\rho_0$ on $\Omega'$. By Hartog’s Lemma (e.g. [8, Theorem 1.31]), for any compact set $K \subset \Omega'$ and any $\epsilon > 0$ there is a $T_\epsilon$ such that

$$\psi_q(z) \leq -1/\rho_0 + \epsilon \text{ for } q \geq T_\epsilon, z \in K.$$ 

Thus $-1/\tilde{\rho}(z) \leq -1/\rho_0$ on $\Omega'$. Since this is true for any domain $\Omega' \subset \Omega \setminus K_f$, we have shown that $\tilde{\rho}(z) \leq \rho_0$ on $\Omega \setminus K_f$.

Recall that $T(z_0,r,f,\infty) \leq \tilde{T}(z_0,r,f,\infty)$ by Proposition 3.3. Thus $\tilde{\rho}(z_0) \geq \rho_0$. On the other hand, by our previous discussion $\rho(z) \leq \rho_0$, and thus $\tilde{\rho}(z_0) = \rho_0$.

We return to $\psi_q \in \mathrm{PSH}(\Omega')$ as above and consider $\psi_q + 1/\rho_0$, which is bounded above by $1/\rho_0$. Note that

$$\limsup_{q \to \infty} \psi_q(z) + 1/\rho_0 = -1/\tilde{\rho}(z) + 1/\rho_0 \leq 0$$

and $\limsup_{q \to \infty} \psi_q(z_0) + 1/\rho_0 = 0$. Thus by [8, Proposition 1.39], the set

$$\{z \in \Omega' : \tilde{\rho}(z) < \rho_0\}$$

is pluripolar. Since for any $z \in \Omega \setminus K_f$ we can find a domain $\Omega' \subset \Omega \setminus K_f$ such that $z_0 \in \Omega'$ and $z \in \Omega'$, we have shown that $\tilde{\rho}(z) = \rho_0$ for $z \in \Omega \setminus (K_f \cup E)$ for some pluripolar set $E$. By Proposition 3.3 there is a pluripolar set $E_1$ such that $\rho(z) = \tilde{\rho}(z)$ for $z \in \Omega \setminus (K_f \cup E_1)$. Then $\rho(z) = \rho_0$ on $\Omega \setminus (K_f \cup E_1 \cup E_2)$. Since $E = E_1 \cup E_2$ is a pluripolar set, we have proved the theorem. □

4. “Black box” operators

4.1. “Black box” operators. We recall from [22] the definition of a “black box” scattering operator, and use much of the notation of [22]. For $R > 0$, denote by $B(R)$ the open ball of radius $R$ in $\mathbb{R}^d$ centered at the origin. Let $\mathcal{H}$ be a complex Hilbert space with an orthogonal decomposition

$$(7) \quad \mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^d \setminus B(R_0)).$$
We denote the corresponding orthogonal projections by $u \mapsto u \upharpoonright_{B(R_0)}$ and $u \mapsto u \upharpoonright_{\mathbb{R}^d \setminus B(R_0)}$. The operators $P : \mathcal{H} \to \mathcal{H}$ which we consider are self-adjoint, unbounded with domain $\mathcal{D} \subset \mathcal{H}$, and the Laplacian on $\mathbb{R}^d \setminus B(R_0)$ in the following sense. If $u \in H^2(\mathbb{R}^d \setminus B(R_0))$ and $u$ vanishes near $B(R_0)$, then $u \in \mathcal{D}$ and $\Delta u = Pu$. Moreover, if $u \in \mathcal{D}$ and $u \upharpoonright_{\mathbb{R}^d \setminus B(R')} = 0$ for some $R' \geq R$, then $(Pu) \upharpoonright_{\mathbb{R}^d \setminus B(R')} = 0$. We also assume that there is a $C \geq 0$ such that

$$(8) \quad P \geq -C$$

and that

$$(9) \quad \mathbb{1}_{B(R_0)}(P + i)^{-1}$$

is a compact operator, where $\mathbb{1}_{B(R_0)}$ stands for the characteristic function of $B(R_0)$.

Finally, we shall make an assumption on the reference operator $P_R^\#$, which we define below, just as it is defined in [22]. For $R_1 > R_0$, let

$$\mathcal{H}_R^\# = \mathcal{H}_{R_0} \oplus L^2(\mathbb{T}_{R_1} \setminus B(R_0))$$

where $\mathbb{T}_{R_1}$ is the flat torus obtained by identifying the sides of \{ $x \in \mathbb{R}^d : |x_i| < R_1$, $i = 1, ..., d$ \}. Let $\chi \in C_\infty_c(B(R_0 + (R_1 - R_0)/2))$ be 1 in a neighborhood of $B(R_0)$. Then set

$$P_{R_1}^\# u = P(\chi u) + \Delta((1 - \chi)u).$$

The domain of $P_{R_1}^\#$ is $\mathcal{D}_{R_1}^\# = \{ u \in \mathcal{H}_{R_1}^\# : \chi u \in \mathcal{D}, \ (1 - \chi)u \in H^2 \}$. This definition is independent of the choice of $\chi$ with such properties. For an operator $Q$, denote by $n_{\epsilon,Q}(r)$ the number of eigenvalues of $Q$ with norm at most $r^2$. Our final assumption (stronger than [22, (1.7)]) is

$$(9) \quad n_{\epsilon,P_{R_1}^\#}(r) = O(r^d)$$

and

$$(10) \quad n_{\epsilon,P_{R_1}^\#}(r) - n_{\epsilon,P_{R_0}^\#}(r - 1) \leq C(1 + r^p),$$

for some constants $p < d$, $C > 0$.

This assumption is not satisfied by some operators of interest in scattering theory, for example, the Laplacian on a finite-volume non-compact hyperbolic surface, which may be placed in the black box framework with $d = 1$. However, scattering-theoretic operators which do not satisfy this assumption tend to be those for which the resonance counting function is already better-understood.

If $\mathcal{H}$ denotes a Hilbert space satisfying (7), we shall denote by $\mathcal{BB}_R(\mathcal{H})$ the set of operators which satisfy all of the conditions above with $R_0$ replaced by $R \geq R_0$ everywhere but in (7), and set

$$\mathcal{BB}(\mathcal{H}) = \bigcup_R \mathcal{BB}_R(\mathcal{H}).$$

Finally, we set

$$(10) \quad \mathcal{MBB}_R(\mathcal{H}) = \{ P \in \mathcal{BB}_R(\mathcal{H}) : \limsup_{r \to \infty} \frac{\log N_p(r)}{\log r} = d \}$$

and

$$\mathcal{MBB}(\mathcal{H}) = \bigcup_R \mathcal{MBB}_R(\mathcal{H}).$$

Throughout this paper, $\mathcal{H}$ denotes a Hilbert space satisfying (7).
4.2. Density. Since we put rather few assumptions on our unbounded “black box” operators, there are not many ways to say if two operators are close. We shall only compare operators which act on the same Hilbert space $H$ (they needn’t have the same domain) and shall use the idea of norm resolvent convergence, e.g. [16, VIII.7].

**Theorem 4.1.** Let $d \geq 3$ be odd, and let $P_0 \in \mathfrak{BB}_R$. Then, for $\delta > 0$, there is a second order self-adjoint differential operator $Q$, with coefficients supported in $\{x \in \mathbb{R}^d : R + \delta < |x| < R + 2\delta\}$, and a sequence $\{t_j\}$, $t_j \downarrow 0$, such that $u_{P-t_jQ}(r)$ has order $d$. In particular, $P_{t_j} = P_0 - t_jQ \in \mathfrak{BB}_R + \mathfrak{BB}$ and $P_j \rightarrow P_0$ in the norm resolvent sense. Thus, $\mathfrak{BB}_R$ is dense in $\mathfrak{BB}$ in the topology compatible with norm resolvent convergence.

This subsection is devoted to the proof of this theorem, which uses Theorem 3.4. In order to construct the function $f$ to which we apply Theorem 3.4, we need an operator acting on $H$ with resonance counting function having order of growth $d$. In Lemma 4.3 we show that there is such an operator.

For $P \in \mathfrak{BB}_R$, denote the scattering matrix associated to $P$ by $S_P(\lambda)$, and, for $\lambda \in \mathbb{R}$, set $\sigma_P(\lambda) = (2\pi i)^{-1} \arg \det S_P(\lambda)$, the scattering phase.

Before beginning the proof, we recall a definition of the scattering matrix. Let $P$ be an operator which coincides with the Laplacian on $\mathbb{R}^d$ when applied to functions supported outside a ball of radius $R$ centered at the origin. Moreover, suppose that if $\chi \in C_c^\infty(\mathbb{R}^d)$ is zero on $B(R)$, then $\chi R_P(\lambda)\chi$ is meromorphic in $\lambda$.

We emphasize that it is not necessary that $P$ be self-adjoint.

We represent the scattering matrix for $P$ in terms of the resolvent. There are several possible ways to do this; we use [14, Proposition 2.1], but see also, for example, [13, Section 2] or [32, Section 3]. Modifying slightly the notation of [14], for $\psi \in C_c^\infty(\mathbb{R}^d)$, let

$$E^\psi(\lambda) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{S}^{d-1})$$

be the operator with Schwartz kernel $\psi(x) \exp(\pm i\lambda(x,\omega))$. For $i = 1, 2, 3$, choose $\chi_i \in C_c^\infty(\mathbb{R}^d)$ so that $\chi_i \equiv 1$ if $|x| < 4 + R$ and $\chi_{i+1} \equiv 1$ on the support of $\chi_i$, $i = 1, 2$. Set

$$A_P(\lambda) = \tilde{c}_d \lambda^{d-2} \mathcal{E}_+^\chi(\lambda)[\Delta, \chi_1]R_P(\lambda)[\Delta, \chi_2]^t \mathcal{E}_-^\chi(\lambda).$$

Here $\tilde{c}_d = i\pi(2\pi)^{-d}$ and $^t \mathcal{E}_-^\chi$ denotes the transpose of $\mathcal{E}_-^\chi$. Then the scattering matrix $S_P(\lambda)$ associated to $P$ is given by

$$S_P(\lambda) = I + A_P(\lambda)$$

and the operator $A_P(\lambda)$ is trace class.

Next we define the operator $Q$ which appears in the statement of Theorem 4.1. Let $\chi_\delta \in C_c^\infty(\mathbb{R}^d)$ be a function such that $0 \leq \chi_\delta \leq 1$, $\chi_\delta(x) = 0$ if $||x| - (R + 3\delta/2)|| > \delta/2$, and $\chi_\delta(x) = 1$ for $R + 3\delta/4 \leq |x| \leq R + 5\delta/4$. Set

$$Q = \chi_\delta \Delta \chi_\delta.$$
Combining (14), (15), and (16), we get

\[ \sigma_{P_t}(\lambda) = \sigma_{P_0}(\lambda) + h(t) \lambda^d + O(\lambda^{\max(d-1,p)}) \]

where

\[ h(t) = (2\pi)^{-d} \frac{\text{vol}(B(1))}{\delta_t} \int_{\supp \chi} ((1 - t\chi_\delta^2)^{-d/2} - 1) dx \]

We note that the \( p \) appearing here is the \( p \) for which the operator \( P_0 \) satisfies (9).

**Proof.** Let \( R_1 > R + 2\delta \). Using (9), [1, Proposition 2.1], and well-known asymptotics of the spectral function of differential operators (e.g. [19]),

\[ n_{e,(P_t)_{R_1}}(\lambda) = n_{e,(P_0)_{R_1}}(\lambda) + h(t) \lambda^d + O(\lambda^{\max(d-1,p)}). \]

Note that any eigenfunctions of \( P_0 \) with positive eigenvalue are 0 on \( \mathbb{R}^d \backslash B(R) \). Thus, they are eigenfunctions of \( P_t \) as well. Similarly, any positive eigenvalues of \( P_t \) are eigenvalues of \( P_0 \) as well. Thus

\[ n_{e,P_0}(\lambda) = n_{e,P_t}(\lambda) + O(1). \]

By [1, Theorem 0.1], for \( t \in [0,1] \),

\[ \sigma_{P_t}(\lambda) = n_{e,(P_t)_{R_1}}(\lambda) - n_{e,P_t}(\lambda) - c_d \frac{\text{vol} R_1}{\delta_t} \lambda^d + O(\lambda^{\max(d-1,p)}). \]

Combining (14), (15), and (16), we get

\[ \sigma_{P_t}(\lambda) = \sigma_{P_0}(\lambda) + h(t) \lambda^d + O(\lambda^{\max(d-1,p)}). \]

\[ \square \]

**Lemma 4.3.** There is a \( t_0, 0 < t_0 < 1 \), such that \( n_{P_t}(r) \) has order \( d \) for \( t_0 < t < 1 \).

**Proof.** We will give the proof by contradiction, so suppose there is a sequence \( t_j \uparrow 1 \) such that \( n_{P_{t_j}}(r) \) has order strictly less than \( d \). Let \( \lambda \in \mathbb{C} \) and \( s_P(\lambda) = \det S_P(\lambda) \). Recall that \( \lambda_0 \) is a zero of \( s_P(\lambda) \) if and only if \( -\lambda_0 \) is a pole of \( s_P(\lambda) \), and the multiplicities agree. Moreover, the poles of \( s_P(\lambda) \) coincide, with multiplicity, to the poles of \( R_P(\lambda) \) at most finitely many exceptions. Then for each such \( t_j \), by results of [31]

\[ s_{P_{t_j}}(\lambda) = e^{ig_j(\lambda)} G_j(\lambda)/G_j(-\lambda). \]

Here \( g_j \) is a polynomial of degree at most \( d \) and \( G_j \) is a canonical product vanishing at the zeros of \( s_{P_{t_j}}(\lambda) \) and has order strictly less than \( d \), by assumption. Then, for some \( \epsilon_j > 0 \) and some \( c_j \),

\[ |G_j(\lambda)| \leq c_j \exp(c_j |\lambda|^{d-\epsilon_j}). \]

By the minimum modulus theorem, there is a \( c_j' \) such that

\[ |G_j(\lambda)| \geq (c_j')^{-1} \exp(-c_j' |\lambda|^{d-\epsilon_j}) \]
when $|\lambda| \leq r$ and $\lambda$ lies outside a family of excluded disks, the sum of whose radii does not exceed $r/2$. But then
\begin{equation}
\log |G_j(r_k e^{i\theta})/G_j(-r_k e^{i\theta})| = O(r_k^{d-\epsilon_j}), \quad 0 \leq \theta \leq 2\pi
\end{equation}
for a sequence $r_k \to \infty$. (The sequence $\{r_k\}$ depends on $t_j$, but we suppress this in our notation.)

By [14, Lemma 4.3], there is a constant $C_0$ independent of $t$ such that
\begin{equation}
|\det S_{P_{t_j}}(\lambda)| \leq C_0 e^{C_0|\lambda|^\alpha}, \quad 0 \leq \arg \lambda \leq \pi/5, \quad |\lambda| \geq 1.
\end{equation}
Using this, (17), and (18), we see that
\begin{equation}
\int_0^\lambda \frac{d}{dt} \log (G_j(t)/G_j(-t)) \, dt = O(\lambda^{d-\epsilon_j})
\end{equation}
for some $\epsilon_j > 0$. Using this and (20),
\begin{equation}
\left| \int_0^\lambda \frac{d}{dt} \log s_{P_{t_j}}(t) \, dt \right| \leq C_0 |\lambda|^d + O(\lambda^{d-\epsilon_j}).
\end{equation}
However, by Lemma 4.2, and using the fact that $\sigma_{P_{t_0}}(\lambda) = O(\lambda^d)$ we can find a $t_0$, $0 < t_0 < 1$, such that for $\lambda \in \mathbb{R}_+$,
\begin{equation}
\frac{1}{i} \int_0^\lambda \frac{s_{P_{t_j}}'(\tau)}{s_{P_{t_j}}(\tau)} \, d\tau \, dt \geq (C_0 + 1)\lambda^d + o(\lambda^d)
\end{equation}
whenever $t_0 < t < 1$. This contradicts (22) when $t_j > t_0$.

Let $d \geq 3$ be odd, and let $P_0 \in \mathfrak{B}_R$. Set $P_z = P_0 - zQ$ where $Q$ is as in (13), and let $s(z, \lambda) = \det S_{P_z}(\lambda)$. Note that there is an open neighborhood $U \subset \mathbb{C}$ of $[0, t_0 + (1 - t_0)/2]$ so that $S_{P_z}(\lambda)$, and thus $s(z, \lambda)$, are meromorphic functions of $(z, \lambda) \in U \times \mathbb{C}$. The following lemma is related to [31, Theorem 7], which considers the scattering determinant for general self-adjoint operators.

**Lemma 4.4.** For $z \in U \setminus K_x$, the order $\rho(z)$ of $\lambda \mapsto s(z, \lambda)$ is at most $d$.

**Proof.** For $z \in [0, t_0 + (1 - t_0)/2]$, this result follows directly from [31, Theorem 7]. By the techniques of [31, Section 4], there is an $m \geq d$ so that $\rho(z) \leq m$ for $z \in U \setminus K_x$. By results of [26], $n_{P_z}(r)$ has order at most $d$ for $z \in U$. Then by Hadamard’s Factorization Theorem, if $\rho(z_1) > d$ then $\rho(z_1)$ must be an integer.

Let
\[ \rho_0 = \sup_{z \in U \setminus K_x} \rho(z). \]
If $\rho_0 \leq d$, then we are done. If $\rho_0 > d$, then $\rho_0$ is an integer and there is a $z_0 \in U \setminus K_x$ such that $\rho(z_0) = \rho_0$. Then by Theorem 3.4, there is a pluripolar set $E \subset U \setminus K_x$ such that $\rho(z) = \rho_0$ for $z \in U \setminus (K_x \cup E)$. By Theorem 7 of [31],
\[ \rho(z) \leq d \] for \( z \in [0, t_0 + (1 - t_0)/2] \). But \( K_s \cap E \) is itself a pluripolar set, and \([0, t_0 + (1 - t_0)/2]\) cannot be contained in any pluripolar set in \( \mathbb{C} \). Thus we must have \( \rho_0 \leq d \).

**Proof of Theorem 4.1.** Let \( P_z \) and \( U \) be as described before Lemma 4.4. Let

\[ h(z) = (2\pi)^{-d} \text{vol}(B(1)) \int_{\text{supp } \chi} ((1 - zx^2)^{-d/2} - 1)dx, \]

which is a holomorphic function of \( z \) for some complex neighborhood of \([0, t_0 + (1 - t_0)/2]\) and agrees with our previous definition of \( h(t) \) when \( t \in [0, t_0 + (1 - t_0)/2] \).

By shrinking \( U \) if necessary, we may assume \( h \) is holomorphic on \( U \). Then set

\[ f(z, \lambda) = s(z, \lambda) \exp(-2\pi i h(z) \lambda^d - 2\pi i \sigma P_0(\lambda)). \]

When \( t_1 \in [0, t_0 + (1 - t_0)/2] \), the poles of \( f(t, \lambda) \) coincide, with multiplicity, with the poles of \( R_{P(t)}(\lambda) \) with at most a finite number of exceptions. Moreover, if \( \lambda_1 \) is a pole of \( f(t_1, \lambda) \), then \(-\lambda_1 \) is a 0 of \( f(t_1, \lambda) \), and the multiplicities agree. By Lemma 4.4, the order of \( \lambda \mapsto f(z, \lambda) \) is at most \( d \).

By Theorem 3.4, there is a pluripolar set \( E \subset U \backslash K_f \) such that \( \lambda \mapsto f(z, \lambda) \) is of order \( d \) for \( z \in U \backslash (K_f \cup E) \). Since the restriction of a pluripolar set \( E \subset \mathbb{C} \) to \( \mathbb{R} \) is of Lebesgue measure 0 [15, Section 3.2], for \( j = 1, 2, 3, \ldots \) we may find \( t_j \in U \backslash (K_f \cup E) \) with \( t_j \in (0, 1/j) \). By Lemma 4.2, \( \int_0^1 \frac{d}{d\tau} \log f(t_j, \tau)d\tau = O(r^{\max(d - 1, p)}) \).

Then by Lemma 2.1, \( n(t_j, r, f, \infty) \) must have order \( d \). Thus \( P_j \in \mathfrak{BBB}_{R+2\delta}. \)

**4.3. Genericity.** In this subsection we give a genericity result for “black box” operators. The statement of the result is less natural than that of the genericity results of Theorems 1.3, 1.5, and 1.7 for more restricted classes of operators. Part of the problem is that for “black box” operators we must use a fairly weak topology, in that we consider a sequence of self-adjoint operators to converge to a self-adjoint operator \( A \) if they converge in the norm resolvent sense. It is, however, possible to have \((A_k + i)^{-1} \to (A + i)^{-1}\) with \( A_k \) self-adjoint but \( A \) not self-adjoint. We consider a more restricted class of operators in order to avoid this type of problem.

For a self-adjoint operator \( A \) let \( Q(A) \) denote the domain of the associated quadratic form. Suppose \( \tilde{P} \) is a self-adjoint operator acting on \( \mathcal{H} \) such that \( \tilde{P} + c_0 I > 0 \), and that \( \text{dom}(\tilde{P}) \) is dense in \( \mathcal{H} \). Let

\[ \mathfrak{BB}_{R,c_0,c_1}(\mathcal{H}, \tilde{P}) = \{ P \in \mathfrak{BB}_R(\mathcal{H}) : Q(\tilde{P}) \subset Q(P), P + c_0 I > 0, (\psi, P\psi) \leq c_1 ((\psi, \tilde{P}\psi) + c_0 (\psi, \psi)) \text{ for } \psi \in Q(\tilde{P}) \}. \]

Set

\[ \mathfrak{BB}(\mathcal{H}, \tilde{P}) = \bigcup_{R > 0, c_0 > 0, c_1 > 0} \mathfrak{BB}_{R,c_0,c_1}(\mathcal{H}, \tilde{P}) \]

and

\[ \mathfrak{BBB}(\mathcal{H}, \tilde{P}) = \mathfrak{BB}(\mathcal{H}, \tilde{P}) \cap \mathfrak{BBB}(\mathcal{H}). \]

As an example, one may take, for a smooth, compact set \( \mathcal{O} \subset \mathbb{R}^d \), \( \mathcal{H} = L^2(\mathbb{R}^d \setminus \mathcal{O}) \) and \( \tilde{P} \) to be the Dirichlet Laplacian on \( \mathbb{R}^d \setminus \mathcal{O} \). Then \( Q(\tilde{P}) = H_0^1(\mathbb{R}^d \setminus \mathcal{O}) \).

For \( R \) large enough that \( \mathcal{O} \subset B(R) \) and \( c_0 > 0, c_1 \geq 0 \), \( \mathfrak{BB}_{R,c_0,c_1}(\mathcal{H}, \tilde{P}) \) contains...
Theorem 4.5. Let $d \geq 3$ be odd, and let $\mathcal{P}$ and $\mathcal{H}$ satisfy the conditions above. The set $\mathfrak{B}(\mathcal{H}, \mathcal{P})$ is Baire typical in the set $\mathfrak{B}(\mathcal{H}, \mathcal{P})$ with the topology compatible with norm resolvent convergence.

To prove the theorem, it remains to show that $\mathfrak{B}(\mathcal{H}, \mathcal{P})$ is a $G_\delta$ set, as Theorem 4.1 shows that it is dense. This will require a number of lemmas.

Lemma 4.6. Suppose $P_k \in \mathfrak{B}_{R_0, c_0, c_1}(\mathcal{H}, \mathcal{P})$ with, for a fixed $R_1 > R$,

$$n_{\mathcal{P}^k} (P_k)_{R_1} (r) - n_{\mathcal{P}^k} (P_k)_{R_1} (r-1) \leq C(1 + r^p)$$

for all $k$, and the sequence $\{(P_k + c_0 + 1)^{-1}\}_{k=1}^{\infty}$ converges in norm. Then there is a self-adjoint operator $P$ so that $P_k \to P$ in the norm resolvent sense and $P \in \mathfrak{B}_{R_0, c_0, c_1}(\mathcal{H}, \mathcal{P})$. Moreover, $n_{\mathcal{P}^k} (P_k)_{R_1} (r) - n_{\mathcal{P}^k} (P_k)_{R_1} (r-1) \leq C(1 + r^p)$ and $n_{\mathcal{P}^k} (P_k)_{R_1} (r) \leq C(1 + r^p)$.

Proof. By [17, Theorem X.65], there is a self-adjoint operator $P$ such that $(P_k + c_0 + 1)^{-1} \to (P + c_0 + 1)^{-1}$ strongly. But then $P + c_0 \geq 0$, and using that $(P_k + c_0 + 1)^{-1} \to (P + c_0 + 1)^{-1}$ in norm and a slight modification of [16, Theorem VIII.19], $P_k \to P$ in the norm resolvent sense.

Next we shall check that $P \in \mathfrak{B}_{R_0, c_0, c_1}(\mathcal{H}, \mathcal{P})$. By [16, Theorem VIII.26], if $u$ is in the domain of $P$ then there are $u_k$ in the domain of $P_k$ such that $u_k \to u$ and $P_k u_k \to Pu$. Then, since $P_k$ is the Laplacian on $\mathbb{R}^d \setminus B(R_0)$, so is $P$. Since $\mathbb{1}_{B(R_0)}(P + i)^{-1}$ is the limit of compact operators, it is compact.

For $R_1 > R_0$, it is possible to construct the resolvent of $P^\#_{R_1}$ using $(P + M)^{-1}$ and the resolvent of the Laplacian on the torus, much like the construction of the resolvent in [22] or in Lemma 4.7 below. Since the same can be done for the resolvent of $(P_k)^\#_{R_1}$, one can see that since $P_k \to P$ in the norm resolvent sense, $(P_k)^\#_{R_1} \to P^\#_{R_1}$ in the norm resolvent sense as well. Using this and [16, Theorem VIII.23], if

$$n_{\mathcal{P}^k} (P_k)_{R_1} (r) - n_{\mathcal{P}^k} (P_k)_{R_1} (r-1) \leq C(1 + r^p)$$

then

$$(25) \quad n_{\mathcal{P}^k} (P_k)_{R_1} (r) - n_{\mathcal{P}^k} (P_k)_{R_1} (r-1) \leq C(1 + r^p)$$

if $r^2$ and $(r-1)^2$ are both not eigenvalues of $P^\#_{R_1}$. If one or both is an eigenvalue, then (25) still holds, since the eigenvalues of $P^\#_{R_1}$ are discrete and

$$\lim_{\epsilon \to 0} (n_{\mathcal{P}^k} (P_k)_{R_1} (r + \epsilon) - n_{\mathcal{P}^k} (P_k)_{R_1} (r + \epsilon - 1)) = n_{\mathcal{P}^k} (P_k)_{R_1} (r) - n_{\mathcal{P}^k} (P_k)_{R_1} (r-1).$$

The inequality $n_{\mathcal{P}^k} (P_k)_{R_1} (r) \leq C(1 + r^d)$ follows similarly.
Finally, \((P_k + c_0 + 1)^{1/2} \psi \rightarrow (P_k + c_0 + 1)^{1/2} \psi\) weakly for \(\psi \in Q(\hat{P})\). Thus
\[
(\psi, P\psi) \leq c_1((\psi, \hat{P}\psi) + c_0(\psi, \psi))
\]
for \(\psi \in Q(\hat{P})\), and \(P \in \mathfrak{B}_R^{c_0, c_1}(H, \hat{P})\). \(\square\)

Recall that \(RP(\lambda) = (P - \lambda^2)^{-1}\), with the upper half plane chosen to be the physical half plane. We shall also need the following lemma.

**Lemma 4.7.** Let \(\chi \in C^\infty_0(\mathbb{R}^d)\) be 1 on \(B(R_0)\). Under the assumptions of Lemma 4.6, \(R_{P_k}(\lambda)\chi \rightarrow R_P(\lambda)\chi\) in norm uniformly on compact sets in \(\lambda\) which do not contain poles of \(R_P\). Moreover, if \(\Pi\) is a projection so that \(R_P(\lambda)\Pi\chi\) is regular at \(\lambda_0\), then
\[
\chi R_{P_k}(\lambda)\Pi \chi \rightarrow \chi R_P(\lambda)\Pi \chi
\]
uniformly on some neighborhood of \(\lambda_0\).

**Proof.** We use a construction which has been used many times in the study of resonances—see, for example, [22, 26].

For \(i = 1, 2, 3\), let \(\chi_i \in C^\infty_0(\mathbb{R}^d)\), with \(\chi_1 = 1\) on \(B(R_0)\) and \(\chi_i \chi_{i+1} = \chi_i\) for \(i = 1, 2\). For \(\lambda_0 \in \mathbb{C}\) with \(\text{Im} \lambda_0 > 0\), set
\[
E_P(\lambda) = \chi_2 R_P(\lambda_0) + (1 - \chi_2) R_\Delta(\lambda)(1 - \chi_1).
\]
Although \(E_P(\lambda)\) depends on the choice of \(\lambda_0\), we suppress this in our notation, as eventually \(\lambda_0\) will be fixed. Then
\[
(P - \lambda^2) E_P(\lambda) \chi_3 = \chi_3(I + F_P(\lambda))
\]
with
\[
F_P(\lambda) = (\lambda_0^2 - \lambda^2)\chi_2 R_P(\lambda_0)\chi_3 + [\Delta, \chi_2](R_P(\lambda_0)\chi_3 - R_\Delta(\lambda)(1 - \chi_1)\chi_3).
\]
For \(\lambda \in \mathbb{C}\), \(F_P(\lambda)\) is a compact operator. By choosing \(\lambda_0\) so that \(\text{Im} \lambda_0 > 0\) is sufficiently large, we can ensure that \(\|F_P(\lambda_0)\| < 1/2\) so that \(I + F_P(\lambda_0)\) is invertible. Then, by analytic Fredholm theory, \((I + F_P(\lambda))^{-1}\) is meromorphic, and
\[
\chi_3 R_P(\lambda)\chi_3 = \chi_3 E_P(\lambda)\chi_3 (I + F_P(\lambda))^{-1}.
\]

We can make the same definitions with \(P_k\) in place of \(P\) everywhere, but using the same fixed \(\lambda_0\) for all. Note that \(F_{P_k}(\lambda) \rightarrow F_P(\lambda)\) in norm, and the convergence is uniform for \(\lambda\) in a compact set. Then, for \(\lambda\) in a compact set which does not include any zeros of \(I + F_P(\lambda)\), \(\chi_3 R_{P_k}(\lambda)\chi_3 \rightarrow \chi_3 R_P(\lambda)\chi_3\) uniformly in norm. Using the fact that \(\chi_3 R_P(\lambda)\chi_3\) and \(\chi_3 R_{P_k}(\lambda)\chi_3\) are meromorphic and so obey a maximum principle on sets which contain no poles, we find that the convergence is uniform on compact sets which avoid the poles of \(\chi_3 R_{P_k}(\lambda)\chi_3\).

Since we can choose the support of \(\chi_3\) to be as large as we want (though finite), this proves the first part of the lemma. The second part follows similarly. \(\square\)
For any $Q \in \mathfrak{B}$, define, for $r \in \mathbb{R}_+$,
\begin{equation}
(26) \quad g_Q(r) = \frac{1}{2\pi i} \int_0^r t^{-1} \int_{-t}^t \frac{s_Q(\tau)}{s_Q(\tau)} d\tau dt + \frac{1}{2\pi} \int_0^{\pi} \log |s_Q(re^{i\theta})| d\theta.
\end{equation}

We shall show in Lemma 4.9 that $g_Q(r)$ is related to the zero-counting function for $s_Q(\lambda)$.

Let $R_0 > 0$, let $\mathcal{H}$ denote a Hilbert space satisfying (7) and let $\tilde{P}$ be an operator as above. For $M, q, j, c > 0$, $0 < p < d$, $R_1 > R$, and $C > 0$, set

\[ A_R(M, q, j, c, R_1, \mathcal{H}, \tilde{P}) = \{ P \in \mathfrak{B} \mathfrak{B}_{R_1, c_1}(\mathcal{H}, \tilde{P}) : \]

\[ n_{e, P_{R_1}}(r) - n_{0, P_{R_1}}(r - 1) \leq C(1 + r^p), \quad n_{e, P_{R_1}}(r) \leq C(1 + r^d) \]

\[ \text{and } g_P(r) \leq M(1 + r^d) \text{ for } 0 \leq r \leq j \}.
\]

**Lemma 4.8.** For $M, q, j, c, R_1, \tilde{P}$, if $r \in [0, j]$ and $P \in A_R(M, q, j, c, R_1, \mathcal{H}, \tilde{P})$, then $\mathcal{P} \in A_R(M, q, j, c, R_1, \mathcal{H}, \tilde{P})$ is closed under the topology compatible with norm resolvent convergence.

**Proof.** We wish to show that if $P_k \to P$ in the norm resolvent sense, and $P_k \in A_{R_0}(M, q, j, c, R_1, \mathcal{H}, \tilde{P})$, then $P \in A_{R_0}(M, q, j, c, R_1, \mathcal{H}, \tilde{P})$. The results of Lemma 4.6 mean that we need only show that $g_P(r)$ satisfies the desired bound for $0 \leq r \leq j$.

Since $P_k$ and $P$ are self-adjoint, $|s_{P_k}(\tau)| = 1 = |s_P(\tau)|$ for $\tau \in \mathbb{R}$. Moreover, using notation as in (11), if $\tau \in \mathbb{R} \setminus \{0\}$, $R_P(\tau) |\Delta, \chi_2|$ is regular since any poles of $R_P(\tau)$ with $\tau \in \mathbb{R} \setminus \{0\}$ correspond to eigenvalues, and the corresponding eigenfunctions must be supported in $\overline{B(R)}$. Thus, using the explicit expression for the scattering matrix (cf (11) and (12)) and Lemma 4.7, $s_{P_k}(\tau) \to s_P(\tau)$ and $s_{P_k}'(\tau) \to s_P'(\tau)$ uniformly for $\tau \in [-j, j]$, so that

\[ \frac{1}{2\pi i} \int_0^r t^{-1} \int_{-t}^t \frac{s_{P_k}'(\tau)}{s_{P_k}(\tau)} d\tau dt \to \frac{1}{2\pi i} \int_0^r t^{-1} \int_{-t}^t \frac{s_P'(\tau)}{s_P(\tau)} d\tau dt \]

when $0 \leq r \leq j$.

Note that for $Q \in \mathfrak{B}$, $\int_0^\pi \log |s_Q(re^{i\theta})| d\theta$ is a continuous function of $r \in [0, \infty)$, even at values of $r$ for which $s_Q(re^{i\theta})$ has a zero or pole for some value of $\theta \in [0, \pi]$. Thus it suffices to prove that

\begin{equation}
(27) \quad \int_0^\pi \log |s_{P_k}(re^{i\theta})| d\theta \to \int_0^\pi \log |s_P(re^{i\theta})| d\theta
\end{equation}

when $0 \leq r \leq j$ and $R_P(re^{i\theta})$ has no poles for $\theta \in [0, 2\pi]$. But for such values of $r$, the explicit representation for the scattering matrix in terms of the resolvent and Lemma 4.7 show that

\[ s_{P_k}(re^{i\theta}) \to s_P(re^{i\theta}) \]

and

\[ \log |s_{P_k}(re^{i\theta})| \to \log |s_P(re^{i\theta})| \]

uniformly for $0 \leq \theta \leq \pi$, proving (27).
Set
\[ B_R(M, q, c_0, c_1, p, C, R_1, \mathcal{H}, \tilde{P}) = \bigcap_{j \in \mathbb{N}} A_{R_0}(M, q, j, c_0, c_1, p, C, R_1, \mathcal{H}, \tilde{P}). \]

Since \( A_{R_0}(M, q, j, c_0, c_1, p, C, R_1, \mathcal{H}, \tilde{P}) \) is closed, so is \( B_R(M, q, c_0, c_1, p, C, R_1, \mathcal{H}, \tilde{P}) \).

**Lemma 4.9.** We have
\[ \mathcal{B}(\mathcal{H}, \tilde{P}) \setminus \mathcal{BB}(\mathcal{H}, \tilde{P}) = \bigcup_{(R, M, l, c_0, l_1, R_1, C) \in \mathbb{N}^8} B_R(M, d - 1/l, c_0, c_1, d - 1/l_1, R_1, C, \mathcal{H}, \tilde{P}). \]

**Proof.** Let \( P \in \mathcal{B}(\mathcal{H}, \tilde{P}) \). Define, for \( r \in [0, \infty) \), \( n_{P,+}(r, \infty) \) to be the number of poles of \( s_P(\lambda) \) in the upper half plane with norm less than or equal to \( r \). Set
\[ N_{P,+}(r, \infty) = \int_0^r \frac{n_{P,+}(s, \infty)}{s} ds. \]

Similarly, set \( n_{P,+}(r, 0) \) to the number of zeros of \( s_P(\lambda) \) in the upper half plane with norm at most \( r \), and \( N_{P,+}(r, 0) = \int_0^r \frac{n_{P,+}(s, 0)}{s} ds \). Note that \( n_{P,+}(r, 0) \) and \( N_{P,+}(r, 0) \) have the same order. Moreover, there are only a finite number of poles of \( R_P(\lambda) \) in the upper half-plane, and the zeros of \( s_P(\lambda) \) correspond, with multiplicity, to the poles of \( s_P(-\lambda) \), and these correspond (with a finite number of exceptions) to the poles of the resolvent. Thus \( N_{P,+}(r, 0) \) and \( n_P(r) \) have the same order.

Using intermediate steps from the proof of [5, Lemma 3.2] and generalizing slightly,

\[ \text{(28)} \]
\[ N_{P,+}(r, 0) - N_{P,+}(r, \infty) = \frac{1}{2\pi i} \int_0^r \mu^{-1} \int_{-\mu}^{\mu} \frac{s_P(\tau)}{s_P(\tau)} d\tau dt + \frac{1}{2\pi} \int_0^{\pi} \log |s_P(re^{i\theta})| d\theta \]

since \( s_P(\tau) = 1 \) for \( \tau \in \mathbb{R} \). From \( \text{(28)} \), \( g_P(r) \) and \( N_{P,+}(r, 0) \) have the same order.

Thus, if the order of \( n_P(r) \) is less than \( d \), then for some \( M, l \in \mathbb{N} \),
\[ g_P(r) \leq M(1 + r^{d-1/l}). \]

Then one can find \( R, c_0, c_1, l_1, R_1, C \in \mathbb{N} \) so that
\[ P \in B_R(M, d - 1/l, c_0, c_1, d - 1/l_1, C, \mathcal{H}, \tilde{P}). \]

On the other hand, if \( P \in B_R(M, d - 1/l, c_0, c_1, d - 1/l_1, C, \mathcal{H}, \tilde{P}) \) for some \( l \in \mathbb{N} \) then \( g_P(r) \) is of order at most \( d - 1/l \) and \( P \in \mathcal{BB}(\mathcal{H}, \tilde{P}) \setminus \mathcal{BB}(\mathcal{H}, \tilde{P}). \)

This completes the proof of Theorem 4.5, since Lemma 4.9 shows that \( \mathcal{BB} \) is a \( G_\delta \) set.
5. SECOND ORDER ELLIPTIC DIFFERENTIAL OPERATORS, AND THE PROOFS OF THEOREMS 1.2 AND 1.3

This section is devoted to the proofs of Theorem 1.2 and 1.3. Theorem 1.2 follows from Theorem 3.4 and [2, Theorem 1.2]. Let $R_0 \in \mathcal{R}$. Our goal is to find, for any $\epsilon > 0$, an operator $\tilde{P} \in \mathcal{MP}_R$ such that $\|P_0 - \tilde{P}\|_W < \epsilon$.

We construct a holomorphic family of operators in $\mathcal{P}_R$ as a first step to obtaining a function to which we can apply Theorem 3.4. Although we could do this in a manner similar to that used in the proof of Theorem 4.1, we prefer to use a simpler method, and will avoid enlarging the region for which the operator differs from the Laplacian. Our construction will require an operator $P_1 \in \mathcal{MP}_R$ such that $\chi(P_1 - \lambda^2)^{-1}\chi$ is regular in a neighborhood of $\lambda = 0$ for any $\chi \in C^\infty_c(R^d)$.

$\chi(x) = 1$ when $|x| \leq R$. We begin by noting that if $V \in L^\infty_{comp}(R^d)$ is supported in $\{x : |x| \leq R\}$, then

$$\chi R\Delta + V(\lambda)\chi = \chi R\Delta(\lambda)\chi(I + VR\Delta(\lambda)\chi)^{-1}\chi.$$ 

Since $d \geq 3$, $\chi R\Delta(\lambda)\chi$ is bounded in a neighborhood of $\lambda = 0$. Thus, by choosing $V$ so that $\|V\|_{L^\infty}$ is sufficiently small, we can ensure that $\chi R\Delta + V(\lambda)\chi$ is holomorphic in a neighborhood of 0. By [2, Theorem 1.2] and since the restriction of a pluripolar set $E \subset \mathbb{C}$ to the real line has Lebesgue measure 0 ([15, Section 3.2]), we can choose $V_1 \in C^\infty_c(B(R))$ so that $\Delta + V_1 \in \mathcal{MP}_R$ and $\|V_1\|_{L^\infty}$ is smaller than any fixed positive number. In particular, we can choose $V_1 \in C^\infty(B(R))$ so that $P_1 = \Delta + V_1 \in \mathcal{MP}_R$ and $R_{P_1}(\lambda)$ is regular near $\lambda = 0$. Then set

$$P_z = zP_1 + (1 - z)P_0.$$ 

Note that for $z \in [0, 1]$, $P_z$ is elliptic and self-adjoint. In fact, there is a neighborhood $U_{0, R} \subset \mathbb{R}$ of $[0, 1]$ on which this is true, and a neighborhood $U_{0, C} \subset \mathbb{C}$ of $[0, 1]$ on which $P_z$ is elliptic.

Let $s(z, \lambda) = \text{det} S_{P_z}(\lambda)$. Then $s(z, \lambda)$ is a meromorphic function of $(z, \lambda) \in U_{0, R} \times \mathbb{C}$, since $S_{P_z}(\lambda)$ is. Moreover, $\lambda_0$ is a zero of $s(z, \lambda)$ if and only if $-\lambda_0$ is a pole of $s(z, \lambda)$.

Set $F(z, x, \xi) = z|\xi|^2 + \sum_{j \leq k} (1 - z)a_{jk}\xi_j\xi_k$ and, for $z \in [0, 1]$, set

$$v(z) = (2\pi)^{-d} \left\{ \text{Vol}\{(x, \xi) : F(x, z, \xi) \leq 1 \text{ and } |x| \leq R\} - \text{Vol}\{(x, \xi) : |\xi|^2 \leq 1 \text{ and } |x| \leq R\} \right\}.$$ 

**Lemma 5.1.** There is a complex connected neighborhood $U_{1, C} \subset U_{0, C} \subset \mathbb{C}$ of $[0, 1]$ such that $v(z)$ has a holomorphic extension from $[0, 1]$ to $U_{1, C}$.

**Proof.** Set $\xi = r\omega$, with $\omega \in S^{d-1}$, and write $\omega = (\omega_1, \omega_2, ..., \omega_d)$. Then

$$F(z, x, \xi) = F(z, x, r\omega) = r^2G(z, x, \omega),$$
with \( G(z, x, \omega) = z + \sum_{j \leq k} (1 - z) a_{jk} \omega_j \omega_k \). Note that for \( z \in [0, 1] \), \( G(z, x, \omega) > 0 \). For \( z \in [0, 1] \),

\[
v(z) = (2\pi)^{-d} \left( \int_{|z| \leq R} \int_{\omega \in S^{d-1}} \int_{r G(z, x, \omega) \leq 1} r^{d-1} dr d\sigma(\omega) dx - C_R \right)
\]

\[
= (2\pi)^{-d} \left( \int_{|z| \leq R} \int_{\omega \in S^{d-1}} \int_0^1 G^{-d/2}(z, x, \omega) r^{d-1} dr d\sigma(\omega) dx - C_R \right)
\]

where \( da \) is the usual measure on \( S^{d-1} \) and

\[
C_R = \text{Vol}\{ (x, \xi) : \ |\xi|^2 \leq 1 \text{ and } |x| \leq R \}.
\]

Since \( G(z, x, \omega) \) is holomorphic in \( z \) and takes on values in \( \mathbb{C} \setminus (-\infty, 0] \) in a complex neighborhood of \( [0, 1] \), we can see that \( v(z) \) has a holomorphic extension to a complex neighborhood \( U_{1, \mathbb{C}} \) of \([0, 1]\), and this neighborhood can be chosen to be connected. \( \square \)

**Proof of Theorem 1.2.** Recall that \( P_z \) is elliptic and self-adjoint for \( z \in [0, 1] \). Thus for \( z_0 \in [0, 1] \), the poles of \( s(z_0, \lambda) \) coincide, with multiplicity, with the poles of \( R_{P_{z_0}}(\lambda) \), with at most a finite number of exceptions. Now set

\[
f(z, \lambda) = e^{-s(z)\lambda^d(2\pi i)} s(z, \lambda).
\]

Just as in Lemma 4.4, one can show that the order of \( \lambda \mapsto f(z, \lambda) \) is at most \( d \), and thus the order of \( \lambda \mapsto f(z, \lambda) \) is at most \( d \).

By our choice of \( V_1 \), \( \chi R_{P_1}(\lambda) \chi \) is regular in a neighborhood of \( \lambda = 0 \). Using (11), (12), and

(29)

\[
R_{P_2}(\lambda) = R_{P_1}(\lambda)(I + (P_z - P_1)R_{P_1}(\lambda))^{-1},
\]

there is a neighborhood of \((1, 0) \in U_{1, \mathbb{C}} \times \mathbb{C}\) in which \( s(z, \lambda) \) is regular. Thus \( 1 \not\in K_f \), where \( K_f \) is as defined in Section 3. Moreover, using Lemma 2.1 and the fact that \( P_1 \in \mathcal{M}_R \), we see that \( f(1, \lambda) \) has order \( d \). By Theorem 3.4, then, there is a pluripolar set \( E \subset U_{1, \mathbb{C}} \setminus K_f \) so that \( \lambda \mapsto f(z, \lambda) \) is of order \( d \) for \( z \in U_{1, \mathbb{C}} \setminus (E \cup K_f) \). Using the fact that the restriction of a pluripolar set \( E \subset \mathbb{C} \) to \( \mathbb{R} \) is of Lebesgue measure 0, [15, Section 3.2], we may find \( \bar{z} \in (U_{1, \mathbb{C}} \setminus (E \cup K_f)) \cap [0, 1] \) such that \( 0 < \bar{z} < \epsilon/(\|P_1 - P_0\|_\Psi + 1) \). Moreover, from [1, Theorem 0.1], and well-known results for asymptotics of eigenvalue counting functions on compact manifolds (eg [19]), \( \int_0^\pi \frac{d}{d\theta} \log f(z, t) dt = O(t^{d-1}) \) for \( z \in [0, 1] \). Then by Lemma 2.1, \( n(\bar{z}, r, f, \infty) \) must have order \( d \). Thus \( \bar{P} = P_{\bar{z}} \in \mathcal{M}_\mathbb{R} \), and by our choice of \( \bar{z} \), \( \|P_0 - \bar{P}\|_\Psi < \epsilon \). \( \square \)

We now turn to proving Theorem 1.3. Since we have shown that \( \mathcal{M}_\mathbb{R} \) is dense in \( \mathcal{P}_R \), it only remains to show that \( \mathcal{M}_\mathbb{R} \subset \mathcal{P}_R \) is a \( G_\delta \) set. The proof is similar to that of Theorem 4.5.
If \( Q \in \mathcal{P}_R \) is given by

\[
Q = \sum_{i \leq j} a_{ij}(x) D_{x_i} D_{x_j} + \sum_j b_j(x) D_{x_j} + V,
\]

where \( D_{x_i} = -i \frac{\partial}{\partial x_i} \), set \( \sigma_2^2(Q, x, \xi) = \sum_{i \leq j} a_{ij}(x) \xi_i \xi_j \). Recall that \( g_Q(R) \) is defined by (26). For \( M, q, j, \alpha > 0 \), set

\[
A_R(M, q, j, \alpha) = \{ Q \in \mathcal{P}_R : \sigma_2^2(Q, x, \xi) \geq \alpha |\xi|^2 \text{ and } g_Q(r) \leq M(1 + r^q) \text{ for } 0 \leq r \leq j \}.
\]

The proof of the following lemma is very similar to the proof of Lemma 4.8.

**Lemma 5.2.** For \( M, q, j, \alpha > 0 \), the set \( A_R(M, q, j, \alpha) \) is closed in the \( C^\infty \) topology.

For \( M, q, \alpha > 0 \), set

\[
B_R(M, q, \alpha) = \bigcap_{j \in \mathbb{N}} A_R(M, q, j, \alpha).
\]

Since \( A_R(M, q, j, \alpha) \) is closed, so is \( B_R(M, q, \alpha) \).

**Lemma 5.3.** We have

\[
(\mathcal{MP}_R)^c = \bigcup_{(M, l, m) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}} B_R(M, d - 1/l, 1/m).
\]

The proof of this lemma follows that of Lemma 4.9, and the lemma concludes the proof of the theorem.

6. **Metric perturbations and the Proofs of Theorems 1.4 and 1.5**

The proof of Theorem 1.4 closely resembles that of Theorems 4.1 and 1.2, though this time we must introduce a family of operators which stay within the class of Laplacians associated to metric perturbations of \( \mathbb{R}^d \) for a range of real values of our parameter \( z \).

**Proof of Theorem 1.4.** Let \( g_0 \in \mathcal{S}_R \) and let \( \epsilon > 0 \). Then let \( g_1 \) be the metric perturbation of \( \mathbb{R}^d \) described in [23, Example 3], with \( \mathbb{R}^d \) scaled if necessary so that \( g_1 \) is the standard Euclidean metric outside a ball of radius \( R \) centered at the origin.\(^2\) Then by [23, Example 3], \( g_1 \in \mathcal{W}_R \). Set \( g_z = (1 - z)g_0 + zg_1 \). For \( z \in [0, 1] \), \( g_z \) is a metric on \( \mathbb{R}^d \), agreeing with the standard metric outside of \( B(R) \). Let \( \Delta_{g_z} \) be the associated Laplacian. Using the expression for \( \Delta_{g_z} \), in local coordinates, one can see that \( \Delta_{g_z} \) can be extended to be a second order elliptic differential operator with coefficients depending holomorphically on \( z \) in some connected neighborhood \( U_C \subseteq \mathbb{C} \) of \( [0, 1] \). We will continue to denote this operator by \( \Delta_{g_z} \), though when \( z \in U_C \setminus [0, 1] \), this may not be the Laplacian associated with a positive definite metric.

\(^2\)Note that our \( R \) is different than the \( R \) appearing in [23, Example 3].
For $z \in [0, 1]$, let $\tilde{g}_z dx$ denote the volume form on $\mathbb{R}^d$ associated with $g_z$. Set

$$v(z) = c_d \int_{|x| \leq R} (\tilde{g}_z - 1) dx$$

where $(2\pi)^dc_d$ is the volume of the unit ball in $\mathbb{R}^d$. Then, just as in Lemma 5.1, $v(z)$ has an extension to an open connected complex neighborhood of $[0, 1] \subset \mathbb{C}$. We will call this neighborhood $U_{\mathbb{C}}$ again, shrinking it if necessary. Now set

$$f(z, \lambda) = e^{-\lambda^4v(z)(2\pi i)s(z, \lambda)}$$

where $s(z, \lambda) = \det S_{\Delta g_z}(\lambda)$. As in the proof of Lemma 4.4, the order of $\lambda \mapsto f(z, \lambda)$ is at most $d$.

Next we will show that $1 \notin K_f$, by showing that $s(z, \lambda)$ is holomorphic in a neighborhood of the point $(1, 0)$. To do this, we first note that since $\Delta g_1$ is a nonnegative, self-adjoint operator on $L^2(\mathbb{R}^d, \tilde{g}_1 dx)$, $R_{\Delta g_1}(\lambda)$ is regular at $\lambda = 0$, and thus is holomorphic in a neighborhood of $\lambda = 0$. Then using (11), (12), and the resolvent equation analogous to (29), we see that $s(z, \lambda)$ is holomorphic in $z$ and $\lambda$ in a neighborhood of $(1, 0)$. Moreover, since $g_1 \in \mathcal{MG}_R$, $s(1, \lambda)$ is of order $d$.

The remainder of the proof follows almost exactly the proof of Theorem 4.1.

Proof of Theorem 1.5. To prove Theorem 1.5, then, we need only show that $\mathcal{MG}_R \subset \mathfrak{G}_R$ is a $G_\delta$ set. Using $\tilde{g} dx$ for the volume form on $\mathbb{R}^d$ associated to $g$, we have $\tilde{g}\Delta g(\tilde{g})^{-1}$ is self-adjoint on $L^2(\mathbb{R}^d)$. Then consider sets

$$A_{\mathfrak{G}, R}(M, q, j, \alpha) = \{ g \in \mathfrak{G}_R : \tilde{g}\Delta g(\tilde{g})^{-1} \in A_R(M, q, j, \alpha) \}$$

where $A_R(M, q, j, \alpha)$ is as in (30). Note that if $g_j \in \mathfrak{G}_R$ with $g_j \to g \in \mathfrak{G}_R$, then $\tilde{g}_j\Delta g_j(\tilde{g}_j)^{-1} \to \tilde{g}\Delta g(\tilde{g})^{-1}$ in the $C^\infty$ topology on $\mathfrak{P}_R$. Proceeding then as in the proof of Theorem 4.5, we see that $\mathcal{MG}_R \subset \mathfrak{G}_R$ is a $G_\delta$ set.

7. Star-shaped obstacles and the Proof of Theorems 1.6 and 1.7

7.1. Preliminaries. In this subsection we lay the groundwork for the proof of Theorem 1.6, the density result for star-shaped obstacles. Throughout the section, we work with the following setup. Let $R > 1$ and let $b \in C^\infty(S^{d-1})$ satisfy $1/R \leq b \leq R$. For $z \in \mathbb{C}$, set

$$b_z = Rz + (1 - z)b$$

and let, for $z \in \mathbb{R}$

$$O_z = \left\{ x \in \mathbb{R}^d : |x| \leq b_z \left( \frac{x}{|x|} \right) \right\}.$$
be defined by

$$φ_2(x) = x + (1 - χ(|x| - R)) \left( b_2 \left( \frac{x}{|x|} \right) - R \right) \frac{x}{|x|}. $$

Note that $φ_2(x) = x$ if $|x| > R + 1$, and that $φ_1 : R^d \setminus \overline{B(R)} \rightarrow R^d \setminus \overline{B(R)}$ is the identity.

**Lemma 7.1.** Let $b, b_2, O_z, φ_2$ be as above. Then for $z \in [0, 1]$

$$φ_2 : R^d \setminus \overline{B(R)} \rightarrow R^d \setminus O_z$$

is a diffeomorphism. Let $P_z$ be the differential operator on $R^d \setminus \overline{B(R)}$ defined, for $z \in [0, 1]$, by

$$(P_z f) \circ φ_2^{-1} = Δ(f(φ_2^{-1})).$$

There is a complex neighborhood $U_0$ of $[0, 1]$ such that for $z \in U_0$ there is a second order elliptic differential operator $P_z$ with coefficients depending smoothly on $x \in R^d \setminus \overline{B(R)}$ and holomorphically on $z \in U_0$, with $P_z = P_z$ for $z \in [0, 1]$.

**Proof.** Note that for $z \in [0, 1]$ and $x \in R^d$,

$$\frac{φ_2(x)}{|φ_2(x)|} = \frac{x}{|x|}.$$

We then show that $φ_2$ is one-to-one for $z \in [0, 1]$ by showing that $|φ_2(rω)|$ is an increasing function of $r > 1$ for any $ω \in S^{d-1}$. Note that for $0 \leq z \leq 1$, $0 < b_2 \leq R$, so that

$$\frac{∂}{∂r} |φ_2(rω)| = 1 - χ'(r - R)(b_2 - R) > 0. $$

It is straightforward to check that $φ_2$ maps $R^d \setminus \overline{B(R)}$ onto $R^d \setminus O_z$. Using (34) and the definition of $φ_2$ one can see that the Jacobian of $φ_2$ is nonsingular for $z \in [0, 1]$, hence $φ_2$ is a diffeomorphism.

Let $x = (x_1, ..., x_d)$ be coordinates on $R^d \setminus \overline{B(R)}$ and, for $z \in U_0$, let $y = (y_1, ..., y_d) = φ_2(x)$ be coordinates on $R^d \setminus O_z$. For $z \in [0, 1]$ we can write

$$\tilde{P}_z f(x) = \sum_{j,k} g_{jk}(z,x) D_{x_k} D_{x_j} f(x) + \sum h_j(z,x) D_{x_j} f(x),$$

with

$$g_{jk}(z,x) = \sum_i \left( \frac{∂(φ_2^{-1})_j}{∂y_i} \frac{∂(φ_2^{-1})_k}{∂y_i} \right) φ_2(x) \quad \text{and} \quad h_j(z,x) = \frac{1}{i} \sum_k \frac{∂^2(φ_2^{-1})_j}{∂y_k^2} φ_2(x).$$

Here $φ_2^{-1}$ is the $j$th component of $φ_2^{-1}$. We recall that the Jacobian of $φ_2$ is nonzero for $z \in [0, 1]$. Thus by the inverse function theorem $g_{jk}$ can be expressed as a rational function with nonzero denominator of derivatives of $φ_2$ with respect to $x$. Since $φ_2$ and $b_2$ depend holomorphically on $z$, $g_{jk}(z,x)$ thus has a holomorphic extension to $z \in U_0 \subset \mathbb{C}$, where $U_0$ is a complex neighborhood of $[0, 1]$. The $h_j(z)$ have a similar expression involving first and second derivatives of $φ_2$, and thus
they too have a holomorphic extension to $U_0$. For $z \in U_0$, using these coefficients in the right hand side of (35) gives us an expression for $P_z$.

Recall that $d$ is odd, let $z \in U_0 \subset \mathbb{C}$ and consider the differential operator $P_z$ on $\mathbb{R}^d \setminus \overline{B(R)}$ with Dirichlet boundary conditions. We shall continue to denote this by $P_z$. The spectrum of the operator $P_z$ may not be contained in $\mathbb{R}$. However, $P_z$ is a compactly supported perturbation of the Laplacian on $\mathbb{R}^d \setminus \overline{B(R)}$ and its continuous spectrum is contained in $[0, \infty)$. In fact, $P_z$ satisfies all the conditions of [26] (see also the related [22]), so that, for $\chi_1 \in C_c^\infty(\mathbb{R}^d)$, $\chi_1 R_{P_z}(\lambda)\chi_1$ has a meromorphic continuation from $\text{Im} \lambda > 0$ to the complex plane. In fact, $\chi_1 R_{P_z}(\lambda)\chi_1$ is meromorphic in $(z, \lambda) \in U_0 \times \mathbb{C}$.

The scattering matrix $S(z, \lambda)$ for $P_z$ can be defined as in (11) and (12). Then $s(z, \lambda) = \det S(z, \lambda)$ is a meromorphic function of $(z, \lambda) \in U_0 \times \mathbb{C}$.

**Lemma 7.2.** If $z \in [0, 1]$, then there is a constant $c_d \neq 0$ such that as $\lambda \to \infty$, $\lambda \in \mathbb{R}$,

$$\frac{1}{i} \log s(z, \lambda) = c_d \text{Vol}(O_z)\lambda^d + O(\lambda^{d-1}).$$

**Proof.** Recall that for $z \in [0, 1]$, $\phi_z : \mathbb{R}^d \setminus \overline{B(R)} \to \mathbb{R}^d \setminus O_z$ is a diffeomorphism which is the identity for $|x|$ sufficiently large. Recalling the definition of $P_z$, the scattering matrix for $P_z$ on $\mathbb{R}^d \setminus \overline{B(R)}$ with Dirichlet boundary conditions is the same as the scattering matrix for the Laplacian on $\mathbb{R}^d \setminus O_z$ with Dirichlet boundary conditions. The asymptotics for the scattering phase, $\frac{1}{i} \log s(z, \lambda)$, are well-known in this setting (e.g. [10, 18] and references).

Let

$$v(z) = \frac{1}{d} \int_{\omega \in S^{d-1}} (Rz + (1 - z)b(\omega))^d d\sigma(\omega)$$

for $z \in U_0$. This is a holomorphic function of $z \in U_0$ which agrees with $\text{Vol}(O_z)$ for $z \in [0, 1]$ (cf. Lemma 5.1).

**Proposition 7.3.** Let $b_z, s(z, \lambda), v(z)$ be as defined above and let

$$f(z, \lambda) = s(z, \lambda) \exp(-ic_d v(z)\lambda^d).$$

Then $\lambda \mapsto f(z, \lambda)$ is of order $d$ for $z \in U_0 \setminus (K_f \cup E)$, where $E$ is a pluripolar set.

**Proof.** Note that $\phi_1$ is the identity mapping on $\mathbb{R}^d \setminus \overline{B(R)}$, so that $P_1$ is the Laplacian on the exterior of the ball of radius $R$. We first show that $1 \notin K_f$, where $K_f$ is as defined in Section 3. The resolvent $R_{P_z}(\lambda)$ is holomorphic in a neighborhood of $\lambda = 0$, so $R_{P_z}(\lambda)$ is holomorphic in $z$ and $\lambda$ in a neighborhood of the point $(1, 0)$. Thus $s(z, \lambda)$ and $f(z, \lambda)$ are holomorphic in a neighborhood of $(1, 0) \in U_0 \times \mathbb{C}$, and $1 \notin K_f$.

Using an argument as in Lemma 4.4, the order of $\lambda \mapsto f(z, \lambda)$ is at most $d$ for $z \in U_0 \setminus K_f$. Since the resonance counting function for the Dirichlet Laplacian on the exterior of the ball has order of growth $d$ ([28] or [24, Theorem 1]), using
Lemma 2.1 and Lemma 7.2, we see that \( f(1, \lambda) \) has order \( d \). Then Theorem 3.4 finishes the proof of the proposition.

7.2. **Proof of Theorem 1.6.** Our proof of Theorem 1.6 uses the notation introduced in the previous subsection.

**Proof of Theorem 1.6.** Let \( R > 1 \), \( \mathcal{O}_0 \subset \mathcal{O}_{ss,R} \) and let \( \epsilon > 0 \). We wish to find \( \mathcal{O}^* \in \mathfrak{M}\mathcal{O}_{ss,R} \) with \( \text{dist}_{ss}(\mathcal{O}, \mathcal{O}^*) < \epsilon \).

Let \( b \in C^\infty(S^{d-1}) \) be such that

\[
\mathcal{O}_0 = \left\{ x \in \mathbb{R}^d : |x| \leq b \left( \frac{x}{|x|} \right) \right\}.
\]

Then necessarily \( 1/R \leq b \leq R \). Define \( b_{\epsilon} \) as in (31), and \( \phi_z, P_z, U_0, s(z, \lambda), \) and \( f(z, \lambda) \) as in Section 7.1. By Proposition 7.3, there is a pluripolar set \( E \) such that \( \lambda \mapsto f(z, \lambda) \) is of order \( d \) for \( z \in U_0 \setminus (K_f \cup E) \). But the restriction of \( E \cup K_f \) to \( \mathbb{R} \) is of Lebesgue measure 0 [15, Section 3.2], so we may find \( z^* \in (U_0 \setminus (K_f \cup E)) \cap [0, 1] \) such that \( 0 < z^* < \epsilon(1 + \|R - b\|_{C^\infty})^{-1} \). Using the fact that \( f(z, \lambda) = s(z, \lambda) \exp(-ic_d(v(z)\lambda^d)) \) and Lemmas 2.1 and 7.2, \( n(r, z^*, f, \infty) = n(r, r^*, s, \infty) \) is of order \( d \). But \( s(z^*, \lambda) \) is the determinant of the scattering matrix for the Dirichlet Laplacian on \( \mathbb{R}^d \setminus \mathcal{O}^* \), where

\[
\mathcal{O}^* = \left\{ x \in \mathbb{R}^d : |x| \leq b_{2^*} \left( \frac{x}{|x|} \right) \right\}.
\]

Thus \( \mathcal{O}^* \in \mathfrak{M}\mathcal{O}_{ss,R} \) and, by our choice of \( z^* \), \( \text{dist}_{ss}(\mathcal{O}^*, \mathcal{O}) < \epsilon \).

7.3. **Proof of Theorem 1.7.** We proceed as in Section 4.3 or 5. For \( \mathcal{O} \in \mathcal{O}_{ss,R} \), set \( S_{\Delta \mathcal{O}}(\lambda) \) to be the scattering matrix associated with the Dirichlet Laplacian on \( \mathbb{R} \setminus \mathcal{O} \), and \( s_{\mathcal{O}}(\lambda) = \det S_{\Delta \mathcal{O}}(\lambda) \). For \( r \in \mathbb{R}_+ \), set

\[
g_{\mathcal{O}}(r) = \frac{1}{2\pi i} \int_0^r t^{-1} \int_{-t}^t \frac{s_{\mathcal{O}}(r)}{s_{\mathcal{O}}(t)} dt \, dt + \frac{1}{2\pi} \int_0^\pi \log |s_{\mathcal{O}}(re^{i\theta})| \, d\theta.
\]

For \( R \in \mathbb{R}_+, M, q, j > 0 \), set

\[
A_{ss,R}(M, q, j) = \{ \mathcal{O} \in \mathcal{O}_{ss,R} : g_{\mathcal{O}}(r) \leq M(1 + r^q) \text{ for } 0 \leq r \leq j \}
\]

in analogy with (30). The proof of the following lemma is similar to that of Lemma 5.2.

**Lemma 7.4.** For \( M, q, j > 0 \), the set \( A_{ss,R}(M, q, j) \) is closed in the \( C^\infty \) topology.

**Proof.** Let \( \mathcal{O}_k \in A_{R}(M, q, j) \), and suppose \( \mathcal{O}_k \to \mathcal{O} \) in the \( C^\infty \) topology. Then clearly \( \mathcal{O} \in \mathcal{O}_{ss,R} \). Suppose

\[
\mathcal{O}_k = \left\{ s \in \mathbb{R}^d : |s| \leq b_k \left( \frac{x}{|x|} \right) \right\}
\]

and

\[
\mathcal{O} = \left\{ s \in \mathbb{R}^d : |s| \leq b \left( \frac{x}{|x|} \right) \right\}.
\]
Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi' \geq 0$, $\chi(t) = 0$ for $t < 0$ and $\chi(t) = 1$ for $t \geq 1$. Define $\psi_k : \mathbb{R}^d \setminus \overline{B(R)} \rightarrow \mathbb{R}^d \setminus O_k$ by

$$\psi_k(x) = x + (1 - \chi(|x| - R)) \left( b_k \left( \frac{x}{|x|} \right) - R \right) \frac{x}{|x|},$$

which is similar to (33). For $f$ defined on $\mathbb{R}^d \setminus \overline{B(R)}$ with Dirichlet boundary conditions, define $Q_k f$ via

$$(Q_k f) \circ \psi_k^{-1} = |J_{\psi_k^{-1}}|^{-1/2} \Delta(|J_{\psi_k^{-1}}|^{1/2}(f \circ \psi_k^{-1})).$$

Here $J_{\psi_k^{-1}}$ is the Jacobian of $\psi_k^{-1}$. With the Dirichlet boundary conditions, $Q_k$ is self-adjoint on $\mathbb{R}^d \setminus \overline{B(R)}$. Define $\psi$ and $Q$ analogously, with $b$ in place of $b_k$.

Using (36) and the fact that $\psi_k$ is the identity outside of a compact set, we have $s_{\Delta_O}(\lambda) = s_{Q_k}(\lambda)$, where $s_{Q_k}$ is the determinant of the scattering matrix associated with $Q_k$. Now the coefficients of the differential operator $Q_k$ approach the corresponding coefficients of the differential operator $Q$ in the $C^\infty$ topology. Thus, following the outline of the proof of Lemma 4.8, we prove the lemma.

For $M, q > 0$, set

$$B_R(M, q) = \bigcap_{j \in \mathbb{N}} A_R(M, q, j).$$

The set $B_R(M, q)$ is closed. The following lemma finishes the proof of Theorem 1.7.

**Lemma 7.5.** For $R > 0$,

$$(\mathfrak{M}_s \Omega)_{ss,R} = \bigcup_{(M,l) \in \mathbb{N} \times \mathbb{N}} B_R(M, d - 1/l).$$

We omit the proof of this lemma, since it follows almost exactly the proof of Lemma 4.9. It is in fact somewhat simpler, since $s_{\Delta_O}(\lambda)$ cannot have any poles in the upper half plane.

**References**


DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MISSOURI
COLUMBIA, MISSOURI 65211
E-MAIL: tjc@math.missouri.edu