Math 426 Homework 2

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1. Construct infinitely many different solutions of the initial value problem

\[ \dot{x} = x^{1/3}, \quad x(0) = 0. \]

Why does the Existence and Uniqueness Theorem for differential equations fail to apply in this case?

2. Prove that every solution of the differential equation \( \dot{x} = \sin^2(x) \) exists for all time; that is for every \( t \in (-\infty, \infty) \). What about solutions of \( \dot{x} = x^4/(x^2 + 1) \)?

3. There is a famous story about someone who wrote a thesis about the nice properties of Hölder functions with Hölder constants larger than one. A function \( f \) such that \( |f(x) - f(y)| \leq L|x - y|^\alpha \) for all \( x \) and \( y \), where \( L \geq 0 \) and \( 0 \leq \alpha \leq 1 \) are constants, is called a Hölder function with Hölder constant \( \alpha \). Such a function is called Lipschitz (as we know) in case \( \alpha = 1 \). What did the student learn (the hard way) at his thesis defense? That is, why is the class of “Hölder functions” with Hölder constants \( \alpha > 1 \) uninteresting?

4. (1) Suppose that \( \eta : \mathbb{R}^n \to \mathbb{R}^n \) is a Lipschitz function with Lipschitz constant \( \alpha \) and \( 0 \leq \alpha < 1 \). Prove that the function \( F : \mathbb{R}^n \to \mathbb{R}^n \) given by \( F(x) = x + \eta(x) \) is bijective. Hint: Use a contraction argument.
(2) Prove that \( F \) is a homeomorphism; that is, \( F \) has a continuous inverse. Hint: Here are some possible approaches. The more abstract approaches are more difficult to execute, but they might teach you something. (a) By quoting a famous theorem, you can conclude that \( F \) has a continuous inverse. What theorem is it? (b) First prove that
$F$ is proper; that is, the inverse image under $F$ of each compact subset of $\mathbb{R}^n$ is compact. Then prove that a continuous bijective proper map on $\mathbb{R}^n$ has a continuous inverse. (c) Prove that the inverse of $F$, which exists because $F$ is bijective, is Lipschitz and therefore continuous.