I tried to be careful not to make a mistake on this exam. But, if you think a problem is incorrectly stated, you should let me know or you can explain in your report what you think is wrong with the problem. Finding errors is good! It shows that you understand what you are doing. I hope you enjoy doing some of the problems on this exam. Good luck.

1. Write a first-order system that is equivalent to the differential equation

\[ \ddot{x} + \dot{x}(x^2 + \dot{x}^2 - 1) + x = 0. \]

Show that the system has a limit cycle. What is the stability type of the limit cycle?

2. Consider the one-dimensional map \( f(x) = r \sin(\pi x) \) as a dynamical system on the unit interval, where the parameter \( r \) satisfies \( 0 < r \leq 1 \). Use computer experiments to show that this map goes through the period doubling route to chaos. Also, use computer experiments to compute an estimate of the Feigenbaum number (see Exercises 10.6.1 and 10.6.2).

3. (a) Recreate Figure 12.5.6. The system is

\[ \ddot{x} + \lambda \dot{x} - x + x^3 = F \cos \omega t. \]

The figure shows the attractor for the Poincaré map corresponding to the parameter values \( \lambda = 0.25 \), \( \omega = 1 \) and \( F = 0.40 \). (b) Set \( \lambda = 0.25 \) and \( \omega = 1 \) and consider \( F \) as a bifurcation parameter. Explore and report on the dynamics of the system as \( F \) varies over the range \( 0 \leq F \leq 1 \). Pick no more than six parameter values that correspond to different dynamics, make pictures and explain in words what each picture is supposed to show.

4. Do problem 12.1.9 on the dynamical system obtained by iteration of the standard map

\[ P(x, y) = (x + y + k \sin x, y + k \sin x). \]
It is very important to make sure that your code evaluates the variables modulo $2\pi$. Your computer window should be $[0, 2\pi] \times [0, 2\pi]$ or $[-\pi, \pi] \times [-\pi, \pi]$. Iterates that fall outside the window are moved back into the window by adding or subtracting an appropriate multiple of $2\pi$.

5. Consider the system

$$\dot{x} = ax + y - x^3, \quad \dot{y} = -x + ay$$

with parameter $a$. (a) Write a pencil and paper argument to show that a stable limit cycle appears for small $a > 0$ and confirm with a computer experiment. (b) As the parameter $a$ is increased, the limit cycle eventually disappears. Use computer experiments to determine the approximate value of $a$ where this bifurcation occurs? Describe the bifurcation in detail including hand-drawn figures if necessary. Hint: Look at all of the rest points and their stable and unstable manifolds as you track the limit cycle.

6. Choose a (nonlinear) dynamical system that has come up in one of your other classes, in a research project, or from some source other than our class or the Strogatz book, describe where your model system comes from, and analyze it using the methods you have learned in this class. Report on your findings.