ON PEPPERONI PIZZAS, HAM SANDWICHES, AND OTHER IMPORTANT MATHEMATICAL PROBLEMS

By Patricia Caldwell

Do your students know when they sit down to a stack of pancakes in the morning that it is possible, with one straight cut of their knife, to cut all 3 pancakes exactly in half? Do they know when they have to share their ham sandwich with a friend that it is possible, with one straight cut of their knife, to cut the sandwich into two pieces so that each piece contains exactly half the ham and half the bread? Do your students know that if they comb their hair so it lays down, the crown on the back of their head is not there just by accident; but, in fact, it must be there? Most importantly, do they know that each of these statements, and more, can be proved mathematically?

I have always felt that one of the problems with mathematics is that you can't talk about it with friends – "nonspecialists" that is. An English student can come home from school and say to his parents, "I learned something interesting in school today. Did you know that in Shakespeare's will he left his *second* best bed to his wife? No one seems to know what happened to his best bed." Or, a chemistry student can say to his friends, "Did you know that 99% of the earth's crust is made up of just 8 elements?" But what does a mathematics student say? The statement "I'll bet you don't know how to add fractions" would probably be followed with a reply of "who cares?" Or a statement such as "I learned how to work with trigonometric identities today" would probably evoke the response "I couldn't possibly understand that. I was never good at math." In this article we will look at some unusual "applications" of mathematics which will not only give your students something to talk about with their friends, but should also help to fix the underlying mathematical concepts in their minds.

It would not surprise anyone if you told them that it is possible to cut one pancake exactly in half with one straight cut. Mathematically we would state this fact is:

Theorem I: If A is any area in the plane, there is a straight line in the plane which bisects A.

In a strict mathematical sense, "area" here means a portion of the plan included in a simple closed curve. But you can easily rely on the student's intuitive notions of area when discussing the theorem. The following "proof" of this theorem is easily understood by any student who has studied the Intermediate Value Theorem. For high school students, I usually give an intuitive geometric "proof" of the Intermediate Value Theorem before giving these results.

"Proof" of Theorem I: First we fix a directed line segment XY in the plane from which to measure angles (see figure 1). Next take any ray XZ which makes

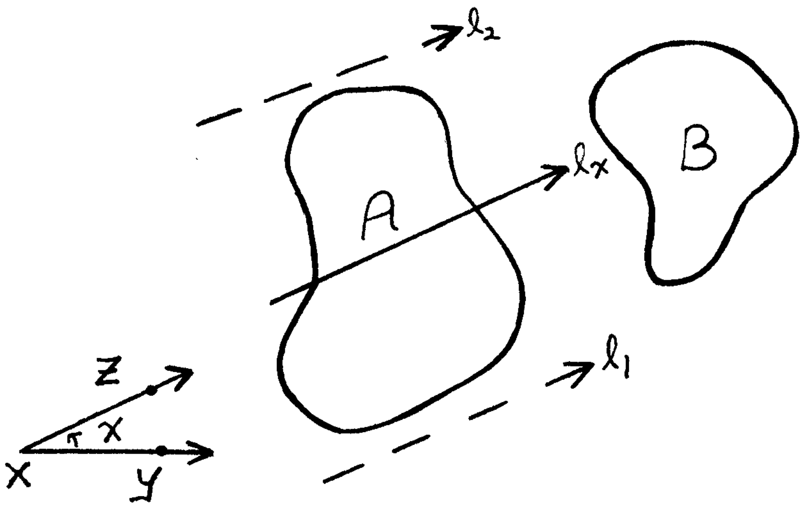

Figure 1

the angle x with XY. Now let ℓ_1 be a directed line with the direction of XZ and lying entirely on one side of A and move this line parallel to itself (and through A) until it is entirely on the other side of A. If we define a function whose value is defined to be the area of A to the left of the line minus the area to the right of the line, then this function is continuous, positive for the position ℓ_1 and negative for the position ℓ_2. Hence, by the Intermediate Value Theorem, it must be zero for some intermediate position ℓ_x. That is, ℓ_x must bisect A.

Now don't miss the opportunity to point out some important lessons to be learned from this "proof." First, and foremost, it is not a proof. We have used the concepts of area to the right and left of the line, without making it clear what this means. We have used a "function" whose domain is undefined and not clear. We have said this function is "continuous" without proving it. Second, you can point out that this is a good example of a stronger theorem proved within a theorem. That is, we have actually shown that if we have a plane area A and a line segment ℓ, then there is a line segment in the plane *parallel to* ℓ which bisects A. In other words, you can turn your plate around to any position and still be able to make one straight line cut *toward yourself* which will cut your pancake exactly in half.

If you want to cut your pancake into 4 pieces of equal area, it would suffice to cut it in half and then use the above theorem to cut each of the halves in half. However, this requires wasted energy since you must make 3 cuts with your knife. Actually, you can cut your pancake into 4 pieces of equal area by making two stright line cuts with your knife. You can even require the cuts to be perpendicular to one another, as the following theorem shows.

Theorem II: If A is an area in the plane, there are two perpendicular lines on the plane which divide A into 4 equal areas.

"Proof" of Theorem II: With the notation of the proof of Theorem I, for each x, we use the proof of Theorem I to find ℓ_x. Now use the theorem again to find the line ℓ_{x+90} which is perpendicular to ℓ_x and also bisects A. (See figure 2). Then we

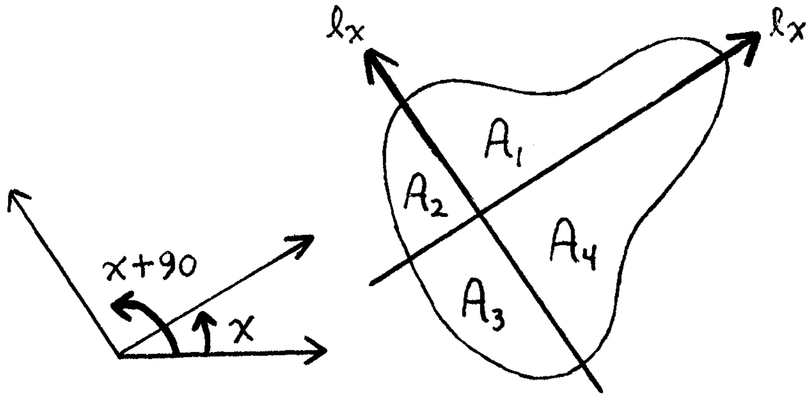

Figure 2

know:

$A_1 + A_2 = A_3 + A_4$

$A_2 + A_3 = A_1 + A_4$

Subtracting the second equation from the first gives

$A_1 - A_3 = A_3 - A_1$

That is, $A_1 = A_3$ and so $A_2 = A_4$. Then to prove the theorem, we must find an angle x so that $A_1 = A_2$ for that x. Since each area A_i, $1 \leqslant i \leqslant 4$ depends upon the angle x, we will write $A_i(x)$ for the areas given in figure 2 for each x. Define a function $f\colon [0, 90°] \to R$ by: $f(x) = A_1(x) - A_2(x)$. Since $f(0) = A_1(0) - A_2(0)$ and $f(90) = A_1(90) - A_2(90) = A_2(0) - A_3(0) = A_2(0) - A_1(0)$, it follows that f has opposite signs at 0 and 90°. Therefore, since f is continuous, there is an x so that $f(x) = 0$. That is, $A_1(x) = A_2(x)$. Hence, ℓ_x and ℓ_{x+90} are the desired lines.

The remarks following Theorem I apply here also, except that our function in this "proof" has a specified domain [0, 90°].

Nobody, however, eats only one pancake for breakfast. And it is not so clear if you have two pancakes on your plate (which do not overlap) that with one straight cut of your knife you can cut both pancakes exactly in half. The following theorem shows that this can also be done:

Theorem III: If A and B are any two areas in the plane, then there exists a straight line in the plane which bisects A and B simultaneously.

As we shall see, the proof of Theorem III is similar to the above. But first note that if the two pancakes were exact circles, we could solve this problem merely by drawing the straight line through their centers. This line will cut across the diameter of each circle and hence will bisect each circle.

"Proof" of Theorem III: Using the notation of the proof of Theorem I, for each $0 \leqslant x \leqslant 180°$, define f(x) to be the area of B to the right of ℓ_x minus the area of B to the left of ℓ_x. (See figure 1) Suppose f(0) is positive.
That is, the line which has the direction of XY and bisects A has more of B to the right of it than to the left of it. If x increases to 180°, then ℓ_{180} has direction YX, bisects A, and is the same as ℓ_0 but oppositely directed (i.e. "left" and "right" are interchanged). Therefore, f(180°) is the same numerically but has the opposite sign of f(0). Again by the Intermediate Value Theorem, $f(\alpha) = 0$ for some $0 \leqslant \alpha \leqslant 180°$. That is, ℓ_α bisects A and B simultaneously.

If we have 3 non-overlapping pancakes on our plate we have a problem. For it is not possible in general to cut 3 pancakes in half with one straight cut of our knife. To see this, we note that the only line which bisects a circle is a diameter of the circle. If we have 3 pancakes on our plate which are exact circles, any line which cuts all 3 in half must go through all 3 centers. That is, if the centers of these 3 circles do not lie on a straight line, then we cannot bisect all three of them with one straight line. Theorems I-III above are often referred to in Topology Courses as "The Pancake Theorems."

There is one problem in all of this. We have assumed that all our pancakes are perfectly flat and to cut them in half we merely need to cut the top surface areas in half. Also, we don't know what happens if they are stacked up. Are we now forced to eat our breakfast without being able to "Mathematically" cut our pancakes in an orderly fashion? No! Luckily for us, the above theorems generalize to three and higher dimensions.

Theorem IV: Given three volumes in space, there exists a single plane which bisects all three simultaneously.

The proof of Theorem IV can also be done using the Intermediate Value Theorem and a little ingenuity, but we will not cover it here. For higher dimensions, the theorem requires more advanced techniques. What does Theorem IV say? For one, it says that if you put 3 pancakes on your plate in the morning (overlapping or not) you can divide all three of them in half (by volume) by making one straight line cut with your knife. Or, more importantly for your friends, if you have a ham sandwich to share, it is possible to make one straight cut through the sandwich so that each piece will contain exactly half of the top slice of bread and half of the bottom slice of bread and half of the ham in your sandwich. For this reason, Theorem IV is sometimes referred to as the "ham-sandwich" theorem. Be careful, however.

If you make this a ham *and* cheese sandwich, the theorem may fail. If your students think this is intuitively clear, remind them that this theorem holds even if they are very sloppy at making a sandwich. That is, even if the top slice of bread is rotated 45° so that its corners are sticking out over the sides of the bottom slice of bread and at the same time the ham is hanging out over one cornor of the bread. For homework, let the students work on the "ham and cheese" problem:

Problem I: Suppose I have a ham and cheese sandwich. Is it possible with one straight cut of my knife to cut the sandwich in half so that each piece has exactly half the ham, half the cheese, and half the bread, but not necessarily requiring that each piece contain exactly half of the upper or lower slices of bread?

It is not difficult to see that this can be done by the above techniques. Basically what is happening in this problem is that we need one dimension to cut the ham in half, a second dimension to cut the cheese in half, and a third dimension to cut the volume of bread in half.

By now the students should be able to solve the important pepperoni pizza problem. Namely, if the pizza has only two pieces of pepperoni on it then they can make one straight line cut with the knife and have two pizzas, each of which contains half the pizza and half of *each* piece of pepperoni. But with three pieces of pepperoni, they may not be able to do it. However, if they have any number of pepperoni and want to cut the pizza so that each part contains half the pizza and half the total amount of pepperoni (but not necessarily half of each piece of pepperoni), this can always be done.

We end by using a somewhat more advanced result on vector fields to prove that you must have a crown on your head. Simply put, a vector field v is a function which assigns to each point x in a region (in the plane or space) a vector vx issuing from it. For example, the velocity vectors of the winds at each point on the surface of the earth form a vector field. If v is a vector field defined on a sphere S in space, we say that v is tangent to S if for each point x in S, the corresponding line segment vx determined by the vector field is tangent to S. We can associate with v a mapping f of S into space by defining f(x) to be the endpoint of the vector whose tail is at the center of the sphere and which is parallel and is equal in length to vx. We say that the vector field v is *continuous* if the function f is continuous. We now have the following theorem:

Theorem V: Let v be a continuous vector field defined over a sphere S and tangent to S. Then there is at least one point x of S to that vx = 0.

Vector field tangent to spheres are often thought of as flows and the theorem merely says that any steady flow on a spherical surface has at least one stationary point. Looked at in terms of our vector field of wind velocities above, the theorem says that at any moment in time, there must be some place on earth where the wind is not blowing. If we consider our "spherical" head, and if we want our hair to lay down in some kind of "continuous" fashion (i.e. we don't want it messed up) then the theorem says there must be at least one "stationary" point on your head. The stationary point is your crown. Of course, there is one problem with this theorem. It does not say that your crown must be on the back of your head. If fact, it may be on your chin. The theorem only says that a "spherical head" with a "continuous flow of hair" must have a crown some place.

I would like to continue but I'm in a hurry to get to the library. I have a large number of people coming to a party in my small apartment tonight and I need to look up a recent amazing result by the mathematician Clifford A. Kottman. Using his theorem, one can prove that in an (infinite dimensional) spherical room of radius 1, it is always possible to put an infinite number of people so that the distance between any two people is strictly greater than 1.

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