Redundancy for localized frames

Radu Balan
University of Maryland, College Park, MD 20742
rvbalan@math.umd.edu
Pete Casazza
University of Missouri, Columbia, MO 65211
pete@math.missouri.edu
Zeph Landau
University of California, Berkeley, CA 94720
landau@eecs.berkeley.edu

January 12, 2009

Abstract

Redundancy is the qualitative property which makes Hilbert space frames so useful in practice. However, developing a meaningful quantitative notion of redundancy for infinite frames has proven elusive. Though quantitative candidates for redundancy exist, the main open problem is whether a frame with redundancy greater than one contains a subframe with redundancy arbitrarily close to one. We will answer this question in the affirmative for ℓ^1-localized frames. We then specialize our results to Gabor multi-frames with generators in $M^1(\mathbf{R}^d)$, and Gabor molecules with envelopes in $W(C, l^1)$. As a main tool in this work, we we answer a longstanding question in finite dimensional frame theory by showing there is a universal function $g(x, y)$ so that for every $\epsilon > 0$, every Parseval frame $\{f_i\}_{i=1}^M$ for an N-dimensional Hilbert space H_N has a subset of fewer than $(1+\epsilon)N$ elements which is a frame for H_N with lower frame bound $g(\epsilon, \frac{M}{N})$. The result of this work is the first meaningful quantitative notion of redundancy for a large class of infinite frames.

The first author was supported by NSF DMS 0807896 and the second author was supported by NSF DMS 0704216

1 Introduction

A basis $\{x_i\}_{i\in I_0}$ for a Hilbert space H (finite or infinite) and an index set I_0 provides a decomposition of any element $x \in H$ as a *unique* linear combination of the basis elements: $x = \sum_{i\in I_0} c_i x_i$. For many applications, this uniqueness of decomposition is the feature that makes bases such a useful structure. However, there are fundamental signal processing issues for which the uniqueness of the coefficients $\{c_i\}_{i\in I_0}$ for a given element $x \in H$ is not a desired quality. These include the following two tasks: a) finding ways to represent elements when some of the coefficients c_i are going to be subject to loss or noise, and b) finding ways to compactly represent a meaningful approximation $x' \approx x$, i.e. finding an approximation $x' = \sum_i c_i' x_i$ that has few non-zero coefficients. For both these tasks, one observes that choosing to express x in terms of a larger set $\{f_i\}_{i\in I}$ that is overcomplete in H has potential advantages. With this setup, any vector $x \in H$ can be written as $\sum_{i\in I} c_i f_i$ in many different ways, and this freedom is advantageous for either of the above tasks. It can allow for a choice of $\{c_i\}_{i\in I}$ with additional structure which can be used in the first task to counter the noise on the coefficients. This same freedom of choice of $\{c_i\}_{i\in I}$ yields many more candidates for a compact meaningful approximation x' of the element x.

These overcomplete sets $\{f_i\}_{i\in I}$ (with some added structure when I is infinite) are known as *frames*. They are defined as follows: let H be a separable Hilbert space and I a countable index set. A sequence $\mathcal{F} = \{f_i\}_{i\in I}$ of elements of H is a *frame* for H if there exist constants A, $B > 0$ such that

$$\forall\, h \in H, \quad A\,\|h\|^2 \le \sum_{i\in I} |\langle h, f_i\rangle|^2 \le B\,\|h\|^2. \tag{1}$$

The numbers A, B are called *lower* and *upper frame bounds*, respectively. If $A = B = 1$ this is a *Parseval frame*. The *frame operator* is the positive, self-adjoint, invertible operator on H given by: $S(\cdot) = \sum_{i\in I}\langle \cdot, f_i\rangle f_i$. It follows that $\{f_i\}_{i\in I}$ is a frame with frame bounds A, B if and only if $a \cdot I \le S \le B \cdot I$.

Frames were first introduced by Duffin and Schaeffer [9] in the context of nonharmonic Fourier series, and today frames play important roles in many applications in mathematics, science, and engineering. We refer to the monograph [8], or the research-tutorial [7] for basic properties of frames.

Central, both theoretically and practically, to the interest in frames has been their overcomplete nature; the strength of this overcompleteness is the ability of a frame to express arbitrary vectors as a linear combination in a "redundant" way (see the opening paragraph for two examples). For infinite dimensional frames, quantifying overcompleteness or redundancy has proven to be challenging. When imagining a measure of redundancy for infinite frames, a wish list of desired properties would include:

1. The redundancy of any frame for the whole space would be greater than or equal to one.
2. The redundancy of a Riesz basis would be exactly one.
3. The redundancy would be additive on unions of frames.
4. Any frame with redundancy bigger than one would contain in it a frame with redundancy arbitrarily close to one.

A model example for what a measure of redundancy should be is given in the very special case of a Gabor Frame generated from a Gaussian function. In general a *Gabor Frame* is defined to be a frame $\mathcal{F}$, generated from time-frequency shifts of a *generator* function $f \in L^2(\mathbf{R})$. Specifically, given $f \in L^2(\mathbf{R})$ along with a subset $\Lambda \subset \mathbf{R}^2$:

$$\mathcal{F} = \{f_\lambda\}_{\lambda \in \Lambda} \text{ where for } \lambda = (\alpha, \beta), \quad f_\lambda(x) = e^{2\pi \alpha x} f(x - \beta).$$

In this case, we say (f, Λ) *generates a Gabor frame* for $L^2(\mathbf{R})$. In the special case where the generator f is a Gaussian function, it has been shown (See [22, 24, 25] and Gröchenig's book [16]) that the lower density of the set Λ is greater than one if and only if $\mathcal{F}$ is a frame for the whole space $L^2(\mathbf{R})$. This result, combined with results from [1, 2], shows that property 4) holds for Gaussian generators. To see this, assume that g is Gaussian, $\Lambda \subset \mathbf{R}^2$ and (g, Λ) generates a Gabor frame for $L^2(\mathbf{R})$. Then, applying the "if" part of the above result we have:

$$1 < D^-(f, \Lambda) \leq D^+(f, \Lambda) < \infty.$$

Now, given $\epsilon > 0$, applying the techniques from [1] (See the *proof* of Theorem 8) we can find a subset $\Sigma \subset \Lambda$ (even of uniform density) so that

$$1 < D^{-}(\Sigma) = D^{+}(\Sigma) < 1 + \epsilon.$$

Again, by the "only if" part of the above result, we have the (g, Σ) generates a Gabor frame for $L^2(\mathbf{R})$.

Separately, over the last 40 years (Since H.J. Landau [21] gave a density condition for Gabor frames with generators certain entire functions), partial progress towards a quantitative notion of redundancy has occurred for both lattice and general Gabor frames. Many works have connected essential features of the frames to quantities related to the density Λ of the associated set of time and frequency shifts (For an extensive account of the development of density theory for Gabor frames see [18] and references therein). As dynamic as these results were, they could not be used to show that the obvious choice for redundancy, namely the reciprocal of the density of Λ, satisfied any version of properties 3) or 4).

Until recently, additional results about redundancy of arbitrary frames or results relating to properties 3) and 4) for Gabor frames have remained elusive. Recent work, however has made significant advances in quantifying redundancy of infinite frames. Progress began with the work in [4, 5] which examined and explored the notion of *excess* of a frame, i.e. the maximal number of frame elements that could be removed while keeping the remaining elements a frame for the same span. This work, however, left open property 4) for frames with infinite excess (which include, for example, Gabor frames that are not Riesz bases). A quantitative approach to a large class of frames with infinite excess (including Gabor frames) was given in [1, 2] which introduced a general notion of a localized frame and, among other results, provided nice quantitative measures associated to this class of frames. From a slightly different angle, the work in [3] quantified overcompleteness for all frames that share a common index set. In [3], a redundancy function for infinite frames was defined. It was shown that this redundancy function, when translated to the localized setting dealt with in [1], corresponded to the density measures described in [3]. Both papers identify this density function as a good candidate for redundancy, show that the function satisfies properties 1)-3) above and remark that a proper notion of redundancy should also possess property 4). A weak partial result related to property 4) was given in [1, 2] where it was shown that for any localized frame $\mathcal{F}$ of redundancy R there exists an $\epsilon > 0$ and a subframe of $\mathcal{F}$ with redundancy $R - \epsilon$.

Here, we show that the redundancy function alluded to in [1, 2] and [3], for l^1 localized frames, has property 4). We show that for any ϵ, every l^1 localized frame with redundancy R has a subframe with redundancy $1+\epsilon$. Specifically we show (See Section 2 for notation and definitions):

Theorem 1.1. *Assume $\mathcal{F} = \{f_i\,;\, i \in I\}$ is a frame for H, $\mathcal{E} = \{e_k\,;\, k \in G\}$ is a l^1-self localized frame for H, with G a discrete countable abelian group, $a : I \to G$ a localization map of finite upper density so that $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized. Then for every $\varepsilon > 0$ there exists a subset $J = J_\varepsilon \subset I$ so that $D^+(a; J) < 1 + \varepsilon$ and $\mathcal{F}[J] = \{f_i; i \in J\}$ is frame for H.*

When specialized to Gabor frames, the result reads:

Theorem 1.2. *Assume $\mathcal{G}(g^1, \ldots, g^n; \Lambda^1, \ldots, \Lambda^n)$ is a Gabor multi-frame for $L^2(\mathbf{R}^d)$ so that $g^1, \ldots, g^n \in M^1(\mathbf{R}^d)$. Then for every $\varepsilon > 0$ there are subsets $J^1_\varepsilon \subset \Lambda^1$, ...,$J^n_\varepsilon \subset \Lambda^n$, so that $\mathcal{G}(g^1, \ldots, g^n; J^1_\varepsilon, \ldots, J^n_\varepsilon)$ is a Gabor multi-frame for $L^2(\mathbf{R}^d)$ and $D^+_B(J^1_\varepsilon \cup \cdots J^n_\varepsilon) \le 1 + \varepsilon$.*

The significance of this work is that it shows, for the first time, that in the case of ℓ^1-localized frames the *density* (See Section 2 for the definition) is really the correct quantitative measure of redundancy. This is the first large class of frames and the first notion of redundancy to satisfy all four of the desired properties listed above.

This work hinges on a fundamental finite dimensional result that is of independent interest. For Parseval frames, the result says that an M-element Parseval frame for H_N contains a subframe of less than $(1+\epsilon)N$ elements with lower frame bound a universal function of ϵ and M/N. The precise statement of the general result is given in Lemma 3.2:

Lemma 3.2 (Finite dimensional removal)**.** *There exists a monotonically increasing function $g : (0,1) \to (0,1)$ with the following property. For any set $\mathcal{F} = \{f_i\}_{i=1}^M$ of M vectors in a Hilbert space of dimension N, and for any $0 < a < \frac{M}{N} - 1$ there exists a subset $\mathcal{F}_a \subset \mathcal{F}$ of cardinality at most $(1+a)N$ so that:*

$$\sum_{f \in \mathcal{F}_a} \langle \cdot, f\rangle f \ge g\left(\frac{a}{2\frac{M}{N} - 1}\right) \sum_{f \in \mathcal{F}} \langle \cdot, f\rangle f \qquad (2)$$

This work is organized as follows. We begin by reviewing the definition of localized frames. In Section 3 we prove the above mentioned fundamental finite dimensional result (Lemma 3.2). We then prove a "truncation" result which is used later to reduce the infinite dimensional case to a sequence of finite dimensional cases. Section 4 contains the proof of Theorem 1.1. We first prove Theorem 1.1 for ℓ^1-localized Parseval frames and then generalize this to arbitrary ℓ^1-localized frames. In Section 5 we apply these results to Gabor Multi-frames with generators in $M^1(\mathbf{R}^d)$, and Gabor molecules with envelopes in $W(C, l^1)$.

2 Notation: localized frames

The idea of localized frames used here was introduced in [1]. (See Gröchenig [15] and Fornasier and Gröchenig [17] for another definition of localization). For this paper, the starting point will be a Hilbert space H, along with two frames for H: $\mathcal{F} = \{f_i \ , \ i \in I\}$ indexed by the countable set I and $\mathcal{E} = \{e_k \ ; \ k \in G\}$ indexed by a discrete countable abelian group G. Here we will assume $G = \mathbf{Z}^d \times \mathbf{Z}_D$ for some integers $d, D \in \mathbf{N}$, where $\mathbf{Z}_D = \{0, 1, 2, \ldots, D-1\}$ is the cyclic group of size D.

We relate the frames $\mathcal{F}$ and $\mathcal{E}$ by introducing a map $a : I \to G$ between their index sets. Following [17, 15, 1] we say $(\mathcal{F}, a, \mathcal{E})$ is l^1 *localized* if

$$\sum_{k \in G} \sup_{i \in I} |\langle f_i, e_{a(i)-k} \rangle| < \infty \tag{3}$$

We shall denote by $\mathbf{r} = (r(g))_{g \in G}$ the localization sequence for $\mathcal{F}$ with respect to $\mathcal{E}$, i.e.

$$r(g) = \sup_{i \in I, k \in G, a(i)-k=g} |\langle f_i, e_k \rangle|.$$

Thus $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized if and only if the localization sequence $\mathbf{r}$ is in $l^1(G)$. That is,

$$\|r\|_1 = \sum_{k \in G} r(k) < \infty \tag{4}$$

Similarily, the set $\mathcal{E}$ is said to be l^1-*self localized* if

$$\sum_{k \in G} \sup_{g \in G} |\langle e_{k+g}, e_g \rangle| < \infty \tag{5}$$

In other words, $\mathcal{E}$ is l^1-self localized if and only if $(\mathcal{E}, i, \mathcal{E})$ is l^1-localized, where $i : G \to G$ is the identity map. We denote by $\mathbf{s} = (s(g))_{g \in G}$ the self-localization sequence of $\mathcal{E}$, that is $s(g) = \sup_{k,l \in G, k-l=g} |\langle e_k, e_l \rangle|$.

An important quantity will be the l^1 norm of the tail of $\mathbf{r}$, namely

$$\Delta(R) := \sum_{|k| \geq R} r_k, \tag{6}$$

and thus if $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized, $\lim_{R\to\infty} \Delta(R) = 0$.

The *upper and lower densities* of a subset $J \subset I$ with respect to the map $a : I \to G$ are defined by

$$D^+(a; J) = \limsup_{N\to\infty} \sup_{k \in G} \frac{|a^{-1}(B_N(k)) \cap J|}{|B_N(0)|} \tag{7}$$

$$D^-(a; J) = \liminf_{N\to\infty} \inf_{k \in G} \frac{|a^{-1}(B_N(k)) \cap J|}{|B_N(0)|} \tag{8}$$

where $B_N(k) = \{g \in G\, ;\, |g - k| \leq N\}$ is the box of radius N and center k in G, and $|Q|$ denotes the number of elements in the set Q. Note that $|B_N(k)| = |B_N(k')|$ for all $k, k' \in G$ and $N > 0$. When $J = I$ we simply call $D^\pm(a; I)$ the *densities of* I, or the *density of the map* a, and we denote them by $D^\pm(I)$ or $D^\pm(a)$. The map a (or, equivalently, the set I) is said to have *finite upper density* if $D^+(I) < \infty$. As proved in Lemma 2 of [1], if a has finite upper density, then there is $K_a \geq 1$ so that

$$|a^{-1}(B_N(k))| \leq K_a |B_N(0)| \tag{9}$$

for all $k \in G$ and $N > 0$. The finiteness of upper density is achieved when frame vectors have norms uniformly bounded away from zero (see Theorem 4 of [1]).

3 Two important lemmas

In this section we will prove two lemmas (Lemma 3.2 and Lemma 3.6) that will be the essential ingredients for the proof of the main result (Theorem 1.1).

3.1 Finite dimensional removal

A longstanding question in finite dimensional frame theory asks whether there is a function $f(\epsilon, \frac{M}{N})$ so that every frame $\{f_i\}_{i=1}^M$ for an N-dimensional Hilbert space $\mathbf{H}_N$ with lower frame bound A, contains a subset $\mathcal{F}$ with $|\mathcal{F}| \leq (1+\epsilon)N$ having lower frame bound $Ag(\epsilon, \frac{M}{N})$? In this section, we give an affirmative answer to this question. First, we give an example showing that this result fails for $\epsilon = 0$.

Example 3.1. *Denote by $\{e_1, \ldots, e_N\}$ an orthonormal basis for H. Let $\mathcal{F}$ consist of $\{e_1, \ldots e_{N-1}\}$ along with K copies of $\frac{1}{\sqrt{K}} e_N$. Thus $\mathcal{F}$ is a Parseval frame with $M = K + N - 1$ elements. However, a subframe with N elements must be the set $\{e_1, \ldots e_{N-1}, \frac{1}{\sqrt{K}} e_N\}$ which has lower frame bound $\frac{1}{K} = \frac{1}{M-N+1}$ which goes to zero as N grows even if the ratio M/N stays fixed.*

However, as we now show, if we allow the subset to be a little fraction larger than N, i.e. of size $(1+\epsilon)N$, then we are able to find a subframe whose lower frame bound does not depend on N but rather on M/N and ϵ:

Lemma 3.2 (Finite dimensional removal)**.** *There exists a monotonically increasing function $g : (0,1) \to (0,1)$ with the following property. For any set $\mathcal{F} = \{f_i\}_{i=1}^M$ of M vectors in a Hilbert space of dimension N, and for any $0 < \epsilon < \frac{M}{N} - 1$ there exists a subset $\mathcal{F}_\epsilon \subset \mathcal{F}$ of cardinality at most $(1+\epsilon)N$ so that:*

$$\sum_{f \in \mathcal{F}_\epsilon} \langle \cdot, f \rangle f \geq g\left(\frac{\epsilon}{2\frac{M}{N} - 1}\right) \sum_{f \in \mathcal{F}} \langle \cdot, f \rangle f \tag{10}$$

The result can be restated in the following form:

Corollary 3.3. *There exists a monotonically increasing function $g : (0,1) \to (0,1)$ with the following property. For any finite frame $\mathcal{F}$ of M elements in an N-dimensional Hilbert space H_N with lower frame bound A, and any $0 < \epsilon < \frac{M}{N} - 1$ there exists a subset $\mathcal{F}' \subset \mathcal{F}$ of cardinality at most $(1+\epsilon)N$ that remains a frame for H_N and has lower frame bound $Ag(\frac{\epsilon}{2M/N-1})$.*

It is easily seen by letting $\frac{K}{N} \to \infty$ in Example 3.1 that the reliance of the lower frame bound on $(\frac{M}{N})^{-1}$ is necessary in Lemma 3.2 and Corollary 3.3.

To prove Lemma 3.2 we will use Lemma 3.5 which is adapted from Theorem 4.3 of Casazza [6] (See Vershynin [26] for a generalization of this result which removes the assumption that the norms of the frame vectors are bounded below.) For the convenience of the reader we recall Theorem 4.3 in [6]:

Theorem 3.4 (Theorem 4.3 in Casazza, [6])**.** *There is a function* $g(v, w, x, y, z) : \mathbf{R}^5 \to \mathbf{R}^+$ *with the following property: Let* $(f_i)_{i=1}^M$ *be any frame for an* N*-dimensional Hilbert space* $\mathbf{H}_N$ *with frame bounds* A, B, $\alpha \leq \|f_i\| \leq \beta$, *for all* $1 \leq i \leq M$, *and let* $0 < \varepsilon < 1$. *Then there is a subset* $\sigma \subset \{1, 2, \ldots, M\}$, *with* $|\sigma| \geq (1-\varepsilon)N$ *so that* $(f_i)_{i\in\sigma}$ *is a Riesz basis for its span with Riesz basis constant* $g(\varepsilon, A, B, \alpha, \beta)$.

We remind the reader that the Riesz basis constant is the larger between the upper Riesz basis bound, and the reciprocal of the lower Riesz basis bound.

Lemma 3.5. *There is a monotonically increasing function* $h : (0, 1) \to (0, 1)$ *with the following property: Let* $\{f_i\}_{i=1}^M$ *be any Parseval frame for an* N*-dimensional Hilbert space* H_N *with* $\frac{1}{2} \leq \|f_i\|^2$, *for all* $1 \leq i \leq M$. *Then for any* $0 < \varepsilon < 1$ *there is a subset* $\sigma \subset \{1, 2, \ldots, M\}$, *with* $|\sigma| \geq (1-\varepsilon)N$ *so that* $\{f_i\}_{i\in\sigma}$ *is a Riesz basis for its span with lower Riesz basis bound* $h(\varepsilon)$.

Proof: The only part of this result which is not proved in Theorem 3.4 is that h may be chosen to be monotonically increasing. So let g satisfy Theorem 4.3 in [6] and define for $0 < \epsilon_0 < 1$:

$$h(\epsilon_0) = \sup_{0<\epsilon\leq\epsilon_0} \frac{1}{g(\epsilon, 1, 1, \frac{1}{\sqrt{2}}, 1)}.$$

Then h is monotonically increasing. Let $\{f_i\}_{i=1}^M$ be any Parseval frame for a N-dimensional Hilbert space H_N with $\frac{1}{2} \leq \|f_i\|^2$ for all $1 \leq i \leq M$ and fix $0 < \epsilon < 1$. There exists a sequence $\{\epsilon_n\}_{n=1}^\infty$ (not necessarily distinct) with $0 < \epsilon_n \leq \epsilon$ so that

$$h(\epsilon) = \lim_{n\to\infty} g(\epsilon_n).$$

By Theorem 3.4, for every $n \in \mathbf{N}$ there is a subset $\mathcal{F}_n = \{f_i\}_{i\in I_n}$ of $\mathcal{F}$ so that

$$|\mathcal{F}_n| \geq (1-\epsilon_n)N,$$

and $\{f_i\}_{i\in I_n}$ is a Riesz basis for its span with lower Riesz basis bound $g(\epsilon_n)$. Since the number of subsets of $\mathcal{F}$ is finite, there exists at least one subset $\mathcal{G} \subset$

$\mathcal{F}$ that appears infinitely often in the sequence $\{\mathcal{F}_n\}_n$. Thus $|\mathcal{G}| \geq (1-\epsilon_n)N$ and $\mathcal{G}$ has a lower frame bound greater than equal to $g(\epsilon_n)$ for n belonging to an infinite subsequence of the positive integers. Taking the limit along this subsequence yields $|\mathcal{G}| \geq lim_{n\to\infty}(1-\epsilon_n)N = (1-\epsilon)N$ and $\mathcal{G}$ has a lower frame bound greater than or equal to

$$\lim_{n\to\infty} g(\epsilon_n) = h(\epsilon_0).$$

□

Proof of Lemma 3.2

Step 1. We first assume that the frame $\{f_i\}_{i=1}^M$ is a Parseval frame for its span H_N, and each vector satisfies $\|f_i\|^2 \leq \frac{1}{2}$. Therefore, by embedding H_N in a M-dimensional Hilbert space and using Neimark's dilation theorem [7] (or the super-frame construction [3]), we find an orthonormal basis $\{e_i\}_{i=1}^M$ and a projection P of rank N so that $f_i = Pe_i$. Let $f_i' = (I-P)e_i$. Then $\{f_i'\}_{i=1}^M$ is a Parseval frame for its span (which is $M-N$-dimensional) and $\|f_i'\|^2 \geq \frac{1}{2}$. Notice that we have for any set of coefficients $(c_i)_{i=1}^M$:

$$\sum_i |c_i|^2 = \|\sum_{i=1}^M c_i e_i\|^2 = \|\sum_i c_i f_i\|^2 + \|\sum_i c_i f_i'\|^2. \tag{11}$$

For a δ that we will specify later, we now apply Theorem 3.5 to the frame $\{f_i'\}_{i=1}^M$ (not $\{f_i\}$) to get a subset $\sigma \in \{1,\ldots,M\}$ with $|\sigma| \geq (1-\delta)(M-N)$ such that $\{f_j'\}_{j\in\sigma}$ is a Riesz basis for its span with lower Riesz bound greater than or equal to $h(\delta)$.

Thus for any set of coefficients $(c_j)_{j\in\sigma}$ we have

$$\|\sum_{j\in\sigma} c_j f_j'\|^2 \geq h(\delta)\sum_j |c_j|^2.$$

Combining this with equation 11 and a choice of $(c_i)_{i=1}^M$ with the property that $c_i = 0$ if $i \notin \sigma$ we have

$$\|\sum_{j\in\sigma} c_j f_j\|^2 \leq (1-h(\delta))(\sum_{j\in\sigma}|c_j|^2). \tag{12}$$

This equation is equivalent to saying that the operator $S_\sigma = \sum_{j\in\sigma}\langle\cdot, f_j\rangle f_j \leq (1-h(\delta))\mathbf{1}$. Therefore, setting $J = \{1,2,\ldots,M\}\backslash\sigma$, we have

$$\mathbf{1} \geq S_J = \sum_{j\in J}\langle\cdot, f_j\rangle f_j = \mathbf{1} - S_\sigma \geq h(\delta)\mathbf{1}.$$

Notice that $|J| \leq M - (1-\delta)(M-N) = N + \delta(M-N) = (1+\delta(\frac{M}{N}-1))N$. Thus any choice of $\delta \leq a/(\frac{M}{N}-1)$ produces a set J of cardinality $|J| \leq (1+a)N$ such that $S_J \geq h(\delta)\mathbf{1}$. Setting $\delta = \frac{a}{2M/N-1}$ and $g = h$ gives the desired result.

Step 2. Assume now that $\{f_i\}_{i=1}^M$ is a Parseval frame without constraints on the norms of f_i. The upper frame bound 1 implies $\|f_i\| \leq 1$, for every $1 \leq i \leq M$. Apply the previous result to the Parseval frame $\{f_{i,1}\}_{i=1}^M \cup \{f_{i,2}\}_{i=1}^M$ where $f_{i,1} = f_{i,2} = \frac{1}{\sqrt{2}} f_i$ for every $1 \leq i \leq M$. Thus we obtain a set $J_1 \subset \{1,2,\ldots,M\} \times \{1,2\}$, $|J_1| \leq (1+a)N$, so that

$$\sum_{(i,k)\in J_1} \langle \cdot, f_{i,k}\rangle f_{i,k} \geq h\left(\frac{a}{\frac{2M}{N}-1}\right)\mathbf{1}$$

Let $J = \{i \ : \ 1 \leq i \leq M, \text{ such that } (i,1) \in J_1 \text{ or } (i,2) \in J_1\}$ Notice $|J| \leq |J_1| \leq (1+a)N$ and

$$\sum_{i\in J} \langle \cdot, f_i\rangle f_i \geq \sum_{(i,k)\in J_1} \langle \cdot, f_{i,k}\rangle f_{i,k} \geq h\left(\frac{a}{2\frac{M}{N}-1}\right)\mathbf{1}$$

which again produces the desired result with $g = h$.

Step 3. For the general case, assume S is the frame operator asssociated to $\{f_i\}_{i=1}^M$. Then $\{g_i := S^{-1/2} f_i\}_{i=1}^M$ is a Parseval frame with the same span. Applying the result of step 2 to this frame, we conclude there exists a subset $J \subset \{1,2,\ldots,M\}$ of cardinality $|J| \leq (1+a)N$ so that $\{g_i\}_{i\in J}$ is frame such that

$$\sum_{i\in J} \langle \cdot, g_i\rangle g_i \geq h\left(\frac{a}{2\frac{M}{N}-1}\right)\mathbf{1}.$$

It follows that

$$\sum_{i\in J} \langle \cdot, f_i\rangle f_i = S^{1/2}\left(\sum_{i\in J} \langle \cdot, g_i\rangle g_i\right) S^{1/2} \geq h\left(\frac{a}{2\frac{M}{N}-1}\right) S$$

which is what we needed to prove. □

3.2 Truncation

In this subsection we assume $\mathcal{E}$ is a l^1 self-localized Parseval frame for H indexed by G, and $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized. We let $\mathbf{r}$ denote the localization sequence of $\mathcal{F}$, and we let $\mathbf{s}$ denote the self-localization sequence of $\mathcal{E}$. Further, we denote by

$$f_{i,R} = \sum_{k \in G, |k-a(i)|<R} \langle f_i, e_k \rangle e_k \tag{13}$$

the *truncated* expansion of f_i with respect to $\mathcal{E}$. Clearly $f_{i,R} \to f_i$ as $R \to \infty$. But does this convergence imply convergence of the corresponding frame operators for $\{f_{i,R}\}_{i \in G}$? The answer is that it does as we now show. Specifically, for a subset $J \subset I$ we denote $\mathcal{F}[J] = \{f_i\}_{i \in J}$ and $\mathcal{F}_R[J] = \{f_{i,R}\ ;\ i \in J\}$. Similarly we denote by S_J and $S_{R,J}$ the frame operators associated to $\mathcal{F}[J]$ and $\mathcal{F}_R[J]$, respectively. The following Lemma shows that the truncated frames well approximate the original frames:

Lemma 3.6. *Choose R_0 so that for all $R \geq R_0$, $\Delta(R) \leq (K_a \|s\|_1)^{-1}$ (See equation 4) and S_J and $S_{R,J}$ as above. Then*

$$\|S_J - S_{R,J}\| \leq E(R), \tag{14}$$

where $E(R) = 3K_a \Delta(R) \|\mathbf{s}\|_1$.

Proof: First denote by $T_J : H \to l^2(J)$, and $T_{R,J} : H \to l^2(J)$ the analysis maps:

$$T_J(x) = \{\langle x, f_i \rangle\}_{i \in J} \quad , \quad T_{R,J}(x) = \{\langle x, f_{i,R} \rangle\}_{i \in J}$$

Since $\mathcal{E}$ is a Parseval frame, $Q : H \to l^2(G)$, $Q(x) = \{\langle x, e_k \rangle\}_{k \in G}$ is an isometry, and

$$\|T_J - T_{R,J}\| = \|(T_J - T_{R,J})^*\| = \|Q(T_J - T_{R,J})^*\|$$

The operator $M = Q(T_J - T_{R,J})^* : l^2(J) \to l^2(G)$ is described by a matrix which we also denote by M. In the canonical bases of $l^2(J)$ and $l^2(G)$, the (k,i) element of M is given by

$$M_{k,i} = \langle f_i - f_{i,R}, e_k \rangle = \sum_{g \in G, |g-a(i)| \geq R} \langle f_i, e_g \rangle \langle e_g, e_k \rangle,$$

and thus

$$|M_{k,i}| \leq \sum_{g\in G, |g-a(i)|\geq R} r(g-a(i))s(g-k) \tag{15}$$

We bound the operator norm of M using Schur's criterion [23, 20]

$$\|M\| \leq max(\sup_{i\in J}\sum_{k\in G}|M_{k,i}|, \sup_{k\in G}\sum_{i\in J}|M_{k,i}|)$$

It follows from (15) that

$$\begin{aligned} \sum_{k\in G}|M_{k,i}| &\leq \Delta(R)\|\mathbf{s}\|_1 \\ \sum_{i\in J}|M_{k,i}| &\leq K_a\Delta(R)\|\mathbf{s}\|_1. \end{aligned}$$

Thus we obtain $\|M\| \leq K_a\Delta(R)\|\mathbf{s}\|_1$ and hence

$$\|T_J - T_{R,J}\| = \|(T_J - T_{R,J})^*\| \leq K_a\Delta(R)\|\mathbf{s}\|_1$$

It follows that

$$\begin{aligned} \|S_J - S_{R,J}\| &= \|(T_J - T_{R,J})^*T_J + (T_{R,J})^*(T_J - T_{R,J})\| \\ &\leq (\|T_J\| + \|T_{R,J}\|)K_a\Delta(R)\|\mathbf{s}\|_1 \\ &\leq 3K_a\Delta(R)\|\mathbf{s}\|_1, \end{aligned}$$

the last inequality coming from $\|T_J\| \leq 1$ and

$$\|T_{R,J}\| \leq \|T_J\| + \|T_{R,J} - T_J\| \leq 1 + K_a\Delta(R)\|\mathbf{s}\|_1 \leq 2,$$

since $\Delta(R) < \frac{1}{K_a\|\mathbf{s}\|_1}$, for $R > R_0$. □

4 Proof of the main result

In this section we prove the main result of the paper, Theorem 1.1.

The core of the proof is contained in subsection 4.1 which proves theorem 1.1 for the special case when both $\mathcal{F}$ and $\mathcal{E}$ are Parseval frames. In subsection 4.2 we show how to generalize this special case.

We begin by giving a brief description of the argument of subsection 4.1. Our starting point is the Parseval frame $\mathcal{F}$ that is localized with respect to another Parseval frame $\mathcal{E}$. Our goal is to produce a subset $\mathcal{F}' \subset \mathcal{F}$ which is a frame for the whole space and which has density not much larger than 1. An outline of the steps is as follows:

1. Using Lemma 3.6, we move from the frame $\mathcal{F}$ to a truncated frame $\mathcal{F}_R$.
2. Based on the localization geometry, we decompose I as the union of disjoint finite boxes $Q_N(k)$, with k taking values in an infinite lattice.
3. For each k, $\mathcal{F}_R[Q_N(k)]$ is a finite dimensional frame. We apply Lemma 3.2 to get subsets $J_{k,N,R}$ of smaller size such that $\mathcal{F}_R[J_{k,N,R}]$ remains a frame with frame operator greater than or equal to a universal constant times the frame operator for $\mathcal{F}_R[Q_N(k)]$. Thus we have constructed a set $J = \cup_k J_{k,N,R}$ for which $\mathcal{F}_R[J]$ is a frame for the whole space.
4. We then use our choice of R along with Lemma 3.6 to conclude that the set $\mathcal{F}[J]$ is also frame for the whole space.
5. Finally, we show that our choice of N is large enough for the frame $\mathcal{F}[J]$ to have small density.

4.1 The case when $\mathcal{F}$ and $\mathcal{E}$ are Parseval

In this subsection we prove the result for the special case of parseval frames.

Lemma 4.1. *Let $\mathcal{F} = \mathcal{F}[I]$ be a Parseval frame for H indexed by I, and let $\mathcal{E}$ be a l^1-self localized Parseval frame for H indexed by the discrete abelian group $G = \mathbf{Z}^d \times \mathbf{Z}_D$ so that $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized with respect to a localization map $a : I \to G$ of finite upper density. Then for every $\varepsilon > 0$ there exists a subset $J = J_\varepsilon \subset I$ so that $D^+(a; J) \leq 1 + \varepsilon$ and $\mathcal{F}' = \mathcal{F}[J]$ is frame for H.*

We begin the proof by recalling some notation. For $k \in G$, $N \in \mathbf{N}$, $B_N(k) = \{g \in G : |g - k| \leq N\}$ is the elements of G in the ball with center k and radius N. Define $Q_N(k) = \{i \in I : |a(i) - k| \leq N\} = a^{-1}(B_N(k))$. Since $D^+(a) < \infty$, there exists $K_a \geq 1$ so that $|a^{-1}(B_N(k))| \leq K_a |B_N(k)|$.

Recall that we assumed $\mathcal{F}$ and $\mathcal{E}$ are Parseval frames for H, $\mathcal{E}$ is l^1-self localized, and $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized. Denote by $\mathbf{r}$ the localization sequence for $(\mathcal{F}, a, \mathcal{E})$, denote by $\mathbf{s}$ the self-localization sequence for $\mathcal{E}$, and recall $E(R) = 3K_a\|\mathbf{s}\|_1 \sum_{k\in G,|k|>R} r(k)$ decays to 0 as $R \to \infty$. Let $g : (0,1) \to (0,1)$ denote the universal function of Lemma 3.2 and let C_ε denote the positive quantity:

$$C_\varepsilon = g\left(\frac{\varepsilon}{2(2K_a - 1)}\right). \tag{16}$$

We now fix $\varepsilon > 0$. For the duration we will fix two large integer numbers R and N as follows. First R is chosen so that

$$E(R) < \frac{C_\varepsilon}{2(1 + C_\varepsilon)} \tag{17}$$

Then N is chosen to be an integer larger than R so that

$$(1 + \frac{\varepsilon}{2})\frac{|B_{N+R}(0)|}{|B_N(0)|} \leq 1 + \varepsilon. \tag{18}$$

Such an N exists since $|B_M(0)| = D(2M + 1)^d$ for $M > D$ and thus $\lim_{N\to\infty} \frac{|B_{N+R}(0)|}{|B_N(0)|} = 1$.

Step 1. Define $\mathcal{F}_R = \{f_{i,R}\,;\, i \in I\}$ to be the truncated frame given by Lemma 3.6 when it is applied to $\mathcal{F}$ and the given R. Let S_R be the frame operator associated to $\mathcal{F}_R$. Notice that since $\mathcal{F}$ is a Parseval frame (and hence its frame operator is $\mathbf{1}$) we have $\|I - S_R\| \leq E(R)$ and consequently

$$(1 + E(R))\mathbf{1} \geq S_R \geq (1 - E(R))\mathbf{1}. \tag{19}$$

Step 2. We let L be the sublattice $(2N\mathbf{Z})^d \times \{0\} \subset G$. For each $k \in L$ and integer M let $V_{M,k} = span\{e_j\,;\, j \in B_M(k)\}$. Notice $dim(V_{M,k}) \leq |B_M(k)|$. Let $r_{k,N,R} = dim\, span\{f_{i,R}\,;\, i \in Q_N(k)\}$. Since $span\,\{f_{i,R}\,;\, i \in Q_N(k)\} \subset V_{N+R,k}$ we obtain $r_{k,N,R} \leq |B_{N+R}(0)|$.

If $|Q_N(k)| \leq (1 + \frac{\varepsilon}{2})|B_{N+R}(0)|$ then set $J_{k,N,R} = Q_N(k)$ so that

$$\sum_{i\in J_{k,N+R}} \langle\cdot, f_{i,R}\rangle f_{i,R} = \sum_{i\in Q_N(k)} \langle\cdot, f_{i,R}\rangle f_{i,R} \geq C_\varepsilon \sum_{i\in Q_N(k)} \langle\cdot, f_{i,R}\rangle f_{i,R} \tag{20}$$

where C_ε is defined in (16).

Assume now that $|Q_N(k)| > (1+\frac{\varepsilon}{2})|B_{N+R}(0)|$. We apply Lemma 3.2 to the set $\{f_{i,R}\,;\, i \in Q_N(k)\}$ with $a = (1+\frac{\varepsilon}{2})\frac{|B_{N+R}(0)|}{r_{k,N,R}} - 1$ and obtain a subset $J_{k,N,R} \subset Q_N(k)$ of size $|J_{k,N,R}| \leq (1+\frac{\varepsilon}{2})|B_{N+R}(k)|$ so that

$$\begin{aligned}
\sum_{i\in J_{k,N.R}} \langle \cdot, f_{i,R}\rangle f_{i,R} \;&\geq\; g\left(\frac{a}{2|Q_N(k)|/r_{k,N,R}-1}\right)\sum_{i\in Q_N(k)}\langle \cdot, f_{i,R}\rangle f_{i,R} && (21)\\
&\geq\; C_\varepsilon \sum_{i\in Q_N(k)} \langle \cdot, f_{i,R}\rangle f_{i,R} && (22)
\end{aligned}$$

where the last inequality follows from the monotonicity of g and the fact that

$$\frac{a}{2|Q_N(k)|/r_{k,N,R}-1} \geq \frac{\varepsilon}{2(2K_a-1)}.$$

In either case

$$|J_{k,N,R}| \leq (1+\frac{\varepsilon}{2})|B_{N+R}(k)| \leq (1+\varepsilon)|B_N(0)|$$

due to (18).

Step 3. Set

$$J_{N,R} = \cup_{k\in L} J_{k,N,R}. \tag{23}$$

Denote by $S_{R,N}$ the frame operator for $\{f_{i,R}\,;\, i\in J_{N,R}\}$. We then have

$$\begin{aligned}
S_{R,N} \;&=\; \sum_{k\in L}\sum_{i\in J_{k,N,R}} \langle \cdot, f_{i,R}\rangle f_{i,R} \\
&\geq\; \sum_{k\in L} C_\varepsilon \sum_{i\in Q_N(k)} \langle \cdot, f_{i,R}\rangle f_{i,R} = C_\varepsilon S_R && (24)\\
&\geq\; C_\varepsilon(1-E(R))\mathbf{1} && (25)
\end{aligned}$$

where the last lower bound comes from (19). This means $\mathcal{F}_{R,N} := \{f_{i,R}\,;\, i \in J_{N,R}\}$ is frame for H with lower frame bound $C_\varepsilon(1-E(R))$.

Step 4. We again apply Lemma 3.6 with $J = J_{N,R}$ to obtain that S_J, the frame operator associated to $\mathcal{F}[J] = \{f_i\,;\, i\in J\}$, is bounded below by

$$S_J \geq S_{R,N} - E(R)\mathbf{1} \geq \left(C_\varepsilon(1-E(R)) - E(R)\right)\mathbf{1} \geq \frac{1}{2}C_\varepsilon \mathbf{1} \tag{26}$$

where the last inequality follows from (17). This establishes that $\mathcal{F}[J]$ is frame for H with lower frame bound at least $\frac{1}{2}C_\varepsilon$.

It remains to show that $J_{N,R}$ has the desired upper density.

Step 5. The upper density of $J = J_{N,R}$ is obtained as follows. First in each box $B_N(k)$, $k \in L$, we have

$$\frac{|a^{-1}(B_N(k)) \cap J|}{|B_N(k)|} = \frac{|J_{k,N,R}|}{|B_N(k)|} \leq (1 + \frac{\varepsilon}{2})\frac{|B_{N+R}(k)|}{|B_N(k)|} \leq 1 + \varepsilon \tag{27}$$

Then, by an additive argument one can easily derive that

$$\limsup_{M\to\infty} \sup_{k\in G} \frac{|a^{-1}(J) \cap B_M(k)|}{|B_M(k)|} \leq 1 + \varepsilon \tag{28}$$

which means $D^+(a; J) \leq 1 + \varepsilon$. □

4.2 Generalizing

We now show how to remove the constraints that both $\mathcal{F}$ and $\mathcal{E}$ are Parseval in Lemma 4.1 . We begin by outlining the argument: starting with the frames $\mathcal{F}$ and $\mathcal{E}$ we show there are canonical Parseval frames $\mathcal{F}^\#$ and $\mathcal{E}^\#$ that have the same localization properties as $\mathcal{F}$ and $\mathcal{E}$. We then apply Lemma 4.1 to these frames to get a subframe of $\mathcal{F}^\#$ that is a frame for the whole space with the appropriate density. Finally, we show that the corresponding subframe of $\mathcal{F}$ has the desired frame and density properties.

A well known canonical construction (see [8]) begins with an arbitrary frame $\mathcal{F} = \{f_i\}$ and produces the canonical Parseval frame

$$\mathcal{F}^\# = \{f_i^\# = S^{-1/2} f_i\}, \tag{29}$$

where S is the frame operator associated to $\mathcal{F}$ (which we recall is defined by $S : H \to H$, $S(x) = \sum_{i\in I} \langle x, f_i \rangle f_i$.) Incidentally, the set

$$\tilde{\mathcal{F}} = \{\tilde{f}_i = S^{-1} f_i\}, \tag{30}$$

is called the *canonical dual frame* and yields a resolution of identity

$$\sum_{i\in I} \langle f, f_i \rangle \tilde{f}_i = \sum_{i\in I} \langle f, \tilde{f}_i \rangle f_i = f, \quad \text{for all } f \in H.$$

In our situation we have two frames $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{E} = \{e_k\}_{k\in G}$ along with $a : I \to G$ such that $(\mathcal{F}, a, \mathcal{E})$ is l^1-localized and $\mathcal{E}$ is a l^1-self localized. As in (29) we define two Parseval frames $\mathcal{F}^\#$ and $\mathcal{E}^\#$ corresponding to $\mathcal{F}$ and $\mathcal{E}$ respectively.

Lemma 2.2 from [15] and Theorem 2 from [1] can be used to show that $\mathcal{F}^\#$ and $\mathcal{E}^\#$ inherit the localization properties of $\mathcal{F}$ and $\mathcal{E}$. Namely

Lemma 4.2. *Given $\mathcal{F}^\#$ and $\mathcal{E}^\#$ as above, if $(\mathcal{F}, a, \mathcal{E})$ is l^1-localized and $\mathcal{E}$ is l^1-self localized then $(\mathcal{F}^\#, a, \mathcal{E}^\#)$ is l^1-localized and $\mathcal{E}^\#$ is l^1-self localized.*

Proof

First, if $\mathcal{E}$ is l^1-self localized then by Theorem 2,(c) in [1] it follows that $\mathcal{E}^\#$ is l^1-self localized. Furthermore, by Theorem 2, (b) in the aforementioned paper it follows that $(\tilde{\mathcal{E}})$ is l^1-self localized, where $\tilde{\mathcal{E}} = \{\tilde{e_k}\}_{k\in G}$ is the canonical dual of $\mathcal{E}$. This implies the existence of a sequence $s \in l^1(G)$ so that

$$|\langle \tilde{e_k}, \tilde{e_j}\rangle| \leq s(k-j) \quad , \quad \text{for all } k, j \in G. \tag{31}$$

Next assume additionally that $(\mathcal{F}, a, \mathcal{E})$ is l^1-localized. This means there exists a sequence $r \in l^1(\mathcal{G})$ so that

$$|\langle f_i, e_k\rangle| \leq r(a(i) - k), \text{ for every } i \in I \text{ and } k \in G. \tag{32}$$

Since $\tilde{e_k} = \sum_{j\in G}\langle \tilde{e_k}, \tilde{e_j}\rangle e_j$ it follows that

$$|\langle f_i, \tilde{e_k}\rangle| = |\sum_{j\in G}\langle f_i, e_j\rangle\langle \tilde{e_j}, \tilde{e_k}\rangle| \leq \sum_{j\in G} r(a(i) - j)s(j-k) =: (r \star s)(a(i) - k)$$

and thus $(\mathcal{F}, a, \tilde{\mathcal{E}})$ is also l^1-localized. By Lemma 3 in [1] it follows that $(\mathcal{F}, a)$ is l^1-self localized.

Again Theorem 2, (b) implies now that $(\tilde{\mathcal{F}}, a)$ is l^1-self localized. Therefore there exists a sequence $t \in l^1(G)$ so that

$$|\langle \tilde{f}_i, \tilde{f}_j\rangle| \leq t(a(i) - a(j)) \quad , \quad \text{for every } i, j \in I. \tag{33}$$

We will show that $(\mathcal{F}, a)$ is l^1-self localized implies that $(\mathcal{F}^\#, a)$ is l^1 localized with respect to $(\mathcal{F}, a)$, meaning that there exists a sequence $u \in l^1(G)$ so that

$$|\langle f_i^\#, f_j\rangle| \leq u(a(i) - a(j)) \quad , \quad \text{for every } i, j \in I \tag{34}$$

Let $G : l^2(I) \to l^2(I)$ be the Gramm operator associated to the frame $\mathcal{F}$, $G = TT^*$, where $T : H \to l^2(I)$ is the analysis operator $T(x) = \{\langle x, f_i\rangle\}_{i\in I}$ and $T^* : l^2(I) \to H$ given by $T^*(c) = \sum_{i\in I} c_i f_i$ is the synthesis operator. Let $\delta_i \in l^2(I)$ denote the sequence of all zeros except for one entry 1 on the i^{th} position. The set $\{\delta_i\}_{i\in I}$ is the canonical orthonormal basis of $l^2(I)$. Since $\mathcal{F}$ is a frame, G is a bounded operator with closed range, and T^* is surjective (onto). Let $G^\dagger$ denote the (Moore-Penrose) pseudoinverse of G. Thus $P = GG^\dagger = G^\dagger G$ is the orthonormal projection onto the range of T in $l^2(I)$. A simple exercise shows that $\tilde{f}_i = T^* G^\dagger \delta_i$, and $f_i^\# = T^*(G^\dagger)^{1/2}\delta_i$. Using the notations from Appendix A of [1], we get $G \in \mathcal{B}_1(I, a)$, the algebra of operators that have l^1 decay. Using Lemma A.1 and then the holomorphic calculus as in the Proof of Theorem 2 of the aforementioned paper, we obtain that G and all its powers G^q, $q > 0$ are in $\mathcal{B}_1(I, a)$. In particular, $G^{1/2} \in \mathcal{B}_1(I, a)$ implying the existence of a sequence $u \in l^1(G)$ so that

$$|\langle G^{1/2}\delta_i, \delta_j\rangle| \leq u(a(i) - a(j))$$

Then:

$$\langle f_i^\#, f_j\rangle = \langle T^*(G^\dagger)^{1/2}\delta_i, T^*\delta_j\rangle = \langle G(G^\dagger)^{1/2}\delta_i, \delta_j\rangle = \langle G^{1/2}\delta_i, \delta_j\rangle$$

which yields (34).

The same proof applied to $(\mathcal{E}, i)$ implies that if $(\mathcal{E}, i)$ is l^1-self localized then $(\mathcal{E}^\#, i, \mathcal{E})$ is l^1-localized (which is to say, equivalently, that $(\mathcal{E}^\#, i)$ is l^1 localized with respect to $(\mathcal{E}, i)$). Explicitely this means there exists a sequence $v \in l^1(G)$ so that

$$|\langle e_k^\#, e_n\rangle| \leq v(k - n) \quad , \quad \text{for every } k, n \in G \tag{35}$$

Putting together (31-35) we obtain:

$$\langle f_i^\#, e_k^\#\rangle = \sum_{j,l\in I} \sum_{m,n\in G} \langle f_i^\#, f_j\rangle\langle \tilde{f}_j, \tilde{f}_l\rangle\langle f_l, e_m\rangle\langle \tilde{e_m}, \tilde{e_n}\rangle\langle e_n, e_k^\#\rangle$$

Hence

$$\begin{aligned} |\langle f_i^\#, e_k^\#\rangle| &\leq \sum_{j,l\in I} \sum_{m,n\in G} u(a(i) - a(j))t(a(j) - a(l))r(a(l) - m)s(m - n)v(n - k) \\ &\leq K_a^2 (u \star t \star r \star s \star v)(a(i) - k) \end{aligned}$$

where K_a is as in (9), and the convolution sequence $u \star t \star r \star s \star v \in l^1(G)$. This means $(\mathcal{F}^\#, a, \mathcal{E}^\#)$ is l^1 localized. $\Box$

We can now prove Theorem 1.1:

Proof of Theorem 1.1

As above we let $\mathcal{F}^\#$ and $\mathcal{E}^\#$ be the canonical Parseval frames associated with $\mathcal{F}$ and $\mathcal{E}$. By Lemma 4.2 we have $(\mathcal{F}^\#, a, \mathcal{E}^\#)$ is l^1 localized and $\mathcal{E}^\#$ is l^1-self localized. Given $\varepsilon > 0$ we apply Lemma 4.1 to get a subset $J \subset I$ such that $D^+(a; J) \leq 1 + \varepsilon$ and $\mathcal{F}^\#[J]$ is a frame for H.

To complete the proof, we now show that $\mathcal{F}[J]$ is also a frame for H. This follows from the following lemma:

Lemma 4.3. *Assume $\mathcal{F} = \{f_i\}_{i \in I}$ is frame for H with frame bounds $A \leq B$. Let $\mathcal{F}^\#$ be the canonical Parseval frame associated to $\mathcal{F}$. If $J \subset I$ is such that $\{f_i^\#\}_{i \in J}$ is frame for H with bounds $A' \leq B'$, then $\mathcal{F}[J] = \{f_i\}_{i \in J}$ is also frame for H with bounds AA' and BB'.*

Proof: Let S be the frame operator associated to $\mathcal{F}$ and so $A\mathbf{1} \leq S \leq B\mathbf{1}$. Now we have the following operator inequality

$$\begin{aligned} AA'\mathbf{1} &\leq A'S = S^{1/2}(A'\mathbf{1})S^{1/2} && (36) \\ &\leq S^{1/2}\left(\sum_{i \in J} \langle \cdot, f_i^\# \rangle f_i^\#\right) S^{1/2} && (37) \\ &\leq S^{1/2}(B'\mathbf{1})S^{1/2} = B'S \leq BB'\mathbf{1}. && (38) \end{aligned}$$

Notice however that the frame operator for $\mathcal{F}[J]$ satisfies

$$\sum_{i \in J} \langle \cdot, f_i \rangle f_i = S^{1/2}\left(\sum_{i \in J} \langle \cdot, f_i^\# \rangle f_i^\#\right) S^{1/2}.$$

Substituting this equality into the middle term of the string of inequalities (37) gives the desired result:

$$AA'\mathbf{1} \leq \sum_{i \in J} \langle \cdot, f_i \rangle f_i \leq BB'\mathbf{1}.$$

$\Box$

5 Application to Gabor

In this section we specialize to Gabor frames and molecules the results obtained in previous section.

First we recall previously known results.

A (generic) *Gabor system* $\mathcal{G}(g;\Lambda)$ generated by a function $g \in L^2(\mathbf{R}^d)$ and a countable set of time-frequency points $\Lambda \subset \mathbf{R}^{2d}$ is defined by

$$\mathcal{G}(g;\Lambda) = \{M_\omega T_x g \ ; \ (x,\omega) \in \Lambda\} = \{e^{2\pi i \langle \omega, t\rangle} g(t-x) \ ; \ (x,\omega) \in \Lambda\} \tag{39}$$

In general we allow Λ to be an irregular set of time-frequency points.

A *Gabor multi-system* $\mathcal{G}(g^1,\ldots,g^n;\Lambda^1,\ldots,\Lambda^n)$ generated by n functions $g^1,\ldots,g^n$ and n sets of time-frequency points $\Lambda^1,\ldots,\Lambda^n$ is simply the union of the corresponding Gabor systems:

$$\mathcal{G}(g^1,\ldots,g^n;\Lambda^1,\ldots,\Lambda^n) = \mathcal{G}(g^1;\Lambda^1) \cup \cdots \cup \mathcal{G}(g^n;\Lambda^n). \tag{40}$$

A *Gabor molecule* $\mathcal{G}(\Gamma;\Lambda)$ associated to an enveloping function $\Gamma : \mathbf{R}^{2d} \to \mathbf{R}$ and a set of time-frequency points $\Lambda \subset \mathbf{R}^{2d}$ is a countable set of functions in $L^2(\mathbf{R}^d)$ indexed by Λ whose short-time Fourier transforms (STFT) have a common envelope of concentration:

$$\begin{aligned} \mathcal{G}(G;\Lambda) \ = \ & \{g_{x,\omega} \ ; \ g_{x,\omega} \in L^2(\mathbf{R}^d) \ : \qquad\qquad (41) \\ & |V_\gamma g_{x,\omega}(y,\xi)| \le \Gamma(y-x,\xi-\omega) \ , \ \forall (x,\omega) \in \Lambda \, , \, \forall (y,\xi) \in \mathbf{R}^{2d}\} \end{aligned}$$

where $\gamma(t) = 2^{d/4} e^{-\pi \|t\|^2}$ and

$$V_\gamma h(y,\xi) = \int e^{-2\pi i \langle \xi, t\rangle} h(t)\gamma(t-y)dt. \tag{42}$$

Remark 5.1. *Note that Gabor systems (and multi-systems) are Gabor molecules, where the common localization function is the absolute value of the short-time Fourier transform of the generating function g, $\Gamma = |V_\gamma g|$ (or the sum of absolute values of STFTs of generating functions $g^1,\ldots,g^n$, $\Gamma = |V_\gamma g^1| + \cdots + |V_\gamma g^n|$).*

When a Gabor system, a Gabor multi-system, or a Gabor molecule, is a frame we shall simply call the set a Gabor frame, a Gabor multi-frame, or a Gabor molecule frame, respectively.

In this section the reference frame $\mathcal{E}$ is going to be the Gabor frame $\mathcal{E} = \mathcal{G}(\gamma; \alpha\mathbf{Z}^d \times \beta\mathbf{Z}^d)$ where γ is the Gaussian window $\gamma(t) = 2^{d/4}e^{-\pi\|t\|^2}$ normalized so that its $L^2(\mathbf{R}^d)$ norm is one, and $\alpha, \beta > 0$ are chosen so that $\alpha\beta < 1$. As is well known (see [22, 24, 25]), for every such α and β, $\mathcal{G}(\gamma; \alpha\mathbf{Z}^d \times \beta\mathbf{Z}^d)$ is frame in $L^2(\mathbf{R}^d)$.

The localization property introduced in Section 2 turns out to be equivalent to a joint concentration in both time and frequency of the generator(s) of a Gabor (multi-)system, or of the envelope of a Gabor molecule. The most natural measures of concentration are given by norms of the *modulation spaces*, which are Banach spaces invented and extensively studied by Feichtinger, with some of the main references being [10, 11, 12, 13, 14]. For a detailed development of the theory of modulation spaces and their weighted counterparts, we refer to the original literature mentioned above and to [16, Chapters 11–13].

For our purpose, two Banach spaces are sufficient: the modulation space M^1 and the *Wiener amalgam space* $W(C, l^1)$.

Definition 5.2. *The* modulation space $M^1(\mathbf{R}^d)$ *(also known as the* Feichtinger algebra S_0*) is the Banach space consisting of all functions f of $L^2(\mathbf{R}^d)$ so that*

$$\|f\|_{M^1} := \|V_\gamma f\|_{L^1} = \int\int_{\mathbf{R}^{2d}} |V_\gamma f(x, \omega)| dx d\omega \ < \ \infty \tag{43}$$

Definition 5.3. *The* Wiener amalgam space $W(C, l^1)$ *over $\mathbf{R}^n$ is the Banach space consisting of continuous functions $F : \mathbf{R}^n \to \mathbf{C}$ so that*

$$\|F\|_{W(C,l^1)} := \sum_{k\in\mathbf{Z}^n} \sup_{t\in[0,1]^n} |F(k+t)| \ < \ \infty \tag{44}$$

Note that the Banach algebra $M^1(\mathbf{R}^d)$ is invariant under the Fourier transform and is closed under both pointwise multiplication and convolution. Furthermore, a function $f \in M^1(\mathbf{R}^d)$ if and only if $V_\gamma f \in W(C, l^1)$ over $\mathbf{R}^{2d}$. In particular the Gaussian window $\gamma \in M^1(\mathbf{R}^d)$.

Consider now a Gabor molecule $\mathcal{G}(\Gamma; \Lambda)$ and define the localization map $a : \Lambda \to \alpha\mathbf{Z}^d \times \beta\mathbf{Z}^d$ via $a(x, \omega) = \left(\alpha\lfloor\frac{1}{\alpha}x\rfloor, \beta\lfloor\frac{1}{\beta}\omega\rfloor\right)$, where $\lfloor\cdot\rfloor$ acts componentwise, and on each component, $\lfloor b\rfloor$ denotes the largest integer smaller than or equal to b.

For any set $J \subset \mathbf{R}^{2d}$, the *Beurling upper and lower density* are defined by

$$D_B^+(J) = limsup_{N\to\infty} \sup_{z\in\mathbf{R}^{2d}} \frac{|\{\lambda \in J \, : \, |\lambda - z| \leq N\}|}{(2N)^{2d}} \tag{45}$$

$$D_B^-(J) = liminf_{N\to\infty} \inf_{z\in\mathbf{R}^{2d}} \frac{|\{\lambda \in J \, : \, |\lambda - z| \leq N\}|}{(2N)^{2d}} \tag{46}$$

The relationship between the upper and lower densities of a subset $J \subset \Lambda$ and the corresponding Beurling densities are given by (see equation (2.4) in [2]):

$$D^+(a;J) = (\alpha\beta)^d D_B^+(J) \tag{47}$$

$$D^-(a;J) = (\alpha\beta)^d D_B^-(J) \tag{48}$$

We are now ready to state the main result of this section from which 1.2 follows as a Corollary:

Theorem 5.4. *Assume $\mathcal{G}(\Gamma;\Lambda) = \{g_\lambda\}_{\lambda\in\Lambda}$ is a Gabor molecule that is frame for $L^2(\mathbf{R}^d)$ with envelope $\Gamma \in W(C,l^1)$. Then for any $\varepsilon > 0$ there exists a subset $J_\varepsilon \subset \Lambda$ so that $\mathcal{G}(\Gamma;J_\varepsilon) = \{g_\lambda\}_{\lambda\in J}$ is frame for $L^2(\mathbf{R}^d)$ and $D_B^+(J_\varepsilon) \leq 1+\varepsilon$.*

Proof: Fix $0 < \varepsilon \leq \frac{1}{2}$. Choose $\alpha, \beta > 0$ so that $(\alpha\beta)^d = 1 - \frac{\varepsilon}{2}$.

First by Theorem 2.d in [2], it follows that $(\mathcal{G}(\gamma, \alpha\mathbf{Z}^d \times \beta\mathbf{Z}^d), i)$ is a l^1-self-localized frame for $L^2(\mathbf{R}^d)$.

Then by Theorem 8.a in [2] it follows that $(\mathcal{G}(\Gamma;\Lambda), a, \mathcal{G}(\gamma, \alpha\mathbf{Z}^d \times \beta\mathbf{Z}^d))$ is l^1-localized. Furthermore, by Theorem 9.a from same reference, the Beurling upper density of Λ must be finite, hence $D^+(a) < \infty$.

Thus the hypotheses of Theorem 1.1 are satisfied and one can find a subset $J_\varepsilon \subset \Lambda$ so that $D^+(a;J_\varepsilon) \leq 1 + \frac{\varepsilon}{4}$. Using 47,

$$D_B^+(J_\varepsilon) = \frac{D^+(a;J_\varepsilon)}{(\alpha\beta)^d} \leq \frac{1+\frac{\varepsilon}{4}}{1-\frac{\varepsilon}{2}} \leq 1+\varepsilon$$

which is what we needed to prove. □

Proof of Theorem 1.2

First note that $\mathcal{G}(g^1,\dots,g^n;\Lambda^1,\dots,\Lambda^n)$ is a Gabor molecule with envelope $\Gamma = |V_\gamma g^1| + \cdots + |V_\gamma g^n|$. Since each $g^1,\dots,g^n \in M^1(\mathbf{R}^d)$ we obtain $\Gamma \in W(C,l^1)$ and the conclusion follows from Theorem 5.4. □

Acknowledgement: The authors would like to thank Chris Heil and Roman Vershynin for many useful discussions and suggestions concerning this article.

References

[1] R. Balan, P.G. Casazza, C. Heil, and Z. Landau. Density, overcompleteness, and localization of frames I: Theory. *J. Fourier Anal. Appl.*, 12(2):105–143, 2006.

[2] R. Balan, P.G. Casazza, C. Heil, and Z. Landau. Density, overcompleteness, and localization of frames II: Gabor frames. *J. Fourier Anal. Appl.*, 12(3):307–344, 2006.

[3] R. Balan and Z. Landau. Measure functions for frames. *J. Funct. Anal.*, 252:630–676, 2007.

[4] Radu Balan, Peter G. Casazza, Christopher Heil, and Zeph Landau. Deficits and excesses of frames. *Adv. Comput. Math.*, 18(2-4):93–116, 2003.

[5] Radu Balan, Peter G. Casazza, Christopher Heil, and Zeph Landau. Excesses of Gabor frames. *Appl. Comput. Harmon. Anal.*, 14(2):87–106, 2003.

[6] P.G. Casazza. Local theory of frames and schauder bases for hilbert space. *Illinois J.our Math.*, 43:291–306, 1999.

[7] P.G. Casazza. The Art of Frame Theory. *Taiwanese J. Math.*, 4:129–201, 2000.

[8] Ole Christensen. *An Introduction to Frames and Riesz Bases.* Birkhäuser, Boston, 2003.

[9] R.J. Duffin and A.C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Am. Math. Soc.*, 72:341–366, 1952. Reprinted in [19].

[10] H.G. Feichtinger. On a new Segal algebra. *Monatsh. Math.*, 92:269–289, 1981.

[11] H.G. Feichtinger. Atomic characterizations of modulation spaces through Gabor-type representations. In *Proc. Conf. Constructive Function Theory*, volume 19 of *Rocky Mountain J. Math.*, pages 113–126, 1989.

[12] H.G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, I. *J. Funct. Anal.*, 86:307–340, 1989. Reprinted in [19].

[13] H.G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, II. *Monatsh. Math.*, 108:129–148, 1989.

[14] H.G. Feichtinger and K. Gröchenig. Gabor frames and time-frequency analysis of distributions. *J. Funct. Anal.*, 146(2):464–495, 1997.

[15] M. Fornasier and K. Gröchenig. Intrinsic localization of frames. *Constr. Approx.*, 22(3):395–415, 2005.

[16] K. Gröchenig. *Foundations of time-frequency analysis.* Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 2001.

[17] K. Gröchenig. Localization of Frames, Banach Frames, and the Invertibility of the Frame Operator. *J. Fourier Anal. Appl.*, 10(2):105–132, 2004.

[18] C. Heil. On the history and evolution of the density theorem for Gabor frames. Technical report, Georgia Institute of Technology, 2006.

[19] C. Heil and D. F Walnut, Eds. Fundamental Papers in Wavelet Theory. Princeton University Press, Princeton, NJ, 2006.

[20] R.V. Kadison and J.R. Ringrose. *Fundamentals of the Theory of Operator Algebras. I.* AMS Graduate Studies in Mathematics 15, 1997.

[21] H.J. Landau. *Necessary density conditions for sampling and interpolation of certain entire functions.* Acta Math. 117: 37-52, 1967.

[22] Yu.I. Lyubarskij. Frames in the Bargmann space of entire functions. In *Entire and subharmonic functions*, volume 11 of *Adv. Sov. Math.*, pages 167–180. American Mathematical Society (AMS), Providence, RI, 1992.

[23] F Riesz and B.S. Nagy. *Functional Analysis.* Dover Publications, 1990.

[24] Kristian Seip. Density theorems for sampling and interpolation in the Bargmann-Fock space. I. *J. Reine Angew. Math.*, 429:91–106, 1992.

[25] Kristian Seip and Robert Wallstén. Density theorems for sampling and interpolation in the Bargmann-Fock space. II. *J. Reine Angew. Math.*, 429:107–113, 1992.

[26] R. Vershynin. Subsequences of frames. *Studia Mathematica*, 145:185–197, 2001.