FUSION FRAMES AND THEORETICAL APPLICATIONS: FOR THE FUSION FRAME WEB PAGE

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The deepest and most difficult question in Fusion Frame Theory is the construction of fusion frames with added properties for specific applications. In frame theory we have a powerful tool introduced by Benedetto and Fickus [1] called *frame potentials* (See [5] for a deep analysis of frame potentials). Frame potentials are a valuable tool for showing the existence of frames with certain specified properties but have the drawback that they do not show how to construct the frames. Recently, Casazza and Fickus [4] have developed a *Fusion Frame Potential* theory. This theory is much more complicated than regular frame potentials and therefore is not as exact at this time. Part of the problem stems from the fact that certain simple examples of fusion frames just do not exist (See [12]). The fusion frame potential theory thus works best if the size of the fusion frame (I.e. The sums of the dimensions of the subspaces in the fusion frame) is quite large comparted to the dimension of the space. This topic needs more development at this time.

The results of Casazza and Kutyniok [7] (See also [8]) show: A family of subspaces $\{W_i\}_{i\in I}$ is a fusion frame for $\mathbb{H}$ with fusion frame bounds C, D if and only if for every choice of vectors $\{f_{ij}\}_{j=1}^{K_i}$ which is a frame for W_i with frame bounds A, B for all $i \in I$, the family $\{f_{ij}\}_{j=1,\ i\in I}^{K_i}$ is a frame for $\mathbb{H}$ with frame bounds AC, BD. So fusion frames come from dividing a frame into subsets which are "good" local frames for the subspaces W_i. For this process to work, we need the local frames to have (uniformly) good lower frame bounds, since these bounds control the computational complexity of reconstruction.

So a natural way to find fusion frames is to take a frame and divide it into subsets which are good frame sequences. This seems like a reasonable approach to fusion frame constructions, but it has been recently shown that this approach is extremely difficult in general. In particular, it is now known that the problem of dividing an arbitrary (even equal norm) Parseval frame into a finite number of subsets each of which has good lower frame bounds is equivalent to one of the deepest and most intractable problems in mathematics: The 1959 *Kadison-Singer Problem* in C^*-Algebras [6, 11, 13]. Trying to simplify the problem does not help. If we consider such a simple case as taking an equal norm Parseval frame $\{f_i\}_{i=1}^{2N}$ for $\mathbb{H}_N$, we already are in trouble.

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The Rado-Horn Theorem (See [9]) tells us [11] that every equal norm Parseval frame $\{f_i\}_{i=1}^{2N}$ for $\mathbb{H}_N$ can be divided into two spanning sets. However, it was recently shown [3] that there does not exist a constant $0 < c < 1$ so that for every equal norm Parseval frame $\{f_i\}_{i=1}^{2N}$ for $\mathbb{H}_N$ we can divide it into two subsets each of which is a frame for $\mathbb{H}_N$ with lower frame bound c. It can be shown that dividing a unit norm frame into Riesz basic sequences (with uniform Riesz basis bounds) will produce a food fusion frame [2]. But being able to do this for arbitrary unit norm frames is also equivalent to KS and is known as the **Feichtinger Conjecture** in frame theory [10, 6, 11] which asserts: Every bounded frame can be divided into a finite number of Riesz basic sequences. It is also known [3] that KS is equivalent to being able to divide every equal norm Parseval frame $\{f_m\}_{m=1}^{2N}$ for $\mathbb{H}_N$ into a finite number of Riesz basic sequences.

So in general, it is a very deep and difficult problem to construct fusion frames from arbitrary frames by dividing the frame into subsets.

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