

**PROBLEMS ON PAVING AND  
THE KADISON-SINGER PROBLEM: FOR THE AIM WEB  
PAGE**

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ABSTRACT. For a background and up to date information about paving see the posted note: *Paving and the Kadison-Singer Problem*. Please send problems which should be posted in this section to Pete Casazza at pete@math.missouri.edu.

1. NOTATION

**Notation 1.1.** Given an orthonormal basis  $\{e_i\}_{i \in I}$  for a Hilbert space  $\mathbb{H}$ , for any  $A \subset I$  we denote by  $P_A$  the **diagonal projection** whose matrix has entries all zero except for the  $(i, i)$ -entries for  $i \in A$  which are all one. For a matrix  $A = (a_{ij})_{i,j=1}^n$  set  $\delta_p = \max\{|a_{ii}| : i = 1, 2, \dots, n\}$ . A **diagonal symmetry** is a diagonal matrix  $(a_{ij})_{i,j=1}^n$  with  $|a_{ii}| = 1$  for  $1 \leq i \leq n$  and  $a_{ij} = 0$  otherwise.

**Notation 1.2.** Given an orthonormal basis  $\{e_i\}_{i \in I}$  for a Hilbert space  $\mathbb{H}$ , a **diagonal decomposition of the identity** ( $k$ -d.d. for short) is a family of diagonal projections with disjoint ranges and

$$\sum_{i=1}^k P_i = I.$$

**Notation 1.3.** If  $A$  is an  $n \times n$  matrix, define

$$\alpha_k(A) = \min\{\max \|P_i A P_i\| : \{P_i\}_{i=1}^k \text{ is a } k\text{-d.d.}\}$$

and if  $A$  is an infinite matrix bounded in the operator norm, define

$$\alpha_k(A) = \inf\{\max \|P_i A P_i\| : \{P_i\}_{i=1}^k \text{ is a } k\text{-d.d.}\}.$$

If  $A \neq 0$ , in both cases define

$$\tilde{\alpha}_k(A) = \frac{\alpha_k(A)}{\|A\|}.$$

## 2. THE PAVING CONJECTURE

**Anderson's Paving Conjecture (PC)** [2]

For every  $\epsilon > 0$ , there is some “universal” positive integer  $k = k(\epsilon)$  so that for every zero-diagonal finite matrix  $A$ , there exists a  $k$ -d.d.  $\{P_i\}_{i=1}^k$  so that for all  $i = 1, 2, \dots, k$  we have

$$(2.1) \quad \|P_i A P_i\| \leq \epsilon \|A\|.$$

When we have the inequality in Equation 2.1 for a matrix  $A$ , we say that  $A$  is  **$(k, \epsilon)$ -pavable**.

Given a fixed matrix  $A$ , if for every  $\epsilon < 1$  there is a  $k \in \mathbb{N}$  so that  $A$  is  $(k, \epsilon)$ -pavable then we say that  $A$  is **pavable**. If we have a class of matrices, we say this **class is pavable** (respectively,  **$(k, \epsilon)$ -pavable**) if every member of the class is pavable (respectively,  $(k\epsilon)$ -pavable).

*Paving* for an arbitrary matrix  $A$  means paving for  $A - E(A)$  where  $E(A)$  denotes the diagonal of  $A$  with respect to a fixed basis. A simple iteration argument shows that PC is equivalent to the existence of a universal  $k$  working for just one fixed  $\epsilon < 1$ .

There are a number of restricted classes of matrices for which paveability with respect to any fixed basis is equivalent to PC:

1. Self-adjoint matrices [5, 6].
2. Unitary operators [5, 6].
3. Positive operators [5, 6].
4. Invertible operators (or invertible operators with zero diagonal) [5, 6]
5. Orthogonal projections [5, 6]
6. Gram matrices [5, 6].
7. Lower (respectively upper) triangular matrices [7].

### 3. PAVING CONJECTURES EQUIVALENT TO PC

**Conjecture 3.1.** *For every zero-diagonal matrix  $A$  (finite or infinite) there is a  $k$  and  $\epsilon < 1$  so that  $A$  is  $(k, \epsilon)$ -pavable. I.e., For every zero-diagonal matrix  $A$  we have  $\tilde{\alpha}_k(A) < 1$  for some  $k$ .*

The equivalence of Conjecture 3.1 and PC is a simple diagonal process. Note that Conjecture 3.1 does not require the universality of  $k$  and the bound below one depends on  $A$ .

**Conjecture 3.2.** *There is a universal  $k$  so that for all zero-diagonal matrices  $A$  (finite and infinite), there is a  $\epsilon < 1$  so that  $A$  is  $(k, \epsilon)$ -pavable. (I.e., For every zero-diagonal matrix  $A$  we have  $\tilde{\alpha}_k(A) < 1$ .)*

**Note:** The rest of the conjectures in this section require universal constants  $k$  and  $\epsilon < 1$  which are independent of  $n \in \mathbb{N}$  and the matrix.

**Conjecture 3.3.** [4] *There exists an  $\epsilon < 1$  and a natural number  $k$  so that all orthogonal projections  $A$  on  $\ell_2^{2n}$  with  $1/2$ 's on the diagonal are  $(k, \epsilon)$ -pavable.*

Note that Conjecture 3.3 does not require zero-diagonal. In [4] it is shown that Conjecture 3.3 fails for  $k = 2$ . I.e. There is no  $\epsilon < 1$  so that this class of projections is  $(2, \epsilon)$ -pavable.

**Conjecture 3.4.** [4] *There is a universal  $k$  and an  $\epsilon < 1$  so that every norm one self-adjoint operator  $U$  with zero diagonal and satisfying  $U^2 = I$  is  $(k, \epsilon)$ -pavable.*

**Conjecture 3.5.** [9] *There exist universal constants  $0 < \delta, \epsilon < 1$  and  $k \in \mathbb{N}$  so that all orthogonal projections  $P$  on  $\ell_2^n$  with  $\delta(P) \leq \delta$  are  $(k, \epsilon)$ -pavable.*

Weaver also posed an equal norm version of this conjecture.

**Conjecture 3.6.** [9] *There exist universal constants  $0 < \delta, \sqrt{\delta} \leq \epsilon < 1$  and  $k \in \mathbb{N}$  so that all orthogonal projections  $P$  on  $\ell_2^n$  with  $\delta(P) \leq \delta$  and  $\|Pe_i\| = \|Pe_j\|$  for all  $i, j = 1, 2, \dots, n$  are  $(k, \epsilon)$ -pavable.*

## 4. RELATED PROBLEMS

**Problem 4.1.** *If PC is true, then are the upper triangular invertibles on  $\ell_2(\mathbb{N})$  path connected? (Larson, Paulsen, Orr, Weiss, Zhang).*

The problem of the connectedness of the upper triangular invertible matrices is well-known in the non-self-adjoint operator community.

**Problem 4.2.** *Are there analogs for Problem 4.1 in other classes of operators such as Laurent, Toeplitz, analytic toeplitz, etc.? (Weiss).*

**Problem 4.3.** *Are the Laurent operators paveable? (Bourgain-Tzafriri [3], Halpern, Kaftal and Weiss [8]).*

**Problem 4.4.** *Are all Laurent operators with  $H^\infty$ -symbol paveable? (Paulsen, Weiss).*

**Problem 4.5.** *Is PC equivalent to PC for some  $k$ -d.d.  $\{P_i\}_{i=1}^k$  where*  

$$|\text{rank } P_i - \text{rank } P_j| \leq 1, \text{ for all } 1 \leq i, j \leq k?$$
*(Casazza, Edidin, Weiss).*

## 5. THE AKEMANN-ANDERSON CONJECTURE

Akemann and Anderson [1] posed a conjecture which would imply a positive solution to KS.

**Conjecture B** ([1], 7.1.3) There exists  $\gamma, \epsilon > 0$  (and independent of  $n$ ) such that for any projection  $P$  with  $\delta_p < \gamma$  there is a diagonal symmetry  $S$  such that  $\|PSP\| < 1 - \epsilon$ .

Conjecture B is still open despite considerable effort having been expended on it. Weaver [10] states that a counterexample to Conjecture B would probably lead to a negative solution to the Paving Conjecture.

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