

# Real Equiangular Frames

(Invited Paper)

Peter G. Casazza  
Department of Mathematics  
The University of Missouri  
Columbia, Missouri 65211–4100  
Email: pete@math.missouri.edu

Dan Redmond  
Department of Mathematics  
The University of Missouri  
Columbia, Missouri 65211–4100  
Email: dredmond@math.missouri.edu

Janet C. Tremain  
Department of Mathematics  
The University of Missouri  
Columbia, Missouri 65211–4100  
Email: janet@math.missouri.edu

**Abstract**—Real equiangular tight frames can be especially useful in practice because of their structure. The problem is that very few of them are known. We will look at recent advances on the problem of classifying the equiangular tight frames and as a consequence give a classification of this family of frames for all real Hilbert spaces of dimension less than or equal to 50.

## I. INTRODUCTION

A family of vectors  $\{f_i\}_{i \in I}$  is a **frame** for a Hilbert space  $\mathbf{H}$  if there are constants  $0 < A \leq B < \infty$  so that for all  $f \in \mathbf{H}$

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2.$$

If  $A = B$  this is an **A-tight frame** and if  $A = B = 1$  this is a **Parseval frame**. If  $\|f_i\| = \|f_j\|$  for all  $i, j \in I$ , this is an **equal norm frame**, and if  $\|f_i\| = 1$  for all  $i \in I$  this is a **unit norm frame**. A unit norm frame with the property that there is a constant  $c$  so that

$$|\langle f_i, f_j \rangle| = c, \quad \text{for all } i \neq j,$$

is called an **equiangular frame** at angle  $c$ . Note that we are calling this the *angle* when really it is the arc cos of the angle.

Frames have traditionally been used in signal processing. Today, frames have a myriad of applications in mathematics and engineering including sampling theory, wavelet theory, wireless communication, signal processing (including sigma-delta quantization), image processing, operator theory, harmonic analysis, pseudodifferential operators, sparse approximation, data transmission with "erasures", filter banks, geophysics, quantum computing, distributed processing and more. What makes frames such a fundamental tool in these areas is their "overcompleteness" which allows representations of vectors which are robust to quantization, resilient to additive noise, give stable reconstruction after erasures and give greater freedom to capture important signal characteristics. For an introduction to frame theory we recommend [5], [8]. For an introduction to frames in time frequency analysis (the mathematics of signal processing) see [10].

Equiangular tight frames first appeared in discrete geometry [16] but today (especially the complex case) have applications in signal processing, communications, coding theory and more [13], [17]. A detailed study of this class of frames was initiated by Strohmer and Heath [17] and Holmes and Paulsen

[14]. Holmes and Paulsen [14] show that equiangular tight frames give error correction codes that are robust against two erasures. Bodmann and Paulsen [4] analyze arbitrary numbers of erasures for equiangular tight frames. Recently, Bodmann, Casazza, Edidin and Balan [3] show that equiangular tight frames are useful for signal reconstruction when all phase information is lost. Recently, Sustik, Tropp, Dhillon and Heath [18] made an important advance on this subject (and on the complex version) which we will discuss in detail later. Other applications include the construction of capacity achieving signature sequences for multiuser communication systems in wireless communication theory [20]. The tightness condition allows equiangular tight frames to achieve the capacity of a Gaussian channel and their equiangularity allows them to satisfy an interference invariance property. Equiangular tight frames potentially have many more practical and theoretical applications. Unfortunately, we know very few of them and so their usefulness is largely untapped. In this note we will look at some recent results in the theory of equiangular tight frames and use this to classify all the equiangular tight frames for Hilbert spaces of dimension less than or equal to 50.

For notation, throughout this paper we will be working with finite dimensional Hilbert spaces of dimension  $N$  and our frames will have  $M$  elements.

## II. EQUIANGULAR LINES

We cannot construct equiangular tight frames until we have the requisite number of equiangular lines. That is, we need a set of  $M$  lines passing through the origin in  $\mathbf{R}^N$  which are **equiangular** in the sense that if we choose a set of unit length vectors  $\{f_m\}_{m=1}^M$ , one on each line, then for  $1 \leq m \neq n \leq M$ ,  $|\langle f_m, f_n \rangle|$  is a constant. These inner products represent the cosine of the acute angle between the lines. The problem of constructing any number (especially, the maximal number) of equiangular lines in  $\mathbf{R}^N$  is one of the most elementary and at the same time one of the most difficult problems in mathematics. After sixty years of research, the maximal number of equiangular lines in  $\mathbf{R}^N$  is known only for 35 dimensions. For a slightly more general view of this topic see Benedetto and Kolesar [2]. This line of research was started in 1948 by Hanntjes [11] in the setting of elliptic geometry where he identified the maximal number of equiangular lines in  $\mathbf{R}^N$  for  $n = 2, 3$ . Later, Van Lint and

Seidel [16] classified the largest number of equiangular lines in  $\mathbf{R}^N$  for dimensions  $N \leq 7$  and at the same time emphasized the relations to discrete mathematics. In 1973, Lemmens and Seidel [15] made a comprehensive study of real equiangular line sets which is still today a fundamental piece of work. Gerzon [15] gave an upper bound for the maximal number of equiangular lines in  $\mathbf{R}^N$ :

*Theorem 2.1 (Gerzon):* If we have  $M$  equiangular lines in  $\mathbf{R}^N$  then

$$M \leq \frac{N(N+1)}{2}.$$

We will see that in most cases there are many fewer lines than this bound gives. Also, P. Neumann [15] produced a fundamental result in the area:

*Theorem 2.2 (P. Neumann):* If  $\mathbf{R}^N$  has  $M$  equiangular lines at angle  $1/\alpha$  and  $M > 2N$ , then  $\alpha$  is an odd integer.

Finally, there is a lower bound on the angle formed by equiangular line sets.

*Theorem 2.3:* If  $\{f_m\}_{m=1}^M$  is a set of norm one vectors in  $\mathbf{R}^N$ , then

$$\max_{m \neq n} |\langle f_m, f_n \rangle| \geq \sqrt{\frac{M-N}{N(M-1)}}. \quad (1)$$

Moreover, we have equality if and only if  $\{f_m\}_{m=1}^M$  is an equiangular tight frame and in this case the tight frame bound is  $\frac{M}{N}$ .

This inequality goes back to Welch [20]. Strohmer and Heath [17] and Holmes and Paulsen [14] give more direct arguments which also yields the "moreover" part. For some reason, in the literature there is a further assumption added to the "moreover" part of Theorem 2.3 that the vectors span  $\mathbf{R}^N$ . This assumption is not necessary. That is, equality in inequality 1 already implies that the vectors span the space [7].

The status of the equiangular line problem at this point is summarized in the following chart [15], [7], [19] where  $N$  is the dimension of the Hilbert space,  $M$  is the maximal number of equiangular lines and these will occur at the angle  $1/\alpha$ .

**Table I: Maximal equiangular line sets**

$N =$	2	3	4	5	6	7	...	13	14
$M =$	3	6	6	10	16	28	...	28	$28 - 30$
$\alpha =$	2	$\sqrt{5}$	3	3	3	3	...	3	5

$N =$	15	16	17	18	19	20
$M =$	36	$\geq 40$	$\geq 48$	$\geq 48$	$72 - 76^*$	$92 - 96^*$
$\alpha =$	5	5	5	5	5	5

$N =$	21	22	23	...	41	42	43
$M =$	126	176	276	...	276	$\geq 276$	344
$\alpha =$	5	5	5	...	5	5	7

The \* in the chart represents two cases which have been reported as solved in the literature but actually are still open.

### III. EQUIANGULAR TIGHT FRAMES: THE FUNDAMENTALS

Naimark (See [6], [12]) showed that Parseval frames come from a unique process.

*Theorem 3.1:* The family  $\{f_m\}_{m=1}^M$  is a Parseval frame for  $\mathbf{R}^N$  if and only if there is an orthogonal projection  $P$  on  $\mathbf{R}^M$  satisfying  $Pe_m = f_m$  for all  $m = 1, 2, \dots, M$  where  $\{e_m\}_{m=1}^M$  is an orthonormal basis for  $\mathbf{R}^N$ .

If  $\{Pe_m\}_{m=1}^M$  is an equal norm equiangular frame for  $\mathbf{R}^N$  then for all  $1 \leq n \neq m \leq M$

$$\langle Pe_m, Pe_n \rangle = -\langle (I-P)e_m, (I-P)e_n \rangle,$$

and  $\{(I-P)e_m\}_{m=1}^M$  is an equal norm Parseval frame. Also, since any tight frame can be renormalized to be a Parseval frame, we have

*Corollary 3.2:* If  $\{f_m\}_{m=1}^M$  is an equiangular tight frame for  $\mathbf{R}^N$  with  $Pe_m = \sqrt{\frac{N}{M}}f_m$ , then  $\{\sqrt{\frac{M}{M-N}}(I-P)e_m\}_{m=1}^M$  is an equiangular tight frame for  $\mathbf{R}^{(M-N)}$ . We call this the **complementary equiangular tight frame**.

It follows that equiangular tight frames come in pairs and if  $M > 2N$  then  $M < 2(M-N)$ . So we only need to classify the equiangular tight frames for  $M > 2N$ .

Certain classes of equiangular tight frames always exist.

*Proposition 3.3:*  $\mathbf{R}^N$  always has an

1.  $M = N$ -element equiangular tight frame at angle 0. (i.e. An orthonormal basis).
2.  $M = (N+1)$ -element equiangular tight frame at angle  $1/N$ . (i.e. The complementary equiangular tight frame to the  $M$ -element equiangular tight frame for  $\mathbf{R}$  given by  $g_m = 1$  for all  $m = 1, 2, \dots, M$ .)

So from now on we will always assume that  $M > N+1$ .

*Corollary 3.4:* To have an equiangular tight frame with  $M$  elements in  $\mathbf{R}^N$  it is necessary that

$$M \leq \min \left\{ \frac{N(N+1)}{2}, \frac{(M-N)(M-N+1)}{2} \right\},$$

or equivalently

$$\frac{2N+1+\sqrt{8N+1}}{2} \leq M \leq \frac{N(N+1)}{2}.$$

*Proof:* The right hand inequality follows from Theorem 2.1 applied to the frame and to its complementary frame. For the left hand inequality we start with

$$M \leq \frac{(M-N)(M-N+1)}{2}$$

This is equivalent to

$$2M \leq M^2 - 2NM + M + N^2 - N.$$

Simplifying, we obtain

$$0 \leq M^2 - (2N+1)M + (N^2 - N) =: f(M). \quad (2)$$

Equality occurs in Equation 2 when

$$M = \frac{2N + 1 \pm \sqrt{(2N + 1)^2 - 4(N^2 - N)}}{2} \quad (3)$$

$$= \frac{2N + 1 \pm \sqrt{4N^2 + 4N + 1 - 4N^2 + 4N}}{2} \quad (4)$$

$$= \frac{2N + 1 \pm \sqrt{8N + 1}}{2} \quad (5)$$

$$= N + \frac{1}{2} \pm \sqrt{2N + \frac{1}{4}}. \quad (6)$$

We can discard the negative part of  $\pm$  in our formula since it would yield that

$$M \leq N + \frac{1}{2} - \sqrt{2N + \frac{1}{4}} < N,$$

which contradicts our general assumption that  $M > N + 1$ . Since the function  $f(M)$  is increasing, we conclude that our inequality is equivalent to

$$M \geq \frac{2N + 1 + \sqrt{8N + 1}}{2}.$$

*Proposition 3.5:* If there is an  $M$  element equiangular tight frame for  $\mathbf{R}^N$  at angle  $1/\alpha$  then

$$\alpha \leq N \leq \alpha^2 - 2.$$

Moreover,  $N = \alpha$  if and only if  $M = N + 1$  and  $N = \alpha^2 - 2$  if and only if  $M = \frac{N(N+1)}{2}$ .

*Proof:* Recall that

$$\alpha^2 = \frac{N(M-1)}{M-N}.$$

Let

$$f(x) = \frac{N(x-1)}{x-N}.$$

Then  $f$  is decreasing and

$$f(N+1) = N^2 \quad \text{and} \quad f\left(\frac{N(N+1)}{2}\right) = N+2.$$

Since  $M \geq N+1$  we have

$$N^2 \geq \alpha^2 = f(M) \geq N+2.$$

#### IV. SOME RECENT RESULTS

A recent important generalization of Theorem 2.2 is due to Sustik, Tropp, Dhillon and Heath [18]:

*Theorem 4.1:* Let  $1 < N < M - 1$ . When  $M \neq 2N$ , a necessary condition for the existence of a real equiangular tight frame is that both of the quantities

$$\alpha = \sqrt{\frac{N(M-1)}{M-N}}, \quad \beta = \sqrt{\frac{(M-N)(M-1)}{N}}$$

are odd integers.

When  $M = 2N$ , it is necessary that  $N$  be an odd integer and that  $(M-1)$  equal the sum of two squares.

Note that the angle  $1/\beta$  is the angle for the complementary frame. In [18] the authors conjecture that the conditions in Theorem 4.1 are also sufficient to have an equiangular tight frame. It is easy to give a counter-example to this conjecture.

*Example 4.2:* If  $M = 64$  in  $\mathbf{R}^{56}$  then

$$\alpha = \sqrt{\frac{N(M-1)}{M-N}} = \sqrt{\frac{56 \cdot 63}{64-56}} = \sqrt{7 \cdot 63} = 21.$$

Also,

$$\beta = \sqrt{\frac{(M-N)(M-1)}{N}} = \sqrt{\frac{8 \cdot 63}{56}} = 3.$$

But,

$$\frac{(M-N)(M-N+1)}{2} = \frac{(64-56)(64-56+1)}{2} = 36,$$

and this is not greater than  $M = 56$  so this class cannot exist by Corollary 3.4.

The following recent result comes from [7]:

*Theorem 4.3:* If  $\mathbf{R}^N$  has an equiangular tight frame with  $M$  elements at angle  $1/\alpha$  then  $M$  is an even integer and  $\alpha$  divides  $M-1$ .

Combining Theorems 4.1 and 4.3 we see that it is possible to simplify the necessary conditions in Theorem 4.1.

*Corollary 4.4:* If we have an equiangular tight frame for  $\mathbf{R}^N$  with  $M$  elements ( $M \neq 2N$ ), then the following are equivalent:

- (1) Both  $\alpha$  and  $\beta$  are odd integers.
- (2)  $M$  is even and  $\alpha$  is an odd integer which divides  $M-1$ . Moreover,  $M = 2N$  if and only if  $\alpha = \beta$ .

*Proof:* The equivalence of (1) and (2) follows immediately from Theorems 4.1, 4.3 and the observation:

$$\alpha\beta = \sqrt{\frac{N(M-1)}{M-N}} \sqrt{\frac{(M-N)(M-1)}{N}} = M-1.$$

For the moreover part, we know that

$$\alpha^2 = \frac{N(M-1)}{M-N} \quad \text{and} \quad \beta^2 = \frac{(M-N)(M-1)}{N}.$$

So  $\alpha^2 = \beta^2$  if and only if  $N^2 = (M-N)^2$ . i.e. If and only if  $M = 2N$ . ■

*Example 4.5:* The assumption  $\alpha$  divides  $M-1$  is necessary in Corollary 4.4.

*Proof:* For example, if  $N = 27$  and  $M = 40$  then  $M$  is even,

$$\alpha = \sqrt{\frac{N(M-1)}{M-N}} = \sqrt{\frac{27 \cdot 39}{13}} = 9,$$

is odd. However,

$$\beta = \sqrt{\frac{(M-N)(M-1)}{N}} = \sqrt{\frac{13 \cdot 39}{27}} = \frac{13}{3},$$

is not an integer. Note also that this satisfies the conditions of our Corollary 3.4 since

$$\begin{aligned} M = 40 &\leq \min\left\{\frac{N(N+1)}{2}, \frac{(M-N)(M-N+1)}{2}\right\} \\ &= \min\left\{\frac{27 \cdot 28}{2}, \frac{13 \cdot 14}{2}\right\} \\ &= \min\{378, 91\} = 91. \end{aligned}$$

*Example 4.6:* The assumption that  $M$  is even is necessary in Corollary 4.4

*Proof:* If  $N = 17$  and  $M = 3 \cdot 17$  then both  $\alpha$  and  $\beta$  are integers, but  $\beta$  is even. That is,

$$\alpha = \frac{N(M-1)}{M-N} = \frac{17 \cdot 50}{51-17} = 25.$$

Also

$$\beta = \frac{(M-1)(M-N)}{N} = \frac{50 \cdot 34}{17} = 100.$$

The following result first appeared in [16] with a different proof.

*Proposition 4.7:* If we have  $M$  equiangular lines spanning  $\mathbf{R}^N$  at angle  $1/\alpha$  with  $\alpha^2 > N$ , then

$$M \leq \frac{\alpha^2 - 1}{\alpha^2 - N} \cdot N,$$

with equality if and only if the corresponding unit vectors form an equiangular tight frame.

*Proof:* By Theorem 2.3,

$$\frac{1}{\alpha} \geq \sqrt{\frac{M-N}{N(M-1)}}.$$

Hence,

$$\alpha^2 \leq \frac{N(M-1)}{M-N}.$$

So

$$\alpha^2 M - \alpha^2 N \leq MN - N,$$

and

$$M(\alpha^2 - N) \leq (\alpha^2 - 1)N$$

Hence, since  $\alpha^2 - N > 0$ ,

$$M \leq \frac{\alpha^2 - 1}{\alpha^2 - N} \cdot N.$$

By Theorem 2.3, we have equality if and only the corresponding unit vectors give an equiangular tight frame. ■

*Proposition 4.8:* Let  $\alpha$  be an integer. Then

$$M = \frac{(\alpha^2 - 1)N}{\alpha^2 - N} \text{ if and only if } \frac{N(M-1)}{M-N} = \alpha^2.$$

*Proof:* We have

$$\frac{N(M-1)}{M-N} = \alpha^2,$$

if and only if

$$NM - N = \alpha^2 M - \alpha^2 N,$$

if and only if

$$(\alpha^2 - 1)N = M(\alpha^2 - N),$$

if and only if

$$M = \frac{(\alpha^2 - 1)N}{\alpha^2 - N}.$$

## V. CLASSIFYING EQUIANGULAR TIGHT FRAMES

Over the years, many authors have made conjectures which were supposed to classify equiangular tight frames. All of these were doomed to failure because we cannot construct equiangular tight frames unless we first have the requisite number of equiangular lines. The following theorem, which summarizes the results presented in this paper, is probably the best that can be said about the classification question since it assumes the required number of equiangular lines exist.

*Theorem 5.1:* The following are equivalent:

- (I) The space  $\mathbf{R}^N$  has an equiangular tight frame with  $M$  elements at angle  $1/\alpha$ .
- (II) We have

$$M = \frac{(\alpha^2 - 1)N}{\alpha^2 - N},$$

and there exist  $M$  equiangular lines in  $\mathbf{R}^N$  at angle  $1/\alpha$ .

Moreover, in this case we have:

- (1)  $\alpha \leq N \leq \alpha^2 - 2$ .
- (2)  $N = \alpha$  if and only if  $M = N + 1$ .
- (3)  $N = \alpha^2 - 2$  if and only if  $M = \frac{N(N+1)}{2}$ .
- (4)  $M = 2N$  if and only if

$$\alpha^2 = 2N - 1 = a^2 + b^2, \quad a, b \text{ integers.}$$

If  $M \neq N + 1, 2N$  then:

- (5)  $\alpha$  is an odd integer.
- (6)  $M$  is even.
- (7)  $\alpha$  divides  $M-1$ .
- (8)  $\beta = \frac{M-1}{\alpha}$  is the angle for the complementary equiangular tight frame.

*Corollary 5.2:* There exists only one equiangular tight frame for  $\mathbf{R}^N$  at a fixed angle  $1/\alpha$ . Hence, no subset of an equiangular tight frame for  $\mathbf{R}^N$  can form an equiangular tight frame for  $\mathbf{R}^N$ .

We have to be careful in interpreting Corollary 5.2. It just says that any subset of an equiangular tight frame cannot be an equiangular tight frame **for the same space**. For example, Tremain [19] has constructed a 28 element equiangular tight frame for  $\mathbf{R}^7$  which contains a 16 element subset which is an equiangular tight frame for  $\mathbf{R}^6$ .

Theorem 5.1 has the advantage that as we discover the maximal number of equiangular lines at an angle  $1/\alpha$  we can quickly discover the equiangular tight frames which can be derived from this. The following table shows how we apply Theorem 5.1 for the angle  $1/5$ . In this table we compute, for  $\alpha = 5$ ,

$$M = \frac{(\alpha^2 - 1)N}{\alpha^2 - N} = \frac{24N}{25 - N},$$

for  $5 \leq N \leq 5^2 - 2 = 23$ . We then have an equiangular tight frame for  $\mathbf{R}^N$  if and only if  $M$  is an even integer and there exist  $M$  equiangular lines in  $\mathbf{R}^N$  at angle  $1/5$ . The two question marks again appear because we are not sure yet if the requisite number of equiangular lines exist. ■

**Table II:** Equiangular tight frames at angle  $1/5$ .

N	$M = \frac{24N}{25-N}$	N	$M = \frac{24N}{25-N}$
5	6	15	36
6	$\frac{24 \cdot 6}{19}$ NO	16	$\frac{24 \cdot 16}{9}$ NO
7	$\frac{24 \cdot 7}{18}$ NO	17	51 NO
8	$\frac{24 \cdot 8}{17}$ NO	18	$\frac{24 \cdot 18}{7}$ NO
9	$\frac{24 \cdot 9}{16}$ NO	19	76?
10	16	20	96?
11	$\frac{24 \cdot 11}{15}$ NO	21	126
12	$\frac{24 \cdot 12}{14}$ NO	22	176
13	26	23	276
14	$\frac{24 \cdot 14}{13}$ NO		

Now we can list all the equiangular tight frames for dimensions less than or equal to 50. The question marks indicate that we do not know the existence yet of the requisite number of equiangular lines. In [18] they give a listing of the equiangular tight frames with fewer than 100 elements.

**Table III:** All equiangular tight frames for dimensions  $N \leq 50$

N	M	$1/\alpha$	N	M	$1/\alpha$
3	6	$1/\sqrt{5}$	25	50	$1/7$
5	10	$1/3$	27	54	$1/\sqrt{53}$
6	16	$1/3$	28	64	$1/7$
7	14	$1/\sqrt{13}$	31	62	$1/\sqrt{61}$
7	28	$1/3$	33	66	$1/\sqrt{65}$
9	18	$1/\sqrt{17}$	35	120	$1/7?$
10	16	$1/5$	36	64	$1/9$
13	26	$1/5$	37	74	$1/\sqrt{73}$
15	30	$1/\sqrt{29}$	37	148	$1/7?$
15	36	$1/5$	41	82	$1/9$
19	38	$1/\sqrt{37}$	41	246	$1/7$
19	76	$1/5?$	42	288	$1/7?$
20	96	$1/5?$	43	86	$1/\sqrt{85}$
21	28	$1/9$	43	344	$1/7$
21	36	$1/7$	45	90	$1/\sqrt{89}$
21	42	$1/\sqrt{41}$	45	100	$1/9$
21	126	$1/5$	45	540	$1/7?$
22	176	$1/5$	46	736	$1/7?$
23	46	$1/\sqrt{45}$	49	98	$1/\sqrt{97}$
23	276	$1/5$			

**Proof for Table III:**

$(N, M, \frac{1}{\alpha})$  represents an  $M$ -element equiangular tight frame existing in  $\mathbf{R}^N$  at angle  $1/\alpha$ .

- (1)  $(3, 6, \frac{1}{\sqrt{5}})$ ,  $(5, 10, 1/3)$ ,  $(6, 16, 1/3)$ ,  $(7, 28, 1/3)$ ,  $(15, 36, 1/5)$ ,  $(21, 126, 1/5)$ ,  $(22, 176, 1/5)$ ,  $(23, 276, 1/5)$ ,  $(41, 246, 1/7)$ ,  $(43, 344, 1/7)$ .

In these cases  $(N, M, 1/\alpha)$  exist by Table I.

- (2)  $(7, 14, \frac{1}{\sqrt{13}})$ ,  $(9, 18, \frac{1}{\sqrt{17}})$ ,  $(13, 26, 1/5)$ ,  $(15, 30, \frac{1}{\sqrt{29}})$ ,  $(19, 38, \frac{1}{\sqrt{37}})$ ,  $(21, 42, \frac{1}{\sqrt{42}})$ ,  $(23, 46, \frac{1}{\sqrt{45}})$ ,  $(25, 50, 1/7)$ ,  $(27, 54, \frac{1}{\sqrt{53}})$ ,  $(28, 64, 1/7)$ ,  $(31, 62, \frac{1}{\sqrt{61}})$ ,  $(33, 66, \frac{1}{\sqrt{65}})$ ,  $(41, 82, 1/9)$ ,  $(43, 86, \frac{1}{\sqrt{85}})$ ,  $(45, 90, \frac{1}{\sqrt{89}})$ ,  $(45, 100, 1/9)$ ,  $(49, 98, \frac{1}{\sqrt{97}})$ .

These  $(N, M, \frac{1}{\alpha})$  exist by know examples from strongly regular graphs (See Sustik, Tropp, Dhillon, Heath [18]).

- (3)  $(10, 16, 1/5)$ ,  $(21, 28, 1/9)$ ,  $(21, 36, 1/7)$ ,  $(36, 64, 1/9)$ .

Also, there is a strongly regular graph  $(73, 36, 17, 18)$  which is know as 73-Paley which shows the existence of  $37, 74, \frac{1}{\sqrt{73}}$ .

These  $(N, M, \frac{1}{\beta})$  are the complementary equiangular tight frames for the frames given in (1).

- (4)  $N = 4, 8, 11, 12, 14, 16, 18, 24, 26, 29, 30, 32, 34, 38, 39, 40, 44, 48$ .

These dimensions have no

$$\alpha = \sqrt{\frac{N(M-1)}{M-N}},$$

an odd integer. The only other case is  $M = 2N$  and these dimensions fail to have equiangular tight frames with  $2N$  vectors by [18] since  $2N - 1$  is not the sum of two squares.

(5) The only other cases where  $\alpha$  is an odd integer are  $(N, M)$  where:

- (a)  $(17, 51)$ .  
 (b)  $(25, 28)$   
 (c)  $(27, 40)$ .  
 (d)  $(33, 45)$ ,  $(33, 55)$ ,  $(33, 99)$ .  
 (e)  $(45, 50)$ ,  $(45, 56)$ .  
 (f)  $(49, 51)$ ,  $(49, 55)$ .  
 (g)  $(50, 64)$ .

(a), (d), (f) fail since  $M$  is not even.  
 (b), (c), (e), (g) fail since they would imply that  $(3, 28)$ ,  $(13, 40)$ ,  $(5, 50)$ ,  $(11, 56)$  and  $(14, 64)$  exist as complementary equiangular tight frames. But these exceed the maximal number of equiangular lines in their dimensions (See Table I).

## VI. SOME OPEN PROBLEMS

In this section we make a few conjectures concerning equiangular line sets.

We have seen that if  $\{f_m\}_{m=1}^M$  is an equiangular tight frame for  $\mathbf{R}^N$  at angle  $1/\alpha$  then  $\alpha \leq N \leq \alpha^2 - 2$ . The corresponding result for equiangular lines should be:

*Conjecture 6.1:* If  $M$  is the maximal number of equiangular lines in  $\mathbf{R}^N$  at angle  $1/\alpha$  and if these lines span  $\mathbf{R}^N$  then  $\alpha \leq N \leq \alpha^2 - 2$ .

Conjecture 6.1 would resolve several issues surrounding Table I. We know that the maximum number of equiangular lines in

$\mathbf{R}^N$  for  $7 \leq N \leq 13$  is 28 and these lines occur at angle  $1/3$ . But we do not know if these lines can span  $\mathbf{R}^N$  for  $N \neq 7$ . A positive solution to Conjecture 6.1 would show that these lines must span a 7-dimensional subspace of  $\mathbf{R}^N$ . A similar result would hold for the 276 equiangular lines in  $\mathbf{R}^N$  for  $23 \leq N \leq 41$  at angle  $1/5$ . The case of  $\mathbf{R}^{14}$  is also quite tangled. It is known [15], [7], [19] that  $\mathbf{R}^{14}$  has a maximum number of equiangular lines at angle  $1/3$  of 28. Again, we do not believe that these lines can span  $\mathbf{R}^{14}$ . However, there are [19] 26 equiangular lines in  $\mathbf{R}^{14}$  at angle  $1/3$  which span  $\mathbf{R}^{14}$ . The maximal number of equiangular lines in  $\mathbf{R}^{14}$  is not known. It can be shown [19] that  $\mathbf{R}^{14}$  also has 28 equiangular lines at angle  $1/5$  and these lines span  $\mathbf{R}^{14}$ . It can also be shown [7] that the maximal number of equiangular lines in  $\mathbf{R}^{14}$  is  $\leq 30$  and that these lines must occur at angle  $1/5$ .

We have seen that if  $\mathbf{R}^N$  has an equiangular tight frame with  $M$  elements, then  $M$  must be even. One can ask the same question for maximal equiangular line sets.

*Conjecture 6.2:* The maximal number of equiangular lines in  $\mathbf{R}^N$  must be even.

We know that an upper bound for the maximal number of equiangular lines in  $\mathbf{R}^N$  is

$$M = \frac{N(N+1)}{2},$$

and that this number can occur only when  $N = \alpha^2 - 2$ . This raises the problem:

*Problem 6.3:* For what values of  $N = \alpha^2 - 2$  with  $\alpha$  an odd integer, does there exist

$$M = \frac{N(N+1)}{2},$$

equiangular lines in  $\mathbf{R}^N$ .

This problem has been verified for  $\alpha = 3, 5$ . Recently Bannai, Munemasa, and Venkov [1] have shown that this conjecture fails for  $\alpha = 7$ . The authors also show that this conjecture fails for infinitely many values of  $\alpha$ . These examples provide further counter-examples to the conjecture that the conditions in Theorem 4.1 are sufficient for the existence of equiangular tight frames. That is, for the extremal cases  $N = \alpha^2 - 2$  with  $\alpha$  an odd integer, and  $M = \frac{N(N+1)}{2}$  we have that

$$\alpha = \sqrt{\frac{N(M-1)}{M-N}}.$$

Also,

$$M = \frac{N(\alpha^2 - 1)}{2} = N \frac{(\alpha - 1)(\alpha + 1)}{2}.$$

So  $M$  is even since both  $\alpha - 1$  and  $\alpha + 1$  are even. By Corollary 4.4, we have that both  $\alpha$  and  $\beta$  are odd integers.

It can be shown [7] that an affirmative answer to Conjecture 6.1 and identifying the class of integers which satisfy Problem 6.3 would go a long way towards classifying all the maximal number of equiangular line sets. But this will have to wait for further progress.

## VII. CONCLUSION

Today, Frame theory has broad applications in mathematics and engineering. Equiangular tight frames are especially useful because of their exact structure. The problem is that we do not have very many examples of equiangular tight frames. We have presented here the *state of the art* for equiangular tight frames. This includes a number of restrictions on their existence which helps explain why there are so few of them. We have also seen that the equiangular tight frame problem is heavily dependent on the real equiangular line problem which, in itself, is one of the deepest and most difficult problems in mathematics and which has yielded a solution for only 35 dimensions in 60 years of research. We have seen that it is possible to classify the equiangular tight frames for dimensions less than or equal to 50. Further developments here require further developments on the real equiangular line problem.

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