Frame expansions in separable Banach spaces

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December 9, 2008

Abstract

Banach frames are defined by straightforward generalization of (Hilbert space) frames. We characterize Banach frames (and X_d-frames) in separable Banach spaces, and relate them to series expansions in Banach spaces. In particular, our results show that we can not expect Banach frames to share all the nice properties of frames in Hilbert spaces.

1 Introduction

Let X denote a separable Banach space and $\{g_i\}$ be a sequence in the dual X^*. A central question is whether we can find a sequence $\{f_i\}$ in X such that the *reconstruction property*

$$f = \sum g_i(f) f_i \tag{1}$$

holds for all $f \in X$.

Banach frames were introduced by Gröchenig as a terminology to express expansions like (1). Banach frames for X are defined with respect to certain sequence spaces X_d (see Definition 1.3). In this paper we characterize pairs of spaces (X, X_d) for which Banach frames (and the related X_d-frames) exist. Furthermore we reveal the connections between Banach frames and the reconstruction property. We also prove that if $\{g_i\} \subset X^*$ is total on X, then we can always find a sequence space X_d such that $\{g_i\}$ is a Banach frame for X w.r.t. X_d. In particular, this leads to a (somewhat unwanted) example of a Banach frame for a Hilbert space, which is not a Hilbert frame.

Our starting point is the concept of p-frames, which was introduced by Aldroubi et al. [1] as a tool to obtain series expansions in shift-invariant spaces. An analysis of p-frames in general Banach spaces appeared in [6]. In order to gain more flexibility, we extend the definition to more general sequence spaces in Definition 1.2.

In the rest of this introduction we state the main definitions. Then, in Section 2 we discuss the main results and their implications. Most proofs, and further remarks, are finally collected in Section 3.

Definition 1.1 *A sequence space X_d is called a BK-space, if it is a Banach space and the coordinate functionals are continuous on X_d, i.e. the relations $x_n = \{\alpha_j^{(n)}\}, x = \{\alpha_j\} \in X_d, \lim_{n\to\infty} x_n = x$ imply $\lim_{n\to\infty} \alpha_j^{(n)} = \alpha_j$ $(j = 1, 2, ...)$.*

Definition 1.2 *Let X be a Banach space and X_d be a BK-space. A countable family $\{g_i\}_{i\in I}$ in the dual X^* is called an X_d-frame for X if*

(i) $\{g_i(f)\} \in X_d, \ \forall f \in X$;

(ii) the norms $\|f\|_X$ and $\|\{g_i(f)\}\|_{X_d}$ are equivalent, i.e. there exist constants $A, B > 0$ such that

$$A||f||_X \leq \|\{g_i(f)\}\|_{X_d} \leq B||f||_X, \ \forall f \in X. \tag{2}$$

A and B are called *X_d-frame bounds*. If at least (i) and the upper condition in (2) are satisfied, $\{g_i\}$ is called an X_d-Bessel sequence for X.

If X is a Hilbert space and $X_d = \ell^2$, (2) means that $\{g_i\}$ is a frame, and in this case it is well known that there exists a sequence $\{f_i\}$ in X such that

$$f = \sum \langle f, f_i \rangle g_i = \sum \langle f, g_i \rangle f_i.$$

Similar reconstruction formulas are not always available in the Banach space setting. This is the reason behind the following definition:

Definition 1.3 *Let X be a Banach space and X_d a sequence space. Given a bounded linear operator $S : X_d \to X$, and an X_d-frame $\{g_i\} \subset X^*$, we say that $(\{g_i\}, S)$ is a Banach frame for X with respect to X_d if*

$$S(\{g_i(f)\}) = f, \ \forall f \in X. \tag{3}$$

Note that (3) can be considered as some kind of "generalized reconstruction formula", in the sense that it tells how to come back to $f \in X$ based on the coefficients $\{g_i(f)\}$. The condition, however, does not imply reconstruction via an infinite series, as we will see later.

The X_d-frame condition implies that we can define an isomorphism

$$U : X \to X_d, \quad Uf := \{g_i(f)\} \;\; f \in X.$$

The extra condition in Definition 1.3 means that S is a left-inverse of U, and thus US is a bounded linear projection of X_d onto the range $R(U)$ of the operator U.

2 The main results

In this section we state the most important results. In order not to interrupt the flow, only very short proofs are included here; the more technical proofs are given in Section 3.

We first characterize the Banach spaces X which have an X_d-frame w.r.t. a given BK-space X_d:

Theorem 2.1 *Let X be a Banach space and X_d a BK-space. Then there exists an X_d-frame for X if and only if X is isomorphic to a subspace of X_d.*

Proof: From the definition, if $\{g_i\}$ is an X_d-frame for a Banach space X then the mapping $U : X \to X_d$ given by $U(f) = \{g_i(f)\}$ is an isomorphism of X into X_d.

For the converse, let X be a subspace of X_d and $\{f_i\}$ the coordinate functionals (which are assumed to be continuous). Let $g_i = f_i|_X$. Then for all $f \in X$, $\{g_i(f)\} = f \in X_d$ and $\|f\|_X = \|\{g_i(f)\}\|_{X_d}$. $\square$

Given an X_d-frame $\{g_i\}$, where X_d is a BK-space for which the canonical unit vectors form a basis, the next result clarifies which extra condition we need in order to ensure that $\{g_i\}$ is a Banach frame. A more detailed result is given in Proposition 3.4.

Proposition 2.2 *Suppose that X_d is a BK-space and that $\{g_i\} \subset X^*$ is an X_d-frame for X. If the canonical unit vectors $\{e_i\}$ form a basis for X_d, then the following conditions are equivalent:*

(i) $R(U)$ is complemented in X_d.

(ii) *There exists a linear bounded operator S, such that $(\{g_i\}, S)$ is a Banach frame for X with respect to X_d.*

(iii) *There exists an X_d^*-Bessel sequence $\{f_i\} \subset X \subseteq X^{**}$ for X^* such that*

$$f = \sum g_i(f) f_i, \quad \forall f \in X.$$

In case X_d does not have the canonical unit vectors as a basis, the reconstruction property might hold without $R(U)$ being complemented in X_d:

Example 2.3 Let $X = c_0, X_d = \ell_\infty$, and $\{g_i\}$ be the canonical unit vector basis of ℓ_1. Then $\{g_i\}$ is an X_d-frame for c_0, and by [10], $R(U) = c_0$ is not complemented in $X_d = \ell_\infty$. However, the reconstruction property holds, e.g., via the canonical unit vector basis $\{e_i\}$ of c_0.

A reformulation of Proposition 2.2 gives a characterization of spaces X possessing Banach frames:

Theorem 2.4 *A Banach space X has a Banach frame with respect to a given sequence space X_d if and only if X is isomorphic to a complemented subspace of X_d.*

Proof: The result follows from Proposition 3.4, but let us show how a Banach frame can be constructed if we assume that X is isomorphic to a complemented subspace of X_d. Let $T : X \to X_d$ be an isomorphism and let $P : X_d \to R(T)$ be a projection of X_d onto $R(T)$. Define $S : X_d \to X$ by $Sx = T^{-1}Px$. Let $\{e_i\}$ be the coordinate functionals of X_d and $y_i = T^* e_i$. For each $x \in X$ we have

$$y_i(x) = T^* e_i(x) = e_i(Tx)$$

Hence, $Tx = \{y_i(x)\}$.

Since T is an isomorphism, it follows that $(\{y_i\}, S)$ is a Banach frame with respect to X_d. $\square$

It is known [3] that every separable Banach space has a Banach frame. We will now describe a way to obtain such a Banach frame; for this we need the following definition.

Definition 2.5 *A family $\{g_i\}_{i \in I} \subset X^*$ is total on X, if*

$$g_i(x) = 0, \forall i \Rightarrow x = 0.$$

By [14], p. 189, when the family $\{g_i\} \subset X^*$ is total on X, the linear space

$$Z_d := \{\{g_i(x)\} \mid x \in X\}, \quad \|\{g_i(x)\}\|_{Z_d} := \|x\|_X \tag{4}$$

is a BK-space, isometrically isomorphic to X.

Every total system $\{g_i\}_{i\in I} \subset X^*$ is a Banach frame for X with respect to the corresponding BK-space Z_d:

Lemma 2.6 *Let $\{g_i\}_{i\in I} \subset X^*$ be a total system. Then there exists an operator $S : Z_d \to X$ such that $(\{g_i\}, S)$ is a Banach frame for X with respect to Z_d.*

Proof: The operator $G : X \to Z_d$ defined by $G(x) := \{g_i(x)\}$ is an isometrical isomorphism between X and Z_d and hence $(\{g_i\}, G^{-1})$ is a Banach frame for X with respect to Z_d. □

Lemma 2.6 has the following consequence, proved in Section 3.

Proposition 2.7 *For every separable Banach space X there exists a total system $\{g_i\}_{i\in I} \subset X^*$ such that the finite sequences are dense in the space Z_d given in* (4)*, and an operator $S : Z_d \to X$ such that $(\{g_i\}, S)$ is a Banach frame for X with respect to Z_d.*

It is well known that there exist separable Banach spaces having no basis; in this light, it is nice to know that there always exist Banach frames. However, the next example demonstrates that the Banach frames might not have the properties we are used to for Hilbert space frames, i.e., we might not gain what we want. In fact, we prove the existence of a Banach frame for a Hilbert space, which is not a frame, and which does not even have the reconstruction property. In order to exclude pathologies like this, it is necessary to exclude BK-spaces like Z_d in (4) from the definition of Banach frames.

Example 2.8 Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$. Consider the family $\{e_i + e_{i+1}\}_{i=1}^{\infty}$, which is complete, but not a frame for $\mathcal{H}$ ([4]). In fact, e_1 does not have a representation as an infinite sum $\sum_{i=1}^{\infty} c_i(e_i + e_{i+1})$. Moreover, there exists no family $\{f_i\} \subset \mathcal{H}$ such that $f = \sum \langle f, e_i + e_{i+1}\rangle f_i$ holds for all $f \in \mathcal{H}$. However, by Lemma 2.6 $\{e_i + e_{i+1}\}_{i=1}^{\infty}$ is a Banach frame for $\mathcal{H}$ with respect to the BK-space

$$\begin{aligned} Z_d = \{\{\langle h, e_i + e_{i+1}\rangle\} \mid h \in \mathcal{H}\} &= \{\{c_i + c_{i+1}\}, \mid \{c_i\} \in \ell^2\}, \\ \|\{c_i + c_{i+1}\}\|_{Z_d} &= \|\{c_i\}\|_{\ell^2}. \end{aligned}$$

For a given sequence $\{g_i\} \subset X^*$ we now present an equivalent condition for the existence of a sequence $\{f_i\} \subset X$ such that $f = \sum_i g_i(f) f_i$ for all $f \in X$.

Proposition 2.9 *Let $\{g_i\} \subset X^*$. The following are equivalent:*

(i) There exists a sequence $\{f_i\} \subset X$ such that $f = \sum_i g_i(f) f_i$, for all $f \in X$.

(ii) There is a BK-space X_d with the canonical unit vectors $\{e_i\}$ as a basis so that $\{g_i\}$ is an X_d-frame for X and an operator $S : X_d \to X$ so that $(\{g_i\}, S)$ is a Banach frame for X with respect to X_d.

If the conditions are satisfied, a choice of $\{f_i\}$ in (i) is $f_i = S(e_i)$.

The importance of this proposition is the following. We know we cannot hope to get reconstruction just from the existence of an X_d-frame since there are spaces with no reconstruction for any family $\{g_i\}$ (i.e. spaces failing the approximation property [2]). So if we are going to be able to use the existence of an X_d-frame for reconstruction, we must have *some* connection between X_d and X. That is, we need some type of operator going back from X_d to X. The above proposition formalizes this fact.

In general, having *reconstruction* is much different from having a basis. Assume for example that $\{f_i\}$ is a basis for a Banach space X, with the biorthogonal sequence $\{g_i\}$. If P is any bounded linear projection on $X, P \neq I$, then for every $f \in P(X)$ we have:

$$f = \sum_i g_i(f) f_i = \sum_i P^*(g_i)(f) P(f_i).$$

That is, the reconstruction property (1) holds with g_i replaced by $P^*(g_i)$ and f_i replaced by $P(f_i)$; however, $\{P(f_i)\}$ is not a basic sequence. Also, the existence of reconstruction families $\{f_i, g_i\}$ does not imply that the space X needs to have any basis at all. For example, there is a Banach space with a basis $\{f_i\}$ (see the discussion following Definition 1.11, page 279 in [2]) and a bounded linear projection P on $\overline{span}\{f_i\}$ so that $P(\overline{span}\{f_i\})$ fails to have a basis (see Proposition 6.7, page 301 of [2]). However, as we saw above $P(\overline{span}\{f_i\})$ does have a countable family of reconstruction functions.

Our final result below expresses the key problem if we want to obtain reconstruction via a given sequence $\{g_i\}$: we can always satisfy the conditions in Proposition 2.9(ii) for a certain choice of X_d, except that the canonical unit vectors might only be a basis for a subspace of X_d.

Theorem 2.10 *If X is a separable Banach space then there is a family $\{g_i\} \subset X^*$ and a separable BK-space X_d containing the canonical unit vectors as a basic sequence so that $\{g_i\}$ is a n X_d-frame for X and $R(U)$ is complemented in X_d.*

3 Proofs and auxiliary results

3.1 General results

We need a general result about continuous linear functionals on X_d; for its proof we refer to [9], page 201. Let us denote the dual space $(X_d)^*$ by X_d^*.

Lemma 3.1 *Let X_d be a BK-space for which the canonical unit vectors $\{e_i\}$ form a Schauder basis. Then the space $Y_d := \{\{h(e_i)\} \mid h \in X_d^*\}$ with the norm $||\{h(e_i)\}||_{Y_d} := ||h||_{X_d^*}$ is a BK-space isometrically isomorphic to X_d^*. Also, every continuous linear functional Φ on X_d has the form*

$$\Phi\{c_i\} = \sum c_i d_i,$$

where $\{d_i\} \in Y_d$ is uniquely determined by $d_i = \Phi(e_i)$, and

$$||\Phi|| = ||\{\Phi(e_i)\}||_{Y_d}.$$

Throughout the paper when we use the dual X_d^* of a BK-space X_d having the canonical unit vectors as a basis, we will identify X_d^* with its isometrically isomorphic BK-space constructed by the above Lemma.

Proposition 3.2 *Let X_d be a BK-space, for which the canonical unit vectors form a basis. Then $\{g_i\} \subset X^*$ is an X_d^*-Bessel sequence for X with bound B if and only if the operator*

$$T : \{d_i\} \to \sum d_i g_i \tag{5}$$

is well-defined (hence bounded) from X_d into X^* and $\|T\| \leq B$.

Proof: First, let $\{g_i\} \subset X^*$ be an X_d^*-Bessel sequence for X with bound B and let $\{e_i\}$ be the canonical unit vector basis of X_d. Define $R : X \to X_d^*$ by $R(f) = \{g_i(f)\}$; then $\|R\| \leq B$. The linear bounded operator $R^* : X_d^{**} \to X^*$ satisfies

$$R^*(e_j)(f) = e_j(R(f)) = g_j(f), \quad \forall f \in X,$$

and thus $R^*e_j = g_j$. Letting $T = R^*|_{X_d}$ we have that $\|T\| \le \|R^*\| = \|R\| \le B$. Finally, $T(\{d_i\}) = T(\sum_i d_i e_i) = \sum_i d_i g_i$.

Now suppose that $T : X_d \to X^*$ given by $T(\{d_i\}) = \sum_i d_i g_i$ is well-defined and thus bounded by the Banach –Steinhaus theorem. Then $T(e_i) = g_i$ and for every $f \in X$ the bounded operator $T^* : X^{**} \to X_d^*$ satisfies

$$T^*(f)(e_i) = f(T(e_i)) = f(g_i).$$

That is, $\{g_i(f)\} = \{T^*(f)(e_i)\}$ which is identified with $T^*(f)$ by Lemma 3.1. So $\{g_i\}$ is an X_d^*-Bessel sequence for X with a bound $\|T^*\| = \|T\| \le B$. □

Corollary 3.3 *Let X_d be a BK-space, whose dual X_d^* has the canonical unit vectors as a basis. If $\{g_i\} \subset X^*$ is an X_d-Bessel sequence for X with bound B then the operator*

$$T : \{d_i\} \to \sum d_i g_i \tag{6}$$

is well-defined (hence bounded) from X_d^* into X^* and $\|T\| \le B$. If X_d is reflexive, the converse is true.

Proof: By Proposition 3.2, $\{g_i\} \subset X^*$ is a X_d^{**}-Bessel sequence for X with bound B if and only if the operator $T : \{d_i\} \to \sum d_i g_i$ is well defined from X_d^* into X^* and $\|T\| \le B$. Clearly, every X_d-Bessel sequence for X with bound B is an X_d^{**}-Bessel sequence for X with bound B, and the converse is true when X_d is reflexive. □

The following result relates X_d-frames to Banach frames and the question of discrete expansions in X and X^*. It extends Proposition 2.2.

Proposition 3.4 *Suppose that X_d is a BK-space and that $\{g_i\} \subset X^*$ is an X_d-frame for X. Then the following conditions are equivalent:*

(i) $R(U)$ is complemented in X_d.

(ii) The operator $U^{-1} : R(U) \to X$ can be extended to a bounded linear operator $V : X_d \to X$.

(iii) There exists a linear bounded operator S, such that $(\{g_i\}, S)$ is a Banach frame for X with respect to X_d.

Also, the condition

(iv) There exists a family $\{f_i\} \subset X$ *such that* $\sum c_i f_i$ *is convergent for all* $\{c_i\} \in X_d$ *and* $f = \sum g_i(f) f_i,\ \forall f \in X.$

implies each of (i)-(iii). If we also assume that the canonical unit vectors $\{e_i\}$ *form a basis for* X_d*, (iv) is equivalent to the above (i)-(iii) and to the following condition (v):*

(v) There exists an X_d^**-Bessel sequence* $\{f_i\} \subset X \subseteq X^{**}$ *for* X^* *such that*

$$f = \sum g_i(f) f_i, \quad \forall f \in X.$$

If the canonical unit vectors form a basis for both X_d *and* X_d^**, (i)-(v) is equivalent to*

(vi) There exists an X_d^**-Bessel sequence* $\{f_i\} \subset X \subseteq X^{**}$ *for* X^* *such that*

$$g = \sum g(f_i) g_i,\ \forall g \in X^*.$$

In each of the cases (v) and (vi), $\{f_i\}$ *is actually an* X_d^**-frame for* X^**.*

Proof: For convenience, we index $\{f_i\}$ and $\{g_i\}$ by the natural numbers. Suppose that X_d is a BK-space. (i)$\Rightarrow$ (ii) is trivial. For the converse, assume (ii) and let $V : X_d \to X$ be a linear bounded extension of U^{-1}. Now consider the bounded operator $P : X_d \to R(U)$ defined by $P = UV$. Using the fact that $VU = I$ (on X), we get $P^2 = P$. For every $f \in X$, we have

$$Uf = UVUf = P(Uf) \in R(P).$$

Hence $R(P) = R(U)$, i.e., the range of U equals the range of a bounded projection. Thus, $R(U)$ is complemented (see [12], p. 127). The equivalence (ii)$\Leftrightarrow$ (iii) is clear.

We now relate the condition (iv) to (i)–(iii). First, assume that (iv) is satisfied. By assumption we can define an operator

$$V : X_d \to X \quad by \quad V : \{c_i\} \to \sum c_i f_i.$$

By the Banach–Steinhaus theorem, V is bounded. Let $\{g_i(f)\} \in R(U)$. Furthermore,

$$V\{g_i(f)\} = \sum g_i(f) f_i = f = U^{-1}Uf = U^{-1}\{g_i(f)\},$$

i.e., V is an extension of U^{-1}. That is, (ii) holds; according to the equivalences proved so far, this means that (i)–(iii) holds.

Assume now that the canonical unit vectors $\{e_i\}$ form a basis for X_d. Assuming that (ii) is satisfied, we will show that (iv) holds. Let $f_i := Ve_i$. Since V is linear and bounded, for all $\{c_i\} \in X_d$ we have

$$\sum_{i=1}^{n} c_i f_i = V(\sum_{i=1}^{n} c_i e_i) \to V\{c_i\}.$$

That is, $\sum c_i f_i$ is convergent. Also, by construction, for all $f \in X$ we have

$$f = VUf = \sum g_i(f) f_i. \tag{7}$$

Thus (iv) holds, as claimed.

Still assuming that the canonical unit vectors $\{e_i\}$ form a basis for X_d, we now prove the equivalence of (iv) and (v). First, assume that (iv) holds. Due to the equivalence of (ii) and (iv), we can (as before) define $f_i := Ve_i$, and the equation (7) is available. By Lemma 3.1, for every $g \in X^*$ we have

$$\{g(f_i)\} = \{gV(e_i)\} \in X_d^* \text{ and } ||\{g(f_i)\}||_{X_d^*} = ||gV|| \le \|V\| \ \|g\|_{X^*},$$

hence $\{f_i\}$, considered as a sequence in X^{**}, is a n X_d^*-Bessel sequence for X^*. Thus, we have proved the claims in (v). On the other hand, if (v) is valid then Proposition 3.2 shows that $\sum c_i f_i$ is convergent for all $\{c_i\} \in X_d$ and hence (iv) holds.

Assume now that the canonical unit vectors form a basis for both X_d and X_d^*; in this case, we want to prove the equivalence of (v) and (vi). We will let B denote a Bessel bound for the X_d-Bessel sequence $\{g_i\}$. Denote the canonical basis for X_d by $\{e_i\}$ and the canonical basis for X_d^* by $\{z_i\}$. Assume that (v) is valid. Let $g \in X^*$; given $n \in \mathbb{N}$,

$$\begin{aligned} ||g - \sum_{i=1}^{n} g(f_i) g_i||_{X^*} &= \sup_{f \in X, ||f||=1} |g(f) - \sum_{i=1}^{n} g(f_i) g_i(f)| \\ &= \sup_{f \in X, ||f||=1} |\sum_{i=1}^{\infty} g(f_i) g_i(f) - \sum_{i=1}^{n} g(f_i) g_i(f)| \\ &= \sup_{f \in X, ||f||=1} |\sum_{i=n+1}^{\infty} g(f_i) g_i(f)| \\ &\le B|| \sum_{i=n+1}^{\infty} g(f_i) z_i|| \to 0 \text{ as } n \to \infty \end{aligned}$$

and hence (vi) holds. Assume now (vi). Let K be an X_d^*-Bessel bound for $\{f_i\}$. For every $g \in X^*$, $\{g(f_i)\}$ belongs to X_d^*, which by Lemma 3.1 is isometrically isomorphic to the space $\{\{G(e_i)\} : G \in X_d^*\}$, and hence $\{g(f_i)\}$ can be identified with $\{G_g(e_i)\}$ for a unique $G_g \in X_d^*$. Then for every $f \in X$,

$$\begin{aligned}
||f - \sum_{i=1}^{n} g_i(f) f_i||_X &= \sup_{g \in X^*, ||g||=1} |g(f) - \sum_{i=1}^{n} g(f_i) g_i(f)| \\
&= \sup_{g \in X^*, ||g||=1} |\sum_{i=1}^{\infty} g(f_i) g_i(f) - \sum_{i=1}^{n} g(f_i) g_i(f)| \\
&= \sup_{g \in X^*, ||g||=1} |\sum_{i=n+1}^{\infty} g(f_i) g_i(f)| \\
&= \sup_{g \in X^*, ||g||=1} |G_g \left(\sum_{i=n+1}^{\infty} g_i(f) e_i \right)| \\
&\leq \sup_{g \in X^*, ||g||=1} ||G_g||\ ||\sum_{i=n+1}^{\infty} g_i(f) e_i|| \\
&= \sup_{g \in X^*, ||g||=1} ||\{g(f_i)\}||\ ||\sum_{i=n+1}^{\infty} g_i(f) e_i|| \\
&\leq K\ ||\sum_{i=n+1}^{\infty} g_i(f) e_i|| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence (v) is valid. Moreover, by a similar calculations as above, for every $g \in X^*$ we have

$$||g|| = \sup_{f \in X, ||f||=1} |g(f)| = \sup_{f \in X, ||f||=1} |\sum g(f_i) g_i(f)| \leq B\ ||\{g(f_i)\}||_{X_d^*},$$

and hence $\{f_i\}$ is an X_d^*-frame for X^*.
□

We now consider certain special choices of the BK-space in the definition of a Banach frame. It turns out that some of them lead to undesired properties in the sense that there exist Banach frames without the properties one usually associates with frames.

3.2 Proof of Proposition 2.7

It is well known (see e.g., [14], p. 219, 226) that every separable Banach space X has a M-basis, i.e., there exists a biorthogonal system $\{x_i, g_i\} \subset X \times X^*$ such that $\{g_i\}$ is total on X and $\overline{\text{span}}\{x_i\} = X$.

Proof of Proposition 2.7: Let $\{x_i, g_i\} \subset X \times X^*$ be a M-basis for X. The operator $G : X \to Z_d$ defined by $G(x) := \{g_i(x)\}$ is an isometrical isomorphism between X and Z_d and hence $(\{g_i\}, G^{-1})$ is a Banach frame for X with respect to Z_d. Since $\{g_i\}$ has a biorthogonal sequence, all the canonical unit vectors, and hence all finite sequences, belong to Z_d. Let now $x \in X$ and fix an arbitrary $\epsilon > 0$.Then there exist $c_{i_1}, c_{i_2}, ..., c_{i_N}$ such that

$$\|x - \sum_{k=1}^{N} c_{i_k} x_{i_k}\| < \epsilon.$$

Then for the finite sequence

$$\{g_i(\sum_{k=1}^{N} c_{i_k} x_{i_k})\}_i = \{\sum_{k=1}^{N} c_{i_k} \delta_{i,i_k}\} = \{0, ..., c_{i_1}, 0, ..., c_{i_N}, 0...\}$$

we have

$$\|\{g_i(x)\} - \{g_i(\sum_{k=1}^{N} c_{i_k} x_{i_k})\}\|_{Z_d} = \|\{g_i(x - \sum_{k=1}^{N} c_{i_k} x_{i_k})\}\|_{Z_d} = \|x - \sum_{k=1}^{N} c_{i_k} x_{i_k}\|_X < \epsilon.$$

Thus the finite sequences are dense in Z_d. □

3.3 Proof of Proposition 2.9

We need a lemma before we give the proof.

Lemma 3.5 *Let $\{f_i\} \subset X \setminus \{0\}$. Then the sequence space*

$$X_d = \{\{c_i\} \ : \ \sum c_i f_i \text{ converges in } X\}$$

with the norm

$$||\{c_i\}||_{X_d} := \sup_N || \sum_{i=1}^{N} c_i f_i||_X \tag{8}$$

is a Banach space, for which the canonical unit vectors form a basis.

Proof: It is well known that X_d is a Banach space (see e.g., [13] p.18). By the definition of X_d, all canonical unit vectors e_i belong to X_d. To show that they form a basis for X_d, it is enough to prove that $\{e_i\}$ is complete and that there exists a constant $C \geq 1$ such that for every $m \geq n$ and scalars $c_1, c_2, ..., c_m$, the inequality $||\sum_{i=1}^n c_i e_i|| \leq C||\sum_{i=1}^m c_i e_i||$ holds. Clearly, for every $m \geq n$ and every $c_1, c_2, ..., c_m$, we have

$$||\sum_{i=1}^n c_i e_i||_{X_d} = \sup_{N\leq n} ||\sum_{i=1}^N c_i f_i||_X \leq \sup_{N\leq m} ||\sum_{i=1}^N c_i f_i||_X = ||\sum_{i=1}^m c_i e_i||_{X_d}.$$

Choose now arbitrary $\{c_i\}$ from X_d and fix arbitrary $\epsilon > 0$. Since $\sum c_i f_i$ converges in X, there exists N_0 such that $||\sum_{i=n+1}^m c_i f_i||_X < \frac{\epsilon}{2}$, $\forall\ m > n > N_0$ and therefore $\sup_{N>n} ||\sum_{i=n+1}^N c_i f_i||_X \leq \frac{\epsilon}{2} < \epsilon$, $\forall\ n > N_0$. Thus, for every $n > N_0$ we have

$$||\{c_i\} - \sum_{i=1}^n c_i e_i||_{X_d} = \sup_{N>n} ||\sum_{i=n+1}^N c_i f_i||_X < \epsilon.$$

Hence $\{e_i\}$ is complete in X_d, which concludes the proof. □

For a given sequence $\{g_i\} \subset X^*$, an equivalent condition for the existence of a sequence $\{f_i\} \subset X$ such that $f = \sum_i g_i(f) f_i$ for all $f \in X$ is given in Proposition 2.9. In case such a representation is possible for a sequence where $f_i \neq 0$, the appearing sequence space equals the one defined in Lemma 3.5, but in the general case a slightly more involved definition is needed:

Proof of Proposition 2.9: (i)⇒ (ii): First we divide the indices $\mathbb{N}$ into two sets:

$$A = \{i : f_i = 0\},$$

and B is the rest of the indices. We define

$$c_0(A) = \{\{c_i\}_{i\in A} : \lim_i c_i = 0\}$$

and norm this space with the sup norm. The canonical unit vectors $\{e_i\}_{i\in A}$ form an unconditional basis for $c_0(A)$. Hence, $\{\frac{1}{i(\|g_i\|+1)} e_i\}_{i\in A}$ is also a basis for $c_0(A)$. Let

$$Z_d := \{\{c_i\}_{i\in A} | \sum_{i\in A} \frac{c_i}{i(\|g_i\|+1)} e_i \text{ converges in } c_0(A)\},$$

with the norm

$$\|\{c_i\}_{i\in A}\|_{Z_d} = \|\sum_{i\in A} \frac{c_i}{i(\|g_i\|+1)} e_i\|_{c_0(A)}.$$

So Z_d has the canonical unit vectors $\{e_i\}_{i\in A}$ as a basis and thus Z_d is a BK-space. Let Y_d be the BK-space defined in Lemma 3.5 for the indices in B. Let $X_d = Y_d \oplus Z_d$ with norm

$$\|y\oplus z\|_{X_d} = \|y\|_{Y_d} + \|z\|_{Z_d}.$$

For every $f \in X$, $\sum_{i\in B} g_i(f) f_i$ converges in X and so $\{g_i(f)\}_{i\in B} \in Y_d$. Also, for every $f \in X$ and every $i \in A$ we have

$$|\frac{g_i(f)}{i(\|g_i\|+1)}| \le |\frac{\|g_i\|\|f\|}{i(\|g_i\|+1)}| \le \frac{\|f\|}{i} \quad \text{and thus} \quad \lim_{i\in A} \frac{g_i(f)}{i(\|g_i\|+1)} = 0,$$

which implies that $\{g_i(f)\}_{i\in A}$ is an element of Z_d. Therefore, for all $f \in X$, $\{g_i(f)\}_{i\in B} \oplus \{g_i(f)\}_{i\in A}$ belongs to X_d.

For the proof of the X_d-frame inequalities we use an idea from [8]. For convenience, assume that $\{g_i\}_{i\in B}$ is indexed by $\mathbb{N}$. Consider the linear bounded operators $S_n : X \to X$, $n \in \mathbb{N}$, defined by $S_n f = \sum_{i=1}^{n} g_i(f) f_i$. For every $f \in X$ we have $f = \sum_{i=1}^{\infty} g_i(f) f_i$ and thus the sequence $\{S_n(f)\}$ is convergent and hence bounded, which implies that $\sup_n ||S_n(f)|| < \infty$. Therefore, by the Uniform Boundedness Principle, $\sup_n ||S_n|| < \infty$, and for every $f \in X$ (and taking into account that $f_i = 0$ for all $i \in A$),

$$||f||_X = \lim_{n\to\infty} ||\sum_{i=1}^{n} g_i(f) f_i||_X \le \sup_n ||\sum_{i=1}^{n} g_i(f) f_i||_X = ||\{g_i(f)\}_{i\in B}||_{Y_d}.$$

So,

$$\|f\|_X \le \|\{g_i(f)\}_{i\in B}\| + \|\{g_i(f)\}_{i\in A}\| = \|\{g_i(f)\}_{i\in B} \oplus \{g_i(f)\}_{i\in A}\|_{X_d}.$$

Also,

$$\begin{aligned} ||\{g_i(f)\}_{i\in B} \oplus \{g_i(f)\}_{i\in A}||_{X_d} &= \|\{g_i(f)\}_{i\in B}\|_{Y_d} + \|\{g_i(f)\}_{i\in A}\|_{Z_d} \\ &= \sup_n ||S_n(f)|| + \sup_{i\in A} |\frac{g_i(f)}{i(\|g_i\|+1)}| \\ &\leq \|f\| \sup_n ||S_n|| + \|f\| \sup_{i\in A} \frac{\|g_i\|}{i(\|g_i\|+1)} \\ &\leq (\sup_n ||S_n|| + 1)||f||_X. \end{aligned}$$

Finally, define $S : X_d \to X$ by $S(e_i) = f_i$ for all i, where $\{e_i\}_{i\in B}$ are the canonical unit vectors in Y_d. For every $\{c_i\}_{i\in B} \oplus \{d_i\}_{i\in A} \in X_d$, we have $\sum_{i\in A} d_i f_i = 0$, $\sum_{i\in B} c_i f_i$ converges in X and

$$\begin{aligned} ||S(\{c_i\}_{i\in B} \oplus \{d_i\}_{i\in A})||_X &= ||\sum_{i\in B} c_i f_i||_X \leq ||\{c_i\}_{i\in B}||_{Y_d} \\ &\leq ||\{c_i\}_{i\in B} \oplus \{d_i\}_{i\in A}||_{X_d}. \end{aligned}$$

Thus S is bounded.

(ii)$\Rightarrow$ (i): This is immediate from Proposition 3.4. □

3.4 Proof of Theorem 2.10

Choose a countable dense subset $\{x_i\}$ of the unit sphere of X. By the Han-Banach theorem, for each i, choose $g_i \in X^*$ with $\|g_i\| = 1$ and $g_i(x_i) = 1$. Thus for every $f \in X$, $\sup_i |g_i(f)| \leq ||f||$. Define $U : X \to \ell_\infty$ by $U(f) = \{g_i(f)\}$. If $x \in X$ with $\|x\| = 1$, there is a sequence of distinct n_i so that $x_{n_i} \to x$. Then the inequalities

$$1 \geq |g_{n_i}(x)| \geq |g_{n_i}(x_{n_i})| - |g_{n_i}(x - x_{n_i})| \geq 1 - ||x - x_{n_i}||$$

imply that

$$\lim_i |g_{n_i}(x)| = 1 \tag{9}$$

and hence $sup_i |g_i(x)| \geq 1$. Thus we obtain

$$||U(f)|| = ||f||, \quad \forall f \in X. \tag{10}$$

Let now X_d be the closed linear span of $R(U)$ and the canonical unit vectors in the sup-norm. Since the canonical unit vectors span c_0, we have them as a basic sequence in X_d. By (10), $\{g_i\}$ is an X_d-frame for X and $R(U)$ is closed in X_d. The Banach space X_d is separable and thus c_0 is complemented in X_d, cf. [10]. Let P be a bounded projection from X_d onto c_0. For every $x \in X$ with $||x|| = 1$, $P(Ux)$ belongs to c_0 and (9) is valid, which implies that $\|(I-P)(Ux)\| \geq 1$. Therefore for every $f \in X$ we have

$$\|(I-P)(Uf)\| \geq ||f|| = ||Uf||.$$

That is, if $T = (I-P)|_{R(U)}$, then $T : R(U) \to (I-P)(R(U))$ is an isomorphism (and hence has closed range). Also, if $y \in (I-P)(X_d)$ then by the definition of X_d, there are sequences $y_n \in R(U)$ and $z_n \in c_0$ so that $y_n + z_n \to y$. Hence,

$$y = (I-P)y = \lim_n (I-P)(y_n + z_n) = \lim_n (I-P)(y_n),$$

i.e. y is in the closure of $(I-P)(R(U))$. Hence, $T(R(U)) = (I-P)(X_d)$ and therefore $T^{-1}(I-P)$ is a projection of X_d onto $R(U)$. $\Box$

Acknowledgment: The authors thank the reviewer for a careful reading of the manuscript and correction of misprints.

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