A FRAME THEORY PRIMER FOR THE KADISON-SINGER PROBLEM

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Abstract. This is a primer on frame theory geared towards the parts of the theory needed for people who want to understand the relationship between the Kadison-Singer Problem and frame theory.

1. Introduction

Hilbert space frames were introduced by Duffin and Schaeffer [23] to address some very deep problems in nonharmonic Fourier series (see [45]). The main property of frames which makes them so useful is their **redundancy**. That is, vectors in the space can have infinitely many representations with respect to the frame but each vector has one natural representation given by the frame coefficients. The role played by redundancy varies with specific applications. One important role is its **robustness**. That is, by speading our information over a wider range of vectors, we are better able to sustain **losses** (called **erasures** in this setting) and still have accurate reconstruction. This shows up in internet coding (for transmission losses), distributed processing (where "sensors" are constantly fading out), modeling the brain (where memory cells are constantly dying out) and a host of other applications. Another advantage of spreading our information over a wider range is to mitigate the effects of noise in our signal or to make it prominant enough so it can be removed as in signal/image processing. Another advantage of redundancy is in areas such as quantum tomography where we need classes of orthonormal bases which have "constant" interactions with one another or we need vectors to form a Parseval frame but have the absolute values of their inner products with all other vectors the same. In speech recognition, we need a vector to be determined by the absolute value of its frame coefficients. This is a very natural frame theory problem since this is impossible for a linearly independent set to achieve. Redundancy is a fundamental issue in this setting.

Since D. Gabor introduced what is now known as time-frequency analysis [25, 29] frames have been traditionally used in signal processing where they are known as **Gabor frames**. For some reason they were not extensively studied outside of this area until the dawn of the wavelet era when they were brought

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back to life by Daubechies, Grossman and Meyer [21]. But today, frames have broad application in mathematics and engineering to a wide variety of areas including sampling theory [1], operator theory [31], harmonic analysis [35], nonlinear sparse approximation [22, 28], pseudodifferential operators [30], wavelet theory [20], wireless communication [43, 44], data transmission with erasures [3, 26], filter banks [4], signal processing [2, 27], image processing [6], geophysics [37], quantum computing [24], distributed processing [14, 41, 42], and much more with new applications arising every year. A good introduction to frames is the book of Christensen [19]. For an introduction to time-frequency analysis see Gröchenig's book [29]. For connections of frames to operator theory see the Memoir of Han and Larson [31]. There are also some good "tutorials" on frames due to Casazza [8, 9] and Heil and Walnut [32] and a section of the book of Young [45] on frames.

2. Frames and Riesz bases

A family of vectors $\{f_i\}_{i\in I}$ in a Hilbert space $\mathbb{H}$ is a **Riesz basic sequence** if there are constants $A, B > 0$ so that for all scalars $\{a_i\}_{i\in I}$ we have:

$$A\sum_{i\in I}|a_i|^2 \le \|\sum_{i\in I} a_i f_i\|^2 \le B\sum_{i\in I}|a_i|^2.$$

We call A, B the **lower and upper Riesz basis bounds** for $\{f_i\}_{i\in I}$. If the Riesz basic sequence $\{f_i\}_{i\in I}$ spans $\mathbb{H}$ we call it a **Riesz basis** for $\mathbb{H}$. So $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathbb{H}$ means there is an orthonormal basis $\{e_i\}_{i\in I}$ so that the operator $T(e_i) = f_i$ is invertible. In particular, each Riesz basis is **bounded**. That is, $0 < \inf_{i\in I}\|f_i\| \le \sup_{i\in I}\|f_i\| < \infty$.

We racall **Parseval's Identity**. If $\{e_i\}_{i\in I}$ is an orthonormal basis for a Hilbert space $\mathbb{H}$, then for all $f \in \mathbb{H}$ we have

$$\|f\|^2 = \sum_{i\in I}|\langle f, f_i\rangle|^2.$$

The notion of a frame is a weakening of Parseval's identity.

Definition 2.1. *A family $\{f_i\}_{i\in I}$ of elements of a (finite or infinite dimensional) Hilbert space $\mathbb{H}$ is called a* **frame** *for $\mathbb{H}$ if there are constants $0 < A \le B < \infty$ (called the* **lower and upper frame bounds***, respectively) so that for all $f \in \mathbb{H}$*

$$A\|f\|^2 \le \sum_{i\in I}|\langle f, f_i\rangle|^2 \le B\|f\|^2. \tag{2.1}$$

If we only have the right hand inequality in Equation 2.1 we call $\{f_i\}_{i\in I}$ a **Bessel sequence with Bessel bound B**. If $A = B$, we call this an A-**tight frame** and if $A = B = 1$, it is called a **Parseval frame**. If all the frame elements have the same norm, this is an **equal norm** frame and if the frame elements are of unit norm, it is a **unit norm frame**. It is immediate that $\|f_i\|^2 \leq B$ (see Proposition 3.1). If also $\inf \|f_i\| > 0$, $\{f_i\}_{i\in I}$ is a **bounded frame**. The numbers $\{\langle f, f_i\rangle\}_{i\in I}$ are the **frame coefficients** of the vector $f \in \mathbb{H}$.

We have not put any restriction on the vectors in a frame. In particular, we can have zero vectors and repeated vectors in the frame. A simple example of a N-tight frame would be to union any N orthonormal bases for the space. Also, the frame vectors do not need to be bounded below in norm. So a simple Parseval frame can be obtained from an orthonormal basis $\{e_i\}_{i\in I}$ by taking

$$e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \cdots$$

Every finite set of vectors $\{f_i\}_{i=1}^n$ can be extended to be a tight frame for the Hilbert space. To see this, for each $1 \leq i \leq n$ construct an orthonormal basis $\{f_{i,j}\}_{j\in I}$ for the Hilbert space containing the vector f_i. The union of these families is an n-tight frame for the space. Parseval frames (especially finite Parseval frames) play a fundamental role in applications of frames. For a listing of the known Parseval frames at this time see [17, 18]. The easiest way to get a frame is to take a Riesz basis for the space and add vectors to it. However, frames do not need to contain subsets which are Riesz bases [10].

If $\{f_i\}_{i\in I}$ is a Bessel sequence, the **synthesis operator** for $\{f_i\}_{i\in I}$ is the bounded linear operator $T : \ell_2(I) \to \mathbb{H}$ given by $T(e_i) = f_i$ for all $i \in I$. The **analysis operator** for $\{f_i\}_{i\in I}$ is T^* and satisfies: $T^*(f) = \sum_{i\in I}\langle f, f_i\rangle e_i$. In particular,

$$\|T^*f\|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2, \quad \text{for all } f \in \mathbb{H},$$

and hence the smallest Bessel bound for $\{f_i\}_{i\in I}$ equals $\|T^*\|^2$. Comparing this to Equation 2.1 we have:

Theorem 2.2. *Let $\mathbb{H}$ be a Hilbert space and $T : \ell_2(I) \to \mathbb{H}$, $Te_i = f_i$ be a bounded linear operator. The following are equivalent:*

1. *$\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$.*
2. *The operator T is bounded, linear, and onto.*
3. *The operator T^* is an (possibly into) isomorphism.*

Moreover, if $\{f_i\}_{i\in I}$ is a basis, then it is a Riesz basis and the Riesz basis bounds are A, B where A, B are the frame bounds for $\{f_i\}_{i\in I}$.

It follows that a Bessel sequence is a Riesz basic sequence if and only if T^* is onto. The **frame operator** for the frame is the positive, self-adjoint invertible operator $S = TT^* : \mathbb{H} \to \mathbb{H}$. That is,

$$Sf = TT^*f = T\left(\sum_{i\in I}\langle f, f_i\rangle e_i\right) = \sum_{i\in I}\langle f, f_i\rangle Te_i = \sum_{i\in I}\langle f, f_i\rangle f_i.$$

The series representation for S converges unconditionally. Also,

$$\langle Sf, f\rangle = \sum_{i\in I} |\langle f, f_i\rangle|^2.$$

It follows that $\{f_i\}_{i\in I}$ is a frame with frame bounds A, B if and only if $A\cdot I \le S \le B\cdot I$. So $\{f_i\}_{i\in I}$ is a Parseval frame if and only if $S = I$. **Reconstruction** of vectors in $\mathbb{H}$ is achieved via the formula:

$$\begin{aligned} f &= SS^{-1}f = \sum_{i\in I}\langle S^{-1}f, f_i\rangle f_i \\ &= \sum_{i\in I}\langle f, S^{-1}f_i\rangle f_i \\ &= \sum_{i\in I}\langle f, f_i\rangle S^{-1}f_i \\ &= \sum_{i\in I}\langle f, S^{-1/2}f_i\rangle S^{-1/2}f_i. \end{aligned}$$

It follows that $\{S^{-1/2}f_i\}_{i\in I}$ is a Parseval frame *equivalent* to $\{f_i\}_{i\in I}$. Two sequences $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ in a Hilbert space are *equivalent* if there is an invertible operator T between their spans with $Tf_i = g_i$ for all $i \in I$. Also, $\{S^{-1}f_i\}_{i\in I}$ is a frame for $\mathbb{H}$ (called the **dual frame** or **canonical dual frame**) with frame bounds $\frac{1}{B}, \frac{1}{A}$.

There is a simple way to tell when two frame sequences are equivalent.

Proposition 2.3. *Let $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$ be frames for a Hilbert space $\mathbb{H}$ with analysis operators T_1 and T_2, respectively. The following are equivalent:*
(1) The frames $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are equivalent.
(2) ker T_1 = ker T_2.

Proof: (1) $\Rightarrow$ (2): If $Lf_i = g_i$ is an isomorphism, then $Lf_i = LT_1e_i = g_i = T_2e_i$ quickly implies our statement about kernels.

(2) $\Rightarrow$ (1): Since $T_i|_{(ker\ T_i)^{\perp}}$ is an isomorphism for $i = 1, 2$, if the kernels are equal, then

$$T_2\left(T_1|_{(ker\ T_2)^{\perp}}\right)^{-1} f_i = g_i$$

is an isomorphism. □

A frame is **exact** if it ceases to be a frame when any one of its elements is removed. A family of vectors $\{f_i\}_{i\in I}$ is ω-independent if $\sum_{i\in I} a_if_i = 0$ implies that $a_i = 0$ for all $i \in I$. We have:

Theorem 2.4. *The following are equivalent for a frame $\{f_i\}_{i\in I}$ in a Hilbert space $\mathbb{H}$:*
(1) $\{f_i\}_{i\in I}$ is a Riesz basic sequence.
(2) $\{f_i\}_{i\in I}$ is exact.
(3) $\{f_i\}_{i\in I}$ is ω-independent.

Proof: We just have to note that a frame is exact if and only if the synthesis operator is one-to-one. Since this operator is bounded, linear and onto, this happens if and only if it is invertible. That is, $\{f_i\}_{i\in I}$ is a Riesz basic sequence. The other equivalence is similar. □

3. Some fundamental properties of frames

We now give some fundamental properties of frames.

Proposition 3.1. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathbb{H}$ with frame bounds A, B. For all $i \in I$, $\|f_i\|^2 \le B$ and $\|f_i\|^2 = B$ iimplies $f_i \perp span_{j\neq i} f_j$. If $\|f_i\|^2 < A$, then $f_i \in \overline{span}_{j\neq i} f_j$.*

Proof: If we replace f in Definition 2.1 by f_i we obtain

$$A\|f\|^2 \le \|f_i\|^4 + \sum_{j\neq i} |\langle f_i, f_j\rangle|^2 \le B\|f_i\|^2. \tag{3.1}$$

It is immediate from here that $\|f_i\|^2 \le B$. Now, assume that $E = \overline{span}_{j\neq i} f_j$ is a proper subspace of $\mathbb{H}$. Putting $P_\perp f_i$ into inequality 3.1 we have:

$$A\|P_\perp f_i\|^2 \le \|P_\perp f_i\|^4.$$

If $\|f_i\|^2 < A$, this would be a contradiction unless $P_\perp f_i = 0$. □

Corollary 3.2. *A Parseval frame $\{f_i\}_{i\in I}$ is an orthonormal basis if and only if $\|f_i\| = 1$ for all $i \in I$.*

Proposition 3.3. *The removal of a vector from a frame either leaves a frame or an incomplete set.*

Proof: As we observed, every frame is equivalent to a Parseval frame. If $\{f_i\}_{i\in I}$ is a Parseval frame, then by Proposition 3.1, for every $i \in I$ either $\|f_i\|^2 = 1$ and $f_i \perp span_{j\neq i} f_j$ or $\|f_i\| < 1$ and $f_i \in \overline{span}_{j\neq i} f_j$. □

Frames which are not Riesz bases have multiple representations for vectors in the space. However, the natural representation of the vector by the frame coefficients is the unique representation of minimal ℓ_2-norm. This is a result of Duffin and Schaeffer [23].

Theorem 3.4. *Let $\{f_i\}_{i\in I}$ be a frame for a Hilbert space $\mathbb{H}$ and $f \in \mathbb{H}$. If $\{a_i\}_{i\in I}$ is any sequence of scalars such that*

$$f = \sum_{i\in I} a_i f_i,$$

then

$$\sum_{i\in I} |a_i|^2 = \sum_{i\in I} |\langle S^{-1} f, f_i\rangle|^2 + \sum_{i\in I} |\langle S^{-1} f, f_i\rangle - a_i|^2.$$

Proof: We have by assumption

$$\sum_{i\in I}\langle S^{-1}f, f_i\rangle f_i = \sum_{i\in I} a_i f_i.$$

Now, taking the inner product of both sides of this equation with $S^{-1}f$ we get

$$\sum_{i\in I} |\langle S^{-1}f, f_i\rangle|^2 = \sum_{i\in I} \overline{\langle S^{-1}f, f_i\rangle} b_i.$$

Now, expanding out the sum above yields

$$\sum_{i\in I} |\langle S^{-1}f, f_i\rangle|^2 + \sum_{i\in I} |\langle S^{-1}f, f_i\rangle - a_i|^2 =$$

$$\sum_{i\in I} |\langle S^{-1}f, f_i\rangle|^2 + \sum_{i\in I} |b_i|^2 + \sum_{i\in I} |\langle S^{-1}f, f_i\rangle|^2 - 2Re\left(\overline{\langle S^{-1}f, f_i\rangle} b_i\right) = \sum_{i\in I} |b_i|^2.$$

□

Another useful result for (infinite, complex) frames is [7]

Theorem 3.5. *Every frame is a multiple of a sum of three orthonormal bases. It is a multiple of a sum of two orthonormal bases if and only if it is a Riesz basis.*

We now make a simple observation about projecting frames.

Theorem 3.6. *If $\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$ with frame bounds A, B and P is any orthogonal projection on $\mathbb{H}$, then $\{Pf_i\}_{i\in I}$ is a frame for $P\mathbb{H}$ with frame bounds A, B.*

Proof: For any $f \in P\mathbb{H}$,

$$\sum_{i\in I} |\langle f, Pf_i\rangle|^2 = \sum_{i\in I} |\langle Pf, f_i\rangle|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2.$$

□

A fundamental result in frame theory is a special case of a result proved by Naimark [38] (and later by Han/Larson [31]). For completeness we include its simple proof.

Theorem 3.7. *A family $\{f_i\}_{i\in I}$ is a Parseval frame for a Hilbert space $\mathbb{H}$ if and only if there is a containing Hilbert space $\mathbb{H} \subset \ell_2(I)$ with an orthonormal basis $\{e_i\}_{i\in I}$ so that the orthogonal projection P of $\ell_2(I)$ onto $\mathbb{H}$ satisfies $P(e_i) = f_i$ for all $i \in I$.*

Proof: The "only if" part is Theorem 3.6. For the "if" part, if $\{f_i\}_{i\in I}$ is a Parseval frame, then the synthesis operator $T : \ell_2(I) \to \mathbb{H}$ is a partial isometry. So T^* is an isometry and we can associate $\mathbb{H}$ with $T^*\mathbb{H}$. Now, for all $i \in I$ and all $g = T^*f \in T^*\mathbb{H}$ we have

$$\langle T^*f, Pe_i\rangle = \langle T^*f, e_i\rangle = \langle f, Te_i\rangle = \langle f, f_i\rangle = \langle T^*f, T^*f_i\rangle.$$

It follows that $Pe_i = T^* f_i$ for all $i \in I$. □

It follows from Theorem 3.7 that there is a unique way to get all Parseval frames with M-elements for a N-dimensional Hilbert space $\mathbb{H}_N$. Namely, take any $M \times M$-unitary matrix (with respect to the orthonormal basis $\{e_i\}_{i=1}^M$) $(a_{ij})_{i,j=1}^M$ and cut it off at the N^{th}-column. Then the corresponding row vectors are a M-element Parseval frame for $\mathbb{H}_N$. That is, if for $j = 1, 2, \cdots, M$ we let

$$f_j = \sum_{i=1}^{N} a_{ij} e_i,$$

then $\{f_j\}_{j=1}^M$ is a Parseval frame for $\mathbb{H}_N$.

Proposition 3.8. *If $\{f_i\}_{i\in I}$ is a frame for a subspace $\mathbf{K}$ of a Hilbert space $\mathbb{H}$ with frame operator S and P is the orthogonal projection of $\mathbb{H}$ onto $\mathbf{K}$, then for all $f \in \mathbb{H}$ we have*

$$Pf = \sum_{i\in I} \langle f, S^{-1} f_i \rangle f_i.$$

Proof: For all $f \in \mathbb{H}$ we have

$$\begin{aligned} Pf &= \sum_{i\in I} \langle Pf, S^{-1} f_i \rangle f_i \\ &= \sum_{i\in I} \langle f, PS^{-1} f_i \rangle f_i \\ &= \sum_{i\in I} f, S^{-1} f_i \rangle f_i. \end{aligned}$$

□

We finish this section with a result which basically states that any bounded operator on a finite dimensional Hilbert space is really just a multiple of a "piece" of a projection from a larger space. This then connects Parseval frame theory to the general theory of bounded operators [12].

Theorem 3.9. *Let $\mathbb{H}_N$ be an n-dimensional Hilbert space with orthonormal basis $\{g_i\}_{i=1}^n$. If $T : \mathbb{H}_N \to \mathbb{H}_N$ is any bounded linear operator with $\|T\| = 1$, then there is a containing Hilbert space $\mathbb{H}_N \subset \ell_2^M$ (M=2n-1) with an orthonormal basis $\{e_i\}_{i=1}^M$ so that the orthogonal projection P from ℓ_2^M onto $\mathbb{H}_N$ satisfies:*

$$Pe_i = Tg_i, \quad \text{for all } i = 1, 2, \cdots, n.$$

Proof: Let S be the frame operator for the Bessel sequence $\{f_i\}_{i=1}^n = \{Tg_i\}_{i=1}^n$ having eigenvectors $\{x_i\}_{i=1}^n$ with respective eigenvalues $\{\lambda_i\}_{i=1}^n$ where $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For $i = 2, 3, \cdots, n$ let $h_i = \sqrt{(1-\lambda_i)} x_i$. Then,

$\{f_i\}_{i=1}^n \cup \{h_i\}_{i=2}^n$ is a Parseval frame for $\mathbb{H}$ since for every $f \in \mathbb{H}$ we have

$$\begin{aligned}\sum_{i=1}^n |\langle f, f_i\rangle|^2 + \sum_{i=2}^n |\langle f, h_i\rangle|^2 &= \langle Sf, f\rangle + \sum_{i=2}^n (1-\lambda_i)|\langle f, x_i\rangle|^2 \\ &= \sum_{i=1}^n \lambda_i |\langle f, x_i\rangle|^2 + \sum_{i=2}^n (1-\lambda_i)|\langle f, x_i\rangle|^2 \\ &= \sum_{i=1}^n |\langle f, x_i\rangle|^2 = \|f\|^2.\end{aligned}$$

Now, by Theorem 3.7, there is a containing Hilbert space ℓ_2^{2n-1} with an orthonormal basis $\{e_i\}_{i=1}^{2n-1}$ so that the orthogonal projection P satisfies: $Pe_i = Tg_i$ for $i = 1, 2, \cdots, n$ and $Pe_i = h_i$ for $i = n+1, \cdots, 2n-1$. □

4. The role of eigenvalues and eigenvectors

Let $\mathbb{H}_N$ be an N-dimensional Hilbert space and let $\{f_i\}_{i=1}^M$ be a frame for $\mathbb{H}_N$. Let S be the frame operator for our frame and let $\{\lambda_j\}_{j=1}^N$ be the eigenvalues for S with respective eigenvectors $\{g_j\}_{j=1}^N$. We now list some of the basic eigenvalue properties.

General frame: The sum of the eigenvalues equals the sum of the squares of the lengths of the frame vectors. i.e.

$$\sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|f_i\|^2.$$

This follows from the fact that for every $1 \le j \le n$, $\sum_{i\in I} |\langle f_i, g_j\rangle|^2 = \lambda_j$. In particular, $\sum_{i\in I} \|f_i\|^2 = \text{trace S}$ ($= N$ if $\{f_i\}_{i\in I}$ is a Parseval frame).

Equal norm frame: If the frame is equal norm then

$$\sum_{j=1}^N \lambda_j = M\|f_1\|^2.$$

Tight frame: If the frame is $A = B$-tight then for every $f \in \mathbb{H}_N$

$$\sum_{i=1}^M |\langle f, f_i\rangle|^2 = A\|f\|^2.$$

Moreover, of T is the synthesis operator for the frame then

$$S = TT^* = A \cdot I_N.$$

Hence,

$$N \cdot A = \sum_{j=1}^{N} \lambda_j = \sum_{i=1}^{M} \|f_i\|^2.$$

Parseval frame: If the frame is Parseval then for every $f \in \mathbb{H}$ we have

$$\sum_{i=1}^{M} |\langle f, f_i \rangle|^2 = \|f\|^2.$$

In this case, the synthesis operator is a partial isometry and

$$S = TT^* = I_N.$$

Also, all the eigenvalues of S equal 1 so

$$N = \sum_{j=1}^{N} \lambda_j = \sum_{i=1}^{M} \|f_i\|^2.$$

Equal norm Parseval frames: If the frame is an equal norm Parseval frame then for all $i = 1, 2, \cdots, M$

$$\|f_i\|^2 = \frac{N}{M}.$$

5. The Schur-Horn Theorem and frames

The Schur-Horn Theorem [34] can be a valuable tool for constructions of frames. The form of it we give here comes from Casazza and Leon [17].

Theorem 5.1. *Let S be a positive self-adjoint operator on a N-dimensional Hilbert space H_N. Let $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N$ be the eigenvalues of S. Fix $M \geq N$ real numbers and $a_1 \geq a_2 \geq \ldots a_M > 0$. The following are equivalent:*

1. *There is a sequence $\{f_i\}_{i=1}^M$ in H_N with frame operator S and $\|f_i\| = a_j$, for all $j = 1, 2, \ldots, M$.*
2. *For every $1 \leq m \leq N$,*

$$\sum_{i=1}^{m} a_i^2 \leq \sum_{i=1}^{m} \lambda_i, \quad \textit{and} \quad \sum_{i=1}^{M} a_i^2 = \sum_{i=1}^{N} \lambda_i. \tag{5.1}$$

Corollary 5.2. *For every $M \geq N$ there is an equal norm Parseval frame for $\mathbb{H}_N$ containing exactly M-elements.*

We also have [11]

Corollary 5.3. *Given an N-dimensional Hilbert space $\mathbb{H}_N$ and a sequence of positive numbers $\{a_i\}_{i=1}^M$ with $a_1 \geq a_2 \geq a_M$, there exists a tight frame $\{f_i\}_{i=1}^M$ for $\mathbb{H}_N$ with $\|f_i\| = a_i$ for all $i = 1, 2, \cdots, M$ if and only if*

$$a_1^2 \leq \frac{1}{N} \sum_{i=1}^{M} a_i^2.$$

Corollary 5.4. *Let S be a positive self-adjoint operator on a N-dimensional Hilbert space H_N. For any $M \geq N$ there is an equal norm sequence $\{f_i\}_{i=1}^M$ in H_N which has S as its frame operator.*

Proof: Let $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N \geq 0$ be the eigenvalues of S. Let

$$a^2 = \frac{1}{M} \sum_{i=1}^{N} \lambda_i. \tag{5.2}$$

Now we check the conditions of Theorem 5.1 to see that there is a sequence $\{f_m\}_{m=1}^M$ in H_N with $\|f_m\| = a$ for all $m = 1, 2, \cdots, M$. We are letting $a_1 = a_2 = \cdots a_M = a$. For the second equality in Theorem 5.1, by Equation 5.2,

$$\sum_{m=1}^{M} \|f_m\|^2 = \sum_{m=1}^{M} a_m^2 = Ma^2 = \sum_{i=1}^{N} \lambda_i. \tag{5.3}$$

For the first inequality in Theorem 5.1, we note that by Equation 5.2 we have that

$$a_1^2 = a^2 = \frac{1}{M} \sum_{i=1}^{N} \lambda_i \leq \frac{1}{N} \sum_{i=1}^{N} \lambda_i \leq \lambda_1.$$

So our inequality holds for $m = 1$. Suppose there is an $1 < m \leq N$ for which this inequality fails and m is the first time this fails. So,

$$\sum_{i=1}^{m-1} a_i^2 = (m-1)a^2 \leq \sum_{i=1}^{m-1} \lambda_i,$$

while

$$\sum_{i=1}^{m} a_i^2 = ma^2 > \sum_{i=1}^{m} \lambda_i.$$

It follows that

$$a_m^2 = a^2 > \lambda_m \geq \lambda_{m+1} \geq \lambda_N.$$

Hence,

$$
\begin{aligned}
Ma^2 = \sum_{m=1}^{M} a_m^2 &\geq \sum_{i=1}^{m} a_i^2 + \sum_{i=m+1}^{N} a_i^2 \\
&> \sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{N} a_i^2 \\
&\geq \sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{N} \lambda_i \\
&= \sum_{i=1}^{N} \lambda_i .
\end{aligned}
$$

But this contradicts Equation 5.3. □

6. The Rado-Horn Theorem and frames

The Rado-Horn Theorem [33, 40] (See Casazza, Kutyniok and Speegle [15] for a generalization) can also be a useful tool for relating frame theory to Kadison-Singer problems. This theorem gives a characterization of those sets of vectors which can be written as the finite union of M linearly independent sets.

Theorem 6.1 (Rado-Horn)**.** *Let I be a countable index set and let $\{f_i : i \in I\}$ be a collection of vectors in a real vector space. There is a partition $\{I_j : j = 1, \cdots, M\}$ such that for each $1 \leq j \leq M$, $\{f_i : i \in I_j\}$ is linearly independent if and only if for all finite $J \subset I$*

$$\frac{|J|}{\dim \operatorname{span}\{f_i : i \in J\}} \leq M. \tag{6.1}$$

The terminology "Rado-Horn Theorem" was introduced in the paper [5]. This theorem was used by Bourgain in his characterization of Sidon sets in $\Pi_{k=1}^{\infty}\mathbb{Z}_p$ [36, 39]. There have also been at least three proofs, all in a similar spirit, of the Rado-Horn Theorem published [17, 33, 40]. Pisier, when discussing a characterization of Sidon sets in $\Pi_{k=1}^{\infty}\mathbb{Z}_p$ states "$\cdots$ d'un lemme d'algébre dû à Rado-Horn dont la démonstration est relativement délicate. [39, p. 704]"

A standard application of the Rado-Horn Theorem to frame theory comes from Casazza and Tremain [12].

Proposition 6.2. *Every equal norm Parseval frame $\{f_i\}_{i=1}^{KN}$ for $\mathbb{H}_N$ can be partitioned into K linearly independent spanning sets.*

Proof: If $J \subset \{1, 2, \cdots, KN\}$, let P_J be the orthogonal projection of $\mathbb{H}_N$ onto *span* $\{f_i\}_{i\in J}$. Since $\{f_i\}_{i=1}^{KN}$ is a equal norm Parseval frame $\sum_{i=1}^{KN} \|f_i\|^2 = KN\|f_1\|^2 = N$. Now,

$$dim(span\ \{f_i\}_{i\in J}) = \sum_{i=1}^{KN} \|P_J f_i\|^2 \geq \sum_{i\in J} \|P_J f_i\|^2 = \sum_{i\in J} \|f_i\|^2 = \frac{|J|}{K}.$$

So the Rado-Horn conditions hold with constant K. If we divide our family of KN vectors into K linearly independent sets, since each set cannot contain more than N-elements, it follows that each has exactly N-elements. □

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