THE KNOWN EQUAL NORM PARSEVAL FRAMES AS OF 2005

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Abstract. Here we list the equal norm Parseval frames for Hilbert spaces as of 2005. We will continue to update this list as new examples become known.

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Contents

1. An introduction to Parseval frames

A family of vectors $\{f_i\}_{i\in I}$ in a Hilbert space $\mathbb{H}$ is a **Riesz basic sequence** if there are constants $A, B > 0$ so that for all scalars $\{a_i\}_{i\in I}$ we have:

$$A\sum_{i\in I}|a_i|^2 \le \|\sum_{i\in I} a_i f_i\|^2 \le B\sum_{i\in I}|a_i|^2. \tag{1}$$

We call $\sqrt{A}, \sqrt{B}$ the **lower and upper Riesz basis bounds** for $\{f_i\}_{i\in I}$. If the Riesz basic sequence $\{f_i\}_{i\in I}$ spans $\mathbb{H}$ we call it a **Riesz basis** for $\mathbb{H}$. So $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathbb{H}$ means there is an orthonormal basis $\{e_i\}_{i\in I}$ so that the operator $T(e_i) = f_i$ is invertible. In particular, each Riesz basis is **bounded**. That is, $0 < \inf_{i\in I}\|f_i\| \le \sup_{i\in I}\|f_i\| < \infty$.

Hilbert space frames were introduced by Duffin and Schaeffer [19] to address some very deep problems in nonharmonic Fourier series (see [39]). A family $\{f_i\}_{i\in I}$ of elements of a (finite or infinite dimensional) Hilbert space $\mathbb{H}$ is called a **frame** for $\mathbb{H}$ if there are constants $0 < A \le B < \infty$ (called the **lower and upper frame bounds**, respectively) so that for all $f \in \mathbb{H}$

$$A\|f\|^2 \le \sum_{i\in I}|\langle f, f_i\rangle|^2 \le B\|f\|^2. \tag{2}$$

A good introduction to frames and Riesz bases is [13]. If we only have the right hand inequality in Equation 2 we call $\{f_i\}_{i\in I}$ a **Bessel sequence with Bessel bound B**. If $A = B$, we call this an A-**tight frame** and if $A = B = 1$, it is called a **Parseval frame**. If all the frame elements have the same norm, this is an **equal norm** frame and if the frame elements are of unit norm, it is a **unit norm frame**. It is immediate that $\|f_i\|^2 \le B$. If also $\inf\|f_i\| > 0$, $\{f_i\}_{i\in I}$ is a **bounded frame**. The numbers $\{\langle f, f_i\rangle\}_{i\in I}$ are the **frame coefficients** of the vector $f \in \mathbb{H}$. If $\{f_i\}_{i\in I}$ is a Bessel sequence, the **synthesis operator** for $\{f_i\}_{i\in I}$ is the bounded linear operator $T : \ell_2(I) \to \mathbb{H}$ given by $T(e_i) = f_i$ for all $i \in I$. The **analysis operator** for $\{f_i\}_{i\in I}$ is T^* and satisfies: $T^*(f) = \sum_{i\in I}\langle f, f_i\rangle e_i$. In particular,

$$\|T^*f\|^2 = \sum_{i\in I}|\langle f, f_i\rangle|^2, \quad \text{for all } f \in \mathbb{H}, \tag{3}$$

and hence the smallest Bessel bound for $\{f_i\}_{i\in I}$ equals $\|T^*\|^2$. Comparing this to Equation 2 we have:

Theorem 1.1. *Let $\mathbb{H}$ be a Hilbert space and $T : \ell_2(I) \to \mathbb{H}$, $Te_i = f_i$ be a bounded linear operator. The following are equivalent:*

(1) *$\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$.*
(2) *The operator T is bounded, linear, and onto.*
(3) *The operator T^* is an (possibly into) isomorphism.*

Moreover, if $\{f_i\}_{i\in I}$ is a Riesz basis, then the Riesz basis bounds are $\sqrt{A},\sqrt{B}$ where A,B are the frame bounds for $\{f_i\}_{i\in I}$.

It follows that a Bessel sequence is a Riesz basic sequence if and only if T^* is onto. The **frame operator** for the frame is the positive, self-adjoint invertible operator $S=TT^*:\mathbb{H}\to\mathbb{H}$. That is,

$$Sf=TT^*f=T\left(\sum_{i\in I}\langle f,f_i\rangle e_i\right)=\sum_{i\in I}\langle f,f_i\rangle Te_i=\sum_{i\in I}\langle f,f_i\rangle f_i. \tag{4}$$

In particular,

$$\langle Sf,f\rangle=\sum_{i\in I}|\langle f,f_i\rangle|^2. \tag{5}$$

It follows that $\{f_i\}_{i\in I}$ is a frame with frame bounds A,B if and only if $A\cdot I\le S\le B\cdot I$. So $\{f_i\}_{i\in I}$ is a Parseval frame if and only if $S=I$. **Reconstruction** of vectors in $\mathbb{H}$ is achieved via the formula:

$$\begin{aligned} f &= SS^{-1}f=\sum_{i\in I}\langle S^{-1}f,f_i\rangle f_i\\ &= \sum_{i\in I}\langle f,S^{-1}f_i\rangle f_i\\ &= \sum_{i\in I}\langle f,f_i\rangle S^{-1}f_i \end{aligned}$$

$$=\sum_{i\in I}\langle f,S^{-1/2}f_i\rangle S^{-1/2}f_i. \tag{6}$$

It follows that $\{S^{-1/2}f_i\}_{i\in I}$ is a Parseval frame *equivalent* to $\{f_i\}_{i\in I}$. Two sequences $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ in a Hilbert space are *equivalent* if there is an invertible operator T between their spans with $Tf_i=g_i$ for all $i\in I$.

Remark 1.2. *Any finite set of vectors $\{f_i\}_{i=1}^M$ in a Hilbert space $\mathbb{H}$ has a frame operator$Sf=\sum_{i=1}^M\langle f,f_i\rangle f_i$ associated with it. S is a positive and self-adjoint operator but is not invertible unless $\{f_i\}_{i=1}^M$ spans $\mathbb{H}$.*

Proposition 1.3. *let $\{f_i\}_{i=1}^M$ be a frame for ℓ_2^N. If $\{g_j\}_{j=1}^N$ is an orthonormal basis of ℓ_2^N consisting of eigenvectors for the frame operator S with respective eigenvalues $\{\lambda_j\}_{j=1}^N$, then for every $1\le j\le N$, $\sum_{i=1}^M|\langle f_i,g_j\rangle|^2=\lambda_j$. In particular, $\sum_{i=1}^M\|f_i\|^2=$ Trace S ($=N$ if $\{f_i\}_{i\in I}$ is a Parseval frame). Furthermore, if $\{f_i\}_{i\in I}$ is an equal norm Parseval frame for ℓ_2^N then $\|f_i\|^2=\frac{N}{M}$.*

Another important result is

Theorem 1.4. *If $\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$ with frame bounds A, B and P is any orthogonal projection on $\mathbb{H}$, then $\{Pf_i\}_{i\in I}$ is a frame for $P\mathbb{H}$ with frame bounds A, B.*

Proof: For any $f \in P(\mathbb{H})$,

$$\sum_{i\in I} |\langle f, Pf_i\rangle|^2 = \sum_{i\in I} |\langle Pf, f_i\rangle|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2.$$

□

A fundamental result in frame theory was proved independently by Naimark and Han/Larson [13, 25]. For completeness we include its simple proof.

Theorem 1.5. *A family $\{f_i\}_{i\in I}$ is a Parseval frame for a Hilbert space $\mathbb{H}$ if and only if there is a containing Hilbert space $\mathbb{H} \subset \ell_2(I)$ with an orthonormal basis $\{e_i\}_{i\in I}$ so that the orthogonal projection P of $\ell_2(I)$ onto $\mathbb{H}$ satisfies $P(e_i) = f_i$ for all $i \in I$.*

Proof: The "only if" part is Theorem 1.4. For the "if" part, if $\{f_i\}_{i\in I}$ is a Parseval frame, then the synthesis operator $T : \ell_2(I) \to \mathbb{H}$ is a partial isometry. So T^* is an isometry and we can associate $\mathbb{H}$ with $T^*\mathbb{H}$. Now, for all $i \in I$ and all $g = T^*f \in T^*\mathbb{H}$ we have

$$\langle T^*f, Pe_i\rangle = \langle T^*f, e_i\rangle = \langle f, Te_i\rangle = \langle f, f_i\rangle = \langle T^*f, T^*f_i\rangle.$$

It follows that $Pe_i = T^*f_i$ for all $i \in I$. □

Theorem 1.5 helps explain why so few classes of equal norm Parseval frames are known. Namely, to get an equal norm Parseval frame we need to find orthogonal projections which map an orthonormal basis to equal norm vectors. There is very little known about such projections, consequently there lies the challenge. There is a universal method for obtaining Parseval frames given in the next lemma (See [10]).

Lemma 1.6. *There is a unique method for constructing Parseval frames in ℓ_2^N. Let U be an $M \times M$, $M \geq N$, unitary matrix,*

$$U = \begin{bmatrix} u_{11} & . & . & . & u_{1M} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ u_{M1} & . & . & . & u_{MM} \end{bmatrix}.$$

Define

$$\begin{bmatrix} \varphi_1 \\ \cdot \\ \cdot \\ \cdot \\ \varphi_M \end{bmatrix} = \begin{bmatrix} u_{11} & \cdot & \cdot & \cdot & u_{1N} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ u_{M1} & \cdot & \cdot & \cdot & u_{MN} \end{bmatrix}, \quad N \leq M.$$

The rows $\{\varphi_i\}_{i=1}^M$ form a Parseval frame for ℓ_2^N.

Another important property of frames comes from [12].

Theorem 1.7. *A family of vectors $\{f_i\}_{i\in I}$ is a frame with frame bounds A and B if and only if the column vectors taken from $\{f_i\}_{i\in I}$ form a Riesz basic sequence with Riesz basis bounds $\sqrt{A}$ and $\sqrt{B}$.*

There is a classification of the sequence of norms of frame vectors which yield a given frame operator which follows from a result of Schur-Horn (See [34]) and rediscovered much later by Casazza and Leon [11].

Theorem 1.8. *Let S be a positive self-adjoint operator on an N-dimensional Hilbert space $\mathbb{H}_N$. Let $\lambda_1 \geq \lambda_2 \geq \ldots.. \geq \lambda_N$ be eigenvalues of S. Fix $M \geq N$ and real numbers $a_1 \geq a_2 \geq \ldots \geq a_M \geq 0$. The following are equivalent:*

(1) There is a frame $\{g_j\}_{j=1}^M$ for $\mathbb{H}_N$ with frame operator S and $\|g_j\|^2 = a_j$ for all $j = 1, 2, \cdots, M$.

(2) For every $1 \leq k \leq N$ we have

$$\sum_{j=1}^{k} a_j^2 \leq \sum_{j=1}^{k} \lambda_j, \tag{7}$$

and

$$\sum_{j=1}^{M} a_j^2 = \sum_{j=1}^{N} \lambda_j. \tag{8}$$

Feng, Wang and Wang [20] (using Householder Transformations) produced a fast and simple algorithm for finding a family of vectors $\{g_j\}_{j=1}^M$ satisfying the requirements in Theorem 1.8.

For some reason the following important corollary of Theorem 1.8 has been overlooked until now.

Corollary 1.9. *Let S be a positive self-adjoint operator on a N-dimensional Hilbert space H_N. For any $M \geq N$ there is an equal norm sequence $\{f_m\}_{m=1}^M$ in H_N which has S as its frame operator.*

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N \geq 0$ be the eigenvalues of S. Let

$$a^2 = \frac{1}{M}\sum_{i=1}^{N} \lambda_i. \tag{9}$$

Now we check the conditions of Theorem 1.8 to see that there is a sequence $\{f_m\}_{m=1}^M$ in H_N with $\|f_m\| = a$ for all $m = 1, 2, \ldots, M$. We are letting $a_1 = a_2 = \ldots a_M = a$. For the second equality in Theorem 1.8, by Equation 9,

$$\sum_{m=1}^{M} \|f_m\|^2 = \sum_{m=1}^{M} a_m^2 = Ma^2 = \sum_{i=1}^{N} \lambda_i. \tag{10}$$

For the first inequality in Theorem 1.8, we note that by Equation 9 we have that

$$a_1^2 = a^2 = \frac{1}{M}\sum_{i=1}^{N} \lambda_i \leq \frac{1}{N}\sum_{i=1}^{N} \lambda_i \leq \lambda_1.$$

So our inequality holds for $m = 1$. Suppose there is an $1 < m \leq N$ for which this inequality fails and m is the first time this fails. So,

$$\sum_{i=1}^{m-1} a_i^2 = (m-1)a^2 \leq \sum_{i=1}^{m-1} \lambda_i,$$

while

$$\sum_{i=1}^{m} a_i^2 = ma^2 > \sum_{i=1}^{m} \lambda_i.$$

It follows that

$$a_m^2 = a^2 > \lambda_m \geq \lambda_{m+1} \geq \lambda_N.$$

Hence,

$$\begin{aligned} Ma^2 = \sum_{m=1}^{M} a_m^2 &\geq \sum_{i=1}^{m} a_i^2 + \sum_{i=m+1}^{N} a_i^2 \\ &> \sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{N} a_i^2 \\ &\geq \sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{N} \lambda_i \\ &= \sum_{i=1}^{N} \lambda_i. \end{aligned}$$

But this contradicts Equation 9. □

We give two more important consequences of Theorem 1.8.

Corollary 1.10. *For every $m \geq n$ there is an equal norm Parseval frame for ℓ_2^n containing exactly m-elements.*

Corollary 1.11. *Given an N-dimensional Hilbert space $\mathbb{H}_N$ and a sequence of positive numbers $\{a_m\}_{m=1}^M$ with $a_1 \geq a_2 \geq \ldots \geq a_M$, there exists a tight frame $\{f_m\}_{m=1}^M$ for $\mathbb{H}_N$ with $\|f_m\| = a_m$ for all $m = 1, 2, \ldots, M$ if and only if*

$$a_1^2 \leq \frac{1}{N} \sum_{m=1}^{M} a_m^2.$$

2. Constructing Tight Frames from sets of vectors

This section will address several methods for constructing tight frames from Bessel sequences of vectors. We also discuss each method's advantages and disadvantages.

Method I: Let $\{f_i\}_{i=1}^{M}$ be a set of norm one vectors in ℓ_2^N. For every $j = 1, ..., M$ let $\{f_{ij}\}_{i=1}^{N}$ be an orthonormal basis for ℓ_2^N with $f_{1j} = f_j$. The family $\{f_{ij}\}_{i\in\{1,...,N\},\, j\in\{1,...,M\}}$ is an A-tight frame with tight frame bound A=M.

Note: Method I shows that every finite set of vectors is part of a tight frame for a Hilbert space. But, this technique has the disadvantage that the tight frame bound is exceptionally large, i.e $A = M$.

Method II: Let $\{f_i\}_{i=1}^{M}$ be a set of vectors in ℓ_2^N. We can add N-1 vectors $\{h_j\}_{j=2}^{N}$ to the family so that $\{f_i\}_{i=1}^{M} \cup \{h_j\}_{j=2}^{N}$ is a tight frame.

Proof of Method II: Let $\{g_j\}_{j=1}^{N}$ be an eigenbasis for the frame operator of $\{f_i\}_{i=1}^{M}$ with respective eigenvalues $\{\lambda_j\}_{j=1}^{N}$, some of which may be zero. Without loss of generality assume $\lambda_1 \geq \lambda_2 \geq ..... \geq \lambda_N$. For $2 \leq j \leq N$, let $h_j = \sqrt{\lambda_1 - \lambda_j}\; g_j$. If S_1 is the frame operator for $\{f_i\}_{i=1}^{k} \cup \{h_j\}_{j=2}^{N}$ then for all $f \in \ell_2^N$

$$\begin{aligned}
S_1 f &= \sum_{i=1}^{M} \langle f, f_i\rangle f_i + \sum_{j=2}^{N} \langle f, h_j\rangle h_j \\
&= \sum_{j=1}^{N} \lambda_j \langle f, g_j\rangle g_j + \sum_{j=2}^{N} \sqrt{\lambda_1 - \lambda_j}\langle f, g_j\rangle \sqrt{\lambda_1 - \lambda_j} g_j \\
&= \sum_{j=1}^{N} \lambda_j \langle f, g_j\rangle g_j + \sum_{j=2}^{N} (\lambda_1 - \lambda_j)\,\langle f, g_j\rangle g_j \\
&= \lambda_1 \langle f, g_1\rangle g_1 + \sum_{j=2}^{N} \lambda_1 \langle f, g_i\rangle g_i \\
&= \lambda_1 \sum_{j=1}^{N} \langle f, g_j\rangle g_i \\
&= \lambda_1 f
\end{aligned}$$

Therefore $\{f_i\}_{i=1}^{M} \cup \{h_j\}_{j=2}^{N}$ is a λ_1-tight frame.

Remark: The advantage to this method is that the upper frame bound is the same as the upper frame bound of the original set of vectors. However, if the original set is an equal norm frame the method does not ensure that the new frame will be an equal norm frame. In general, even if $\{f_i\}_{i=1}^M$ is an equal norm frame for ℓ_2^N we can't make $\{f_i\}_{i=1}^M \cup \{h_j\}_{j=2}^N$ tight be adding N-1 vectors of the same norm as f_i.

The next method for producing equal norm tight frames comes from [2].

Method III: Let $\{f_i\}_{i\in I}$ be a unit norm Bessel sequence in $\mathbb{H}$ with Bessel bound B. There is a unit norm family $\{g_j\}_{j\in J}$ so that $\{f_i\}_{i\in I} \cup \{g_j\}_{j\in J}$ is a unit norm tight frame with tight frame bound $\lambda \leq B+2$.

Proof of Method III: This proof is done in the finite dimensional case. The infinite dimensional case follows by a similar argument using the results of [9]. Let $\{f_i\}_{i=1}^M$ be a unit norm Bessel sequence with Bessel bound B in an N-dimensional Hilbert space $\mathbb{H}_N$. Let S be the frame operator for this family and let $\lambda_1 \geq \lambda_2 \geq ..... \geq \lambda_N \geq 0$ be the eigenvalues with respective eigenvectors $\{e_i\}_{i=1}^N$ for S. So B=λ_1. If we consider $N(\lambda_1+1+\epsilon)-M$, we see that this equals $N\lambda_1+N-M$ if $\epsilon=0$ and it equals $N\lambda_1+2N-M$ If $\epsilon=1$. In particular, there is an $0\leq\epsilon\leq 1$ so that $N(\lambda_1+1+\epsilon)-M=K\geq N$ where $K\in\mathbb{N}$. Now let S_0 be the positive self-adjoint operator on $\mathbb{H}_N$ given by

$$S_0\left(\sum_{i=1}^N c_ie_i\right) = \sum_{i=1}^N \left[(\lambda_1+1+\epsilon)-\lambda_i\right] c_ie_i. \tag{1}$$

So S_0 is a positive self-adjoint operator on $\mathbb{H}_N$ with eigenvectors $\{e_i\}_{i=1}^N$ having respective eigenvalues $\{\lambda_1+1+\epsilon-\lambda_i\}_{i=1}^N$ (which are now in increasing order). Since each of these eigenvalues is greater than 1, letting $a_i=1$ for $i=1,2,...,K$ we immediately have the first inequality given in (2) of Theorem 1.8. Also,

$$\sum_{j=1}^N \left[(\lambda_1+1+\epsilon)-\lambda_i\right] = N(\lambda_1+1+\epsilon) - \sum_{i=1}^N \lambda_i = N(\lambda_1+1+\epsilon)-M=K.$$

The last equality above follows from the fact that

$$\sum_{i=1}^M \|f_i\|^2 = M = \sum_{i=1}^N \lambda_i.$$

Applying Theorerm 1.8, there is a family of unit norm vectors $\{g_i\}_{i=1}^K$ in $\mathbb{H}_N$ having S_0 for its frame operator. If follows that $\{f_i\}_{i=1}^M \cup \{g_i\}_{i=1}^N$ is a unit norm frame for $\mathbb{H}_N$ having frame operator $S+S_0$. But $S+S_0$ has eigenvetors $\{e_i\}_{i=1}^N$ with respective eigenvalues

$$[(\lambda_1+1+\epsilon)-\lambda_i]+\lambda_i=\lambda+1+\epsilon=:\lambda$$

So our unit norm frame is tight with tight frame bound $\lambda\le\lambda_1+2$. □

For our next method, we first need to recall a standard result.

Proposition 2.1. *If S is a positive, self-adjoint bounded operator on $\mathbb{H}$, then $\{S^{1/2}e_i\}_{i\in I}$ is a sequence of vectors with frame operator S for any orthonormal basis $\{e_i\}_{i\in I}$. In particular, if S is also invertible, then we conclude that there is a Riesz basis for $\mathbb{H}$ having frame operator S.*

Proof. For any $f\in\mathbb{H}$ we have:

$$\begin{aligned}\sum_{i\in I}\langle f,S^{1/2}e_i\rangle S^{1/2}e_i &= S^{1/2}\left(\sum_{i\in I}\langle S^{1/2}f,e_i\rangle e_i\right)\\ &= S^{1/2}\left(S^{1/2}f\right)=Sf.\end{aligned}$$

□

Method IV: Given a Bessel sequence of vectors $\{f_i\}_{i\in I}$ in $\mathbb{H}$ with Bessel bound B and frame operator S, let $\{g_j\}_{j\in J}$ be a family of vectors which has $BI-S$ as its frame operator. Then $\{f_i\}_{i\in I}\cup\{g_i\}_{i\in I}$ is a frame for $\mathbb{H}$ with frame operator $S+(BI-S)=BI$. i.e. This is a tight frame.

Massey and Ruiz [32] generalized Theorem 1.8 to the case where we want to add vectors of prescribed norms to a given family of vectors and end up with a tight frame. There are also variations of this result in [32] including the infinite dimensional case.

Theorem 2.2 (Massey and Ruiz)**.** *Given vectors $\{f_i\}_{i\in I}$ in $\mathbb{H}_N$ with frame operator S having trace α and eigenvalues $\{\lambda_j\}_{j=1}^N$ and a non-increasing sequence $\{a_i\}_{i=1}^M$ of positive real numbers, there is a sequence of vectors $\{g_i\}_{i=1}^M$ in $\mathbb{H}_N$ with $\|g_i\|^2=a_i$ and $\{f_i\}_{i\in I}\cup\{g_i\}_{i=1}^M$ is a tight frame if and only if*

$$\frac{1}{N}\left(\sum_{i=1}^M a_i+\alpha\right)\ge\lambda_1,$$

and

$$\frac{1}{N}\left(\sum_{i=1}^M a_i+\alpha\right)\ge\frac{1}{k}\sum(a_i+\lambda_{n-i+1}),\quad 1\le k\le min\{N,M\}.$$

3. Harmonic Frames

In this section we will define three types of harmonic frames and show each type is an equal norm Parseval frame. For results on harmonic frames we refer the reader to [10, 22, 36, 40].

3.1. Real Harmonic Frames.

Theorem 3.1. *The family* $\{\varphi_i\}_{i=0}^{M-1}$ *is an orthonormal basis for* $\mathbb{R}^M$ *where for M =2k+1*

(1)
$$\begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{M-3} \\ \varphi_{M-2} \\ \varphi_{M-1} \end{bmatrix} = \sqrt{\frac{2}{M}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ 1 & \cos 2\pi\frac{1}{M} & \cos 2\pi\frac{2}{M} & \cdots & \cos 2\pi\frac{(M-1)}{M} \\ 0 & \sin 2\pi\frac{1}{M} & \sin 2\pi\frac{2}{M} & \cdots & \sin 2\pi\frac{M-1}{M} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos 2\pi\frac{k}{M} & \cos 2\pi\frac{2k}{M} & \cdots & \cos 2\pi\frac{(k(M-1))}{M} \\ 0 & \sin 2\pi\frac{k}{M} & \sin 2\pi\frac{2k}{M} & \cdots & \sin 2\pi\frac{(k(M-1))}{M} \end{bmatrix}$$

and for M=2k

(2)
$$\begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{M-3} \\ \varphi_{M-2} \\ \varphi_{M-1} \end{bmatrix} = \sqrt{\frac{2}{M}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ 1 & \cos 2\pi\frac{1}{M} & \cos 2\pi\frac{2}{M} & \cdots & \cos 2\pi\frac{(M-1)}{M} \\ 0 & \sin 2\pi\frac{1}{M} & \sin 2\pi\frac{2}{M} & \cdots & \sin 2\pi\frac{M-1}{M} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos 2\pi\frac{k}{M} & \cos 2\pi\frac{2k}{M} & \cdots & \cos 2\pi\frac{(k(M-1))}{M} \\ 0 & \sin 2\pi\frac{(k-1)}{M} & \sin 2\pi\frac{2(k-1)}{M} & \cdots & \sin 2\pi\frac{((k-1)(M-1))}{M} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Proof. Let $\{\varphi_j\}_{j=0}^{M-1}$ be defined above. First, we want to show that each φ_j has norm one.
For j=1, $\|\varphi_j\|^2 = \frac{2}{M} \times \frac{M}{2} = 1$. If $j \neq 1, j < M-1$, is odd

$$\begin{aligned}
\|\varphi_j\|^2 &= \frac{2}{M}\sum_{j=0}^{M-1}\cos^2 2\pi\frac{kj}{M} \\
&= \frac{2}{M}\frac{1}{2}\sum_{j=0}^{M-1}\left(1+\cos 2\pi 2\frac{kj}{M}\right) \\
&= \frac{2}{M}\left(\frac{M}{2}+\frac{1}{2}\sum_{j=0}^{M-1}\cos 2\pi 2\frac{kj}{M}\right) \\
&= \frac{2}{M}\left(\frac{M}{2}+\frac{1}{2}Re\sum_{j=0}^{M-1}\left(\omega^{2k}\right)\right),\ \omega = e^{\frac{2\pi i}{M}} \\
&= \frac{2}{M}\left(\frac{M}{2}+\frac{1}{2}Re\left(\frac{1-\left(\omega^{2k}\right)^j}{1-\omega^{2k}}\right)\right) \\
&= 1.
\end{aligned}$$

Similarly, if $j < M-1$ is even

$$\begin{aligned}
\|\varphi_j\|^2 &= \frac{2}{M}\sum_{j=0}^{M-1}\sin^2 2\pi\frac{kj}{M} \\
&= \frac{2}{M}\left(\frac{M}{2}-\frac{1}{2}\sum_{j=0}^{M-1}\cos 2\pi 2\frac{kj}{M}\right) \\
&= 1.
\end{aligned}$$

For the case when j=M-1, if M is odd then it falls under the previous case and if M is even then clearly $\|\varphi_{N-1}\| = 1$. So we have that $\|\varphi_j\| = 1$ for all j.

It remains to show that the φ_j's are orthogonal. Let $\omega = e^{\frac{2\pi i}{M}}$. For k even

$$\begin{aligned}
\langle\varphi_0,\varphi_k\rangle &= \frac{2}{M\sqrt{2}}\sum_{j=0}^{M-1}\cos 2\pi\frac{kj}{M} \\
&= \frac{2}{M\sqrt{2}}Re\sum_{j=0}^{M-1}\left(\omega^k\right)^j \\
&= 0.
\end{aligned}$$

For ℓ odd

$$\begin{aligned}
\langle \varphi_0, \varphi_\ell \rangle &= \frac{2}{M\sqrt{2}} \sum_{j=0}^{M-1} \sin 2\pi \frac{\ell j}{M} \\
&= \frac{2}{M\sqrt{2}} Im \sum_{j=0}^{M-1} \left(\omega^{\ell}\right)^j \\
&= 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
\langle \varphi_k, \varphi_\ell \rangle &= \frac{2}{M} \sum_{j=1}^{M-1} \cos \frac{2\pi j\,(k-\ell)}{M} \sin \frac{2\pi j\,(k-\ell)}{M} \\
&= \frac{1}{M} \sum_{j=1}^{M-1} \sin \frac{2\pi j 2\,(k-\ell)}{M} \\
&= \frac{1}{M} Im \sum_{j=1}^{M-1} \left(\omega^{2(k-\ell)}\right)^j \\
&= \frac{1}{M} Im \left[\sum_{j=1}^{M-1} \left(\omega^{2(k-\ell)}\right)^j - 1\right] \\
&= \frac{1}{M} Im\,(0-1) \\
&= 0. \quad \square
\end{aligned}$$

$\square$

By Lemma 1.6 if we take any N-columns, $N < M$, from the matrices given in Theorem 3.1, the corresponding row vectors form a Parseval frame for ℓ_2^N called a **(real) harmonic frame**. Similarly, we could take any N-columns from the transpose of these matrices, then the corresponding row vectors form a Parseval frame for ℓ_2^N.

3.2. **Complex Harmonic Frames.** In this subsection we look at the complex versions of the harmonic frames.

Theorem 3.2. *The family* $\{\varphi_i\}_{i=0}^{M-1}$ *in* $\mathbb{C}^M$ *is an orthonormal basis for* $\mathbb{C}^M$ *where for* $\omega = e^{\frac{2\pi i}{M}}$

$$\begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{M-2} \\ \varphi_{M-1} \end{bmatrix} = \sqrt{\frac{1}{M}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \cdots & \omega^{M-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(M-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{(M-2)} & \omega^{2(M-2)} & \cdots & \omega^{(M-2)(M-1)} \\ 1 & \omega^{(M-1)} & \omega^{2(M-1)} & \cdots & \omega^{(M-1)(M-1)} \end{bmatrix}.$$

Proof. Let $\{\varphi_j\}_{j=0}^{M-1}$ and ω be defined as above. It is obvious that each φ_j has norm one. Now we want to show that the rows are orthogonal. For $\ell \neq j$ we have

$$\begin{aligned} \langle \varphi_k, \varphi_\ell \rangle &= \frac{1}{M} \sum_{k=0}^{M-1} \omega^{jk} \overline{\omega^{\ell k}} \\ &= \frac{1}{M} \sum_{k=0}^{M-1} \omega^{jk} \omega^{-\ell k} \\ &= \frac{1}{M} \sum_{k=0}^{M-1} \omega^{k(j-\ell)} \\ &= \frac{1}{M} \left(\frac{1 - \left(\omega^{j-\ell}\right)^M}{1 - \omega^{j-\ell}} \right) \\ &= \frac{1}{M} \left(\frac{1 - \left(\omega^{M}\right)^{j-\ell}}{1 - \omega^{j-\ell}} \right) \\ &= \frac{1}{M} \left(\frac{1 - 1^{j-\ell}}{1 - \omega^{j-\ell}} \right) = 0 \quad \square \end{aligned}$$

□

3.3. **General Harmonic Frames.** The material in this section are due to Casazza and Kovačević and come from [10].

Definition 3.3. *Fix* $M \geq N$, $|c| = 1$, *and* $\{b_i\}_{i=1}^N$ *with* $|b_i| = \frac{1}{\sqrt{M}}$. *Let* $\{c_i\}_{i=1}^N$ *be distinct* M^{th} *roots of* c, *and for* $0 \leq k \leq M-1$ *let* $\varphi_k = \left(c_1^k b_1, c_2^k b_2, \cdots, c_N^k b_N\right)$. *Then* $\{\varphi_i\}_{i=0}^{M-1}$ *is a* general harmonic frame *for* $\mathbb{C}^N$.

Proposition 3.4. *Every general harmonic frame is unitarily equivalent to a frame of the form* $\left\{c^k \psi_k\right\}_{k=0}^{M-1}$, *where* $|c| = 1$ *and* $\{\psi_k\}_{k=0}^{M-1}$ *is a harmonic frame.*

Proposition 3.5. *Let* $\{\psi_k\}_{k=0}^{M-1}$ *be a harmonic frames and let* $|c| = 1$. *Then* $\left\{c^k\varphi_k\right\}_{k=0}^{M-1}$ *is equivalent to* $\{\psi_k\}_{k=0}^{M-1}$ *if and only if* c *is an* M^{th} *root of unity and there is a permutation* σ *of* $\{1, 2, \cdots, N\}$ *so that* $\varphi_{kj} = \psi_{k\sigma(j)}$, *for all* $0 \le k \le M-1$ *and all* $1 \le j \le N$. *A general harmonic frame is equivalent to a harmonic tight frame if and only if it equals a harmonic tight frame.*

Proposition 3.6. *The family* $\left\{c^k\psi_k\right\}_{k=0}^{M-1}$ *is a general harmonic frame for* $\mathbb{H}_N$ *if and only if there is a vector* $\varphi_0 \in \mathbb{H}_N$ *with* $\|\varphi_0\|^2 = \frac{N}{M}$, *an orthonormal basis* $\{e_i\}_{i=1}^{N}$ *for* $\mathbb{H}_N$ *and a unitary operator* U *on* $\mathbb{H}_N$ *with* $Ue_i = c_ie_i$, *with* $\{c_i\}_{i=1}^{N}$ *distinct* M^{th} *roots of some* $|c| = 1$ *so that* $\varphi_k = U^k\varphi_0$, *for all* $0 \le k \le M-1$.

Theorem 3.7. *Let* U *be a unitary operator on* $\mathbb{H}_N$, $\varphi_0 \in \mathbb{H}_N$ *and assume* $\left\{U^k\varphi_0\right\}_{k=0}^{M-1}$ *is a equal norm Parseval frame for* $\mathbb{H}_N$. *Then* $U^M = cI$ *for some* $|c| = 1$ *and* $\left\{U^k\varphi_0\right\}_{k=0}^{M-1}$ *is a general harmonic frame. That is, the general harmonic frames are the only equal norm Parseval frames generated by a group of unitary operators with a single generator.*

A listing of all harmonic frames and their properties can be found in [36].

3.4. **Maximal Robustness to Erasures.**

The results in this section are due to Kovačević and Puschel can be found in [33].

We call a frame given by a matrix F **maximally robust to erasures** if every $n \times n$ submatrix of F is invertible. If U is an $m \times m$ matrix and F is constructed by keeping all columns with indices in the set $I \subset \{0, 1, \ldots, m\}$, then we write

$$F = U[I].$$

We will work with the **discrete Fourier transform** defined by

$$DFT_m = \frac{1}{\sqrt{m}}\left[\omega_m^{k\ell}\right]_{0\le k,\ell\le m}, \quad \omega_m = e^{2\pi i/m}.$$

We have from [33]

Theorem 3.8. *For* $n \le m$,

$$F = DFT_m[0, 1, \ldots, n],$$

is an equal norm Parseval frame which is maximally robust to erasures.

The above frames are complex. There are also real versions of these frames given in [33]. Define for odd $n = 2k+1$,

$$U_n = \begin{bmatrix} 1 & & \\ & I_k & -iJ_k \\ & J_k & iI_k \end{bmatrix} \tag{3}$$

where J_k is I_k with the columns in reversed order.

Theorem 3.9. *Let* $0 \leq k < \frac{m-1}{2}$ *and* $n = 2k+1$. *Then*

$$\begin{aligned} F &= DFT_m[0,1,\ldots,k,m-k,m-k+1,\ldots,m-1]U_n \\ &= \left[[cos\ \frac{2j\ell\pi}{n}]_{0\leq j<m, 0\leq \ell \leq k} [-sin\ \frac{2j\ell\pi}{n}]_{0\leq j<m, 1\leq \ell\leq k} \right]. \end{aligned}$$

is an equal norm (real) tight frame with maximal robustness to erasures.

For even n, we first define

$$\widetilde{DFT_m} = \left[\omega_m^{(k+\frac{1}{2})\ell}\right]_{0\leq k,\ell\leq m},$$

and

$$V_n = \begin{bmatrix} I_k & -iJ_k \\ J_k & iI_k \end{bmatrix} \tag{4}$$

Theorem 3.10. *Let* $1 \leq k \leq \frac{m}{2}$ *and let* $n = 2k$. *Then*

$$\begin{aligned} F &= \widetilde{DFT_m}\,[0,1,\ldots,k-1,m-k,m-k+1,\ldots m-1]\,V_n \\ &= \left[[cos\ \frac{2(j+\frac{1}{2})\ell\pi}{n}]_{0\leq j<m, o\leq \ell<k} [-sin\ \frac{2(j+\frac{1}{2})\ell\pi}{n}]_{0\leq j<m, 1\leq \ell\leq k} \right]. \end{aligned}$$

is a equal norm (real) tight frame which is maximally robust to erasures.

4. Structured Parseval Frames

4.1. **Using Tight Frames to Construct New Tight Frames.** Fix $N, K, M \in \mathbb{N}$ with $K \leq N$. Let

$$\mathcal{K} = \{A \subset \{1, 2, \ldots, N\} : |A| = K\}.$$

For every $A \in \mathcal{K}$, let $\{f_i^A\}_{i=1}^M$ be a unit norm tight frame for $\mathbb{H}_K$ (so the tight frame bound is M/K). For every $A \in \mathcal{K}$ with $A = \{i_1 < i_2 < \cdots i_K\}$ define $T_A : \mathbb{H}_N \to \mathbb{H}_K$ by $T_A(f)(j) = f(i_j)$.

Proposition 4.1. *The family*

$$\{T_A^* f_i^A\}_{i=1, A\in\mathcal{K}}^M$$

is a unit norm tight frame for $\mathbb{H}_N$ *with tight frame bound*

$$\frac{M}{N}\binom{N}{K}.$$

Proof. Each T_A^* is an isometric embedding. Moreover, if $f \in \mathbb{H}_N$ we have

$$\begin{aligned}
\sum_{A\in\mathcal{K}} \sum_{i=1}^M |\langle f, T_A^* f_i^A\rangle|^2 &= \sum_{A\in\mathcal{K}} \sum_{i=1}^M |\langle T_A f, f_i^A\rangle|^2 \\
&= \sum_{A\in\mathcal{K}} \frac{M}{K} \|T_A f\|^2 \\
&= \frac{M}{K} \sum_{A\in\mathcal{K}} \sum_{i\in A} |F(i)|^2 \\
&= \frac{M}{K} \sum_{i=1}^N \sum_{A\in\mathcal{K},\ i\in A} |f(i)|^2 \\
&= \frac{M}{K} \sum_{i=1}^N \binom{N}{K} |f(i)|^2 \\
&= \frac{M}{K} \binom{N}{K} \sum_{i=1}^N |f(i)|^2 \\
&= \frac{M}{N} \binom{N}{K} \|f\|^2. \quad \square
\end{aligned}$$

$\square$

4.2. **M-Circle and M-Semicircle Frames.** The results in this section are from Bodmann and Paulsen [4]. Bodmann and Paulsen introduced the notion of frame paths to construct M-circle and M-semicircle frames and an alternate frame path definition of real harmonic frames.

Definition 4.2. *A continuous map* $f:[a,b] \to \mathbb{R}^M$ *(respectively,* $\mathbb{C}^M$*) is called a* **uniform frame path** *iff* $\|f(t)\| = 1$ *for all* t *and there are infinitely many choices of* N *such that* $F_N = \{f\left(a + \frac{b-a}{N}\right), f\left(a + \frac{2(b-a)}{N}\right), \cdots, f(b)\}$ *is an equal norm* $\frac{N}{M}$*-tight frame for* $\mathbb{R}^M$ *(respectively,* $\mathbb{C}^M$*). We call any such* F_N *a* **frame obtained by regualar sampling of** f*.*

Examples of such frames would include real and complex harmonic frames [4].

In this section we discuss M-circle frame paths in $\mathbb{R}^M$, $M > 2$, where the image of the frame path is the union of M circles. Let $\{e_i\}_{i=1}^M$ be the canonical orthonormal basis for $\mathbb{R}^M$. The image of $\{e_i\}_{i=1}^M$ will be the union of unit circles in the $e_1 - e_2$-plane, $e_2 - e_3$-plane,$\cdots$,$e_{M-1} - e_M$ plane. For the cases we will consider let $N = 4M$. To define the continuous path one needs only to see that it is possible to traverse this union of M circles in a continuous manner, passing through each quarter circle exactly once. Since the intersection of these circles occur at the 2M points, $\pm e_1, \cdots, \pm e_M$, to define the path it is enough to make clear the order in which one passes through the above points.

Proposition 4.3. *When* $M > 2$ *is even, the following path traverses each of the quarter circles exactly once,*

$$\begin{array}{ccccccccc}
+e_1 & \to & +e_2 & \to & +e_3 & \to & \cdots & +e_M & \to \\
-e_1 & \to & -e_2 & \to & -e_2 & \to & \cdots & -e_M & \to \\
+e_1 & \to & -e_2 & \to & +e_3 & \to & \cdots & -e_M & \to \\
-e_1 & \to & +e_2 & \to & -e_3 & \to & \cdots & +e_M & \to \\
+e_1 & & & & & & & &
\end{array}$$

where the sign in the third and fourth rows alternates.

Proposition 4.4. *When* $M > 1$ *is odd, the following path traverses each of the quarter circles exactly once,*

$$\begin{array}{ccccccccc}
+e_1 & \to & +e_2 & \to & +e_3 & \to & \cdots & +e_M & \to \\
-e_1 & \to & -e_2 & \to & -e_2 & \to & \cdots & -e_M & \to \\
-e_1 & \to & +e_2 & \to & -e_3 & \to & \cdots & -e_M & \to \\
+e_1 & \to & -e_2 & \to & +e_3 & \to & \cdots & +e_M & \to \\
+e_1 & & & & & & & &
\end{array}$$

where the sign in the third and fourth rows alternates.

Definition 4.5. *Now, using the above ordering we will define a piecewise smooth map $f : [0, 4M] \to \mathbb{R}^M$ so that on ith interval $[i-1, i]$ the image of f traces the ith quarter circle given in the above ordering. So in the case that $M \geq 4$ is even this is accomplished be setting*

$$f(t) = \begin{cases} \left(\cos \frac{\pi t}{2}, \sin \frac{\pi t}{2}, 0, \cdots, 0\right) & 0 \leq t \leq 1 \\ \left(0, \cos \frac{\pi(t-1)}{2}, \sin \frac{\pi(t-1)}{2}, 0, \cdots, 0\right) & 1 \leq t \leq 2 \\ \left(0, \cdots, 0, \cos \frac{\pi(t-M+1)}{2}, \sin \frac{\pi(t-M+1)}{2}\right) & M-1 \leq t \leq M \\ \left(-\sin \frac{\pi(t-M+1)}{2}, 0, \cdots, \cos \frac{\pi(t-M+1)}{2}\right) & M \leq t \leq M+1 \\ \left(\sin \frac{\pi(t-4M+1)}{2}, 0, \cdots, \cos \frac{\pi(t-4M+1)}{2}\right) & 4M-1 \leq t \leq 4M \end{cases} \tag{1}$$

[4].

Theorem 4.6. *(The Circle Frames). If $n \in \mathbb{N}$, $N = 4nM$ with $M \geq 3$ and let $f : [0, 4M] \to \mathbb{R}^M$ denote the M-circle path defined above, then $\{f\left(\frac{4Mj}{N}\right) : j = 1, \cdots, N\}$ is an equal norm $\frac{N}{M}$-tight frame for $\mathbb{R}^M$.*

Theorem 4.7. *(The Semicircle Frames). If $f : [0, 1] \to \mathbb{R}^2$ be defined by $f(t) = (\cos(\pi t), \sin(\pi t))$, then for any $N > 2$, the set $F_N = \{f\left(\frac{j}{N}\right) : 1 \leq j \leq N\}$ is an equal norm, $\frac{N}{2}$-tight frame for $\mathbb{R}^2$.*

The construction of the **M-semicircles path** will be similar to the construction of the M-circles path in that the construction of the map $f : [0, 2M] \to \mathbb{R}^M$ is identical. The construction of these paths differs because to construct the M-semicircles path we need only choose a path that exhausts a connected semicircle on each of the M-circles.

Proposition 4.8. *When $M > 2$ is even, the following path traverses a connected semicircle on each of the M circles exactly once,*

$$\begin{array}{lllllllll} +e_1 & \to & +e_2 & \to & +e_3 & \to & \cdots & +e_M & \to \\ -e_1 & \to & +e_2 & \to & -e_3 & \to & \cdots & +e_M & \to \\ +e_1 \end{array}$$

When $M > 1$ is odd, the following path traverses a connected semicircle on each of the M circles exactly once,

$$\begin{array}{lllllllll} +e_1 & \to & +e_2 & \to & +e_3 & \to & \cdots & +e_M & \to \\ -e_1 & \to & +e_2 & \to & -e_3 & \to & \cdots & -e_M & \to . \\ +e_1 \end{array}$$

[4]

Theorem 4.9. *If $n \in \mathbb{N}$, $N = 2nM$ with $M \geq 3$ and $f : [0, 2M] \to \mathbb{R}^M$ denote the M-semicircle path defined above, then F_N is an equal norm $\frac{N}{M}$-tight frame obtained by regular sampling.*

5. Frames of Translates

Translates of a single function play a fundamental role in frame theory, time-frequency analysis, sampling theory and more [1, 16, 17].

There is a simple classification of which functions give (tight) frames of translates. For this we need a definition. For $x \in \mathbb{R}$ we define *translation by* x by

$$\tau_x : L^2(\mathbb{R}) \to L^2(\mathbb{R}), (\tau_x f)(y) = f(y - x),\ y \in \mathbb{R}.$$

We first introduce some notation. For a function $\phi \in L^1(\mathbb{R})$ we denote by $\hat{\phi}$ the *Fourier transform of* ϕ

$$\hat{\phi}(\xi) = \int \phi(x) e^{-2\pi i \xi x}(x)dx.$$

As usual the definition of the Fourier transform extends to an isometry $\phi \to \hat{\phi}$ on $L^2(\mathbb{R})$.

Now suppose $\phi \in L^2(\mathbb{R})$ and that $b > 0$. Let us identify the circle $\mathbb{T}$ with the interval $[0, 1)$ via the standard map $\xi \to e^{2\pi i \xi}$. We define the function $\Phi_b : \mathbb{T} \to \mathbb{R}$ by

$$\Phi_b(\xi) = \sum_{n \in \mathbb{Z}} |\hat{\phi}(\frac{\xi + n}{b})|^2.$$

Note that $\Phi_b \in L^1(\mathbb{T})$.

For any $n \in \mathbb{Z}$ we note that

$$\langle \tau_{nb}\phi, \phi \rangle = \langle e^{-2\pi i n \xi b}\hat{\phi}, \hat{\phi} \rangle = \frac{1}{b}\int_0^1 \Phi_b(\xi) e^{-2\pi i n \xi} d\xi = \frac{1}{b}\hat{\Phi}_b(n).$$

We now have a classification of (tight) frames of translates from [8] (See also [3]).

Theorem 5.1. *If $\phi \in L^2(\mathbb{R})$, and $b > 0$ then:*

(1) $(\tau_{nb}\phi)_{n\in\mathbb{Z}}$ is an orthonormal sequence if and only if

$$\Phi_b(\gamma) = b \quad a.e.$$

(2) $(\tau_{nb}\phi)_{n\in\mathbb{Z}}$ is a Riesz basic sequence with frame bounds $A,\ B$ if and only if

$$bA \le \Phi_b(\gamma) \le bB \quad a.e.$$

(3) $(\tau_{nb}\phi)_{n\in\mathbb{Z}}$ is a frame sequence with frame bounds $A,\ B$ if and only if

$$bA \le \Phi_b(\gamma) \le bB \quad a.e.$$

on $\mathbb{T} \setminus N_b$ where $N_b = \{\xi \in \mathbb{T} : \Phi_b(\xi) = 0\}$.

We also mention a surprising result from [8].

Theorem 5.2. *Let $I \subset \mathbb{Z}$ be bounded below, $a > 0$ and $g \in L^2(\mathbb{R})$. Then $\{T_{na}g\}_{n\in I}$ is a frame sequence if and only if it is a Riesz basic sequence.*

6. Gabor Frames

An excellent reference for time-frequency analysis is Gröchening's book [24].

Given a function $g \in L^2(\mathbb{R}^d)$, for any $f \in L^2(\mathbb{R}^d)$ we define **translation of f by** $x \in \mathbb{R}^d$ and **modulation of f by** $y \in \mathbb{R}^d$ respectively as

$$T_x(f)(t) = f(t-x) \quad \text{and} \quad M_y(f)(t) = e^{2\pi i y \cdot t} f(t).$$

Definition 6.1. *Given a non-zero* **window function** $g \in L^2(\mathbb{R}^d)$ *and lattice parameters* $\alpha, \beta > 0$*, the set of time-frequency shifts*

$$\mathcal{G}(g, \alpha, \beta) = \{T_{\alpha k} M_{\beta n} g\}_{k,n \in \mathbb{Z}^d}$$

is called a **Gabor system**. *If this family forms a frame for* $L^2(\mathbb{R}^d)$*, it is called a* **Gabor frame** *or* **Weyl-Heisenberg frame**.

It is a very deep question when g, α, β yields a Gabor frame [24]. It can be shown [24] that the frame operator for a Gabor frame commutes with translation and modulation which yields:

Theorem 6.2. *For any Gabor frame* $\mathcal{G}(g, \alpha, \beta)$ *with frame operator* S*, the family* $\mathcal{G}(S^{-1/2}, \alpha, \beta)$ *is an equal norm Parseval frame for* $L^2(\mathbb{R}^d)$*.*

Now let us look at finite Gabor systems. Let $\omega = e^{2\pi i/n}$. The **translation operator** T is the unitary operator on $\mathbb{C}^n$ given by

$$Tx = T(x_0, x_1, \ldots, x_{n-1}) = (x_{n-1}, x_0, x_1, \ldots, x_{n-2}),$$

and the **modulation operator** M is the unitary operator defined by

$$Mx = M(x_0, x_1, \ldots, x_{n-1}) = (\omega^0 x_0, \omega^1 x_1, \ldots, \omega^{n-1} x_{n-1}).$$

Given a vector $g \in \mathbb{C}^n$, the **finite Gabor system with window function g** is the family

$$\left\{M^\ell T^k g\right\}_{\ell, k \in \mathbb{Z}_n}.$$

In the discrete case, Gabor systems always form tight frames. Although this has been folklore for quite some time, the first formal proof we have seen is in [31].

Theorem 6.3. *For any* $0 \neq g \in \mathbb{C}^n$*, the collection* $\{M^\ell T^k g\}_{\ell,k \in \mathbb{Z}_n}$ *is an equal norm tight frame for* $\mathbb{C}^n$ *with tight frame bound* $n^2\|f\|^2$*.*

Lawrence, Pfander and Walnut [31] examined the linear independence of discrete Gabor systems.

Theorem 6.4. *If* n *is prime, there is a dense open set* E *of full measure (i.e. The Lebesgue measure of* $\mathbb{C}^n \setminus E$ *is* 0*) in* $\mathbb{C}^n$ *such that for every* $f \in E$*, every subset of the Gabor system* $\{M^\ell T^k g\}_{\ell,k \in \mathbb{Z}_n}$ *containing* n*-elements is linearly independent.*

7. Wavelet Frames

For an introduction to wavelets we recommend the books [26, 37].

Definition 7.1. *An orthonormal wavelet is a function* $\psi \in L^2(\mathbb{R})$ *such that the system* $\{\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k), \quad j,k \in \mathbb{Z}$ *is an orthonormal basis for* $L^2(\mathbb{R})$.

Since wavelets are orthonormal bases they are also unit norm Parseval frames. There is a complete characterization of wavelet functions [26, 37].

Theorem 7.2. *A function* $\psi \in L^2(\mathbb{R})$ *is an orthonormal wavelet if and only if* $\|\psi\|_2 = 1$,

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}\left(2^j\xi\right)|^2 = 1, \quad for\ a.e.\ \xi \in \mathbb{R} \tag{1}$$

and

$$\sum_{j\geq 0} \hat{\psi}\left(2^j\xi\right)\overline{\hat{\psi}\left(2^j\left(\xi+q\right)\right)} = 0, \quad for\ a.e.\ \xi \in \mathbb{R} \tag{2}$$

whenever q *is an odd integer and* $\hat{\psi}(\xi) = \int_{\mathbb{R}}^{M} \psi(x)\, e^{-2\pi i\xi x}dx$.[26]

If we relax the requirement that translates be orthogonal, we have a generalization of wavelets to **wavelet frames**. That is, we only require our system of functions to form a frame for $L^2(\mathbb{R})$. Again, there is a classification of all functions giving Parseval wavelet frames due to Hernandez, Labate and Weiss [27].

Theorem 7.3. *Let* P *be a countable indexing set,* $\{g_p\}_{p\in P}$ *a collection of functions in* $L^2(\mathbb{R}^M)$ *and* $\{C_p\}_{p\in P}$ *be the corresponding collection of matrices in* $GL_M(\mathbb{R})$. *Suppose that*

$$L(f) = \sum_{p\in P}\sum_{m\in\mathbb{Z}^M}\int_{supp\hat{f}} |\hat{f}\left(\xi + C_p^I m\right)|^2 \frac{1}{|detC_p|}|\hat{g}_p(\xi)|^2 d\xi < \infty \tag{3}$$

for all $f \in D = \{f \in L^2\left(\mathbb{R}^M\right) : \hat{f} \in L^\infty\left(\mathbb{R}^M\right)$ *and* $supp\hat{f}$ *is compact*$\}$, *where* $C_p^I = (C_p^t)^{-1}$. *The system* $\{T_{C_p k} g_p \,|k \in \mathbb{R}^M, p \in P\}$, *where for* $y \in \mathbb{R}^M$ $T_y f = f(\cdot - y)$, *is an equal norm Parseval frame for* $L^2(\mathbb{R}^M)$ *if and only if*

$$\sum_{p\in P_\alpha} \frac{1}{|detC_p|}\overline{\hat{g}\left(\xi\right)}\hat{g}_p\left(\xi+\alpha\right) = \delta_{\alpha,0}\ for\ a.e\ \xi \in \mathbb{R}^M, \tag{4}$$

for each $\alpha \in \Lambda$, *where* δ *is the Kronecker delta for* $\mathbb{R}^M$.

8. Filter Bank Frames

Results on filter bank frames can be found in [15, 30]. The results are due to Casazza, Chebira, and Kovačević [7].

Definition 8.1. *Given $k, N, M \in \mathbb{N}$, a* **Filter bank frame** *for $\ell_2(\mathbb{Z})$ is a frame for $\ell_2(\mathbb{Z})$, say $\{\varphi_m\}_{m\in\mathbb{Z}}$, satisfying the following:*

1. For $0 \le i \le kN-1$ and $j \notin \{0, 1, \ldots, kN-1\}$ we have that $\varphi(j) = 0$.

2. For $j = 0, 1, \ldots, M-1$ and $i \in \mathbb{Z}$, $\varphi_{iM+j} = T_{iN}\varphi_j$, where T_{iN} is translation by iN.

Notation 8.2. *Throughout this section we will let $\{e_i\}_{i\in\mathbb{Z}}$ be the natural orthonormal basis for $\ell_2(\mathbb{Z})$.*

Proposition 8.3. *Let $N < L-1$ be natural numbers and let $\{\varphi_j\}_{j=0}^{M-1}$ be a frame for the span of $\{e_i\}_{i=0}^{L-1}$ with frame bounds A, B. Let $\varphi_{iM+j} = T_{iN}\varphi_j$ for all $0 \le j \le M-1$ and all $i \in \mathbb{Z}$. Then $\{\varphi_i\}_{i\in\mathbb{Z}}$ is a frame for $\ell_2(\mathbb{Z})$ with frame bounds $A\lfloor \frac{L}{N} \rfloor$, $B\lceil \frac{L}{N} \rceil$.*

Proof. Let $\varphi \in \ell_2(\mathbb{Z})$ and compute

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} |\langle \varphi, \varphi_m \rangle|^2 &= \sum_{i\in\mathbb{Z}} \sum_{j=0}^{M-1} |\langle \varphi, \varphi_{iM+j} \rangle|^2 \\
&= \sum_{i\in\mathbb{Z}} \sum_{j=0}^{M-1} |\langle \varphi, T_{iN}\varphi_j \rangle|^2 \\
&\le \sum_{i\in\mathbb{Z}} B \sum_{n=1}^{iN+L-1} |\varphi(n)|^2 \\
&\le B \left\lceil \frac{L}{N} \right\rceil \sum_{n\in\mathbb{Z}} |\varphi(n)|^2 \\
&= B \left\lceil \frac{L}{N} \right\rceil \|\varphi\|^2.
\end{aligned}$$

The lower frame bound follows similarly. □

Next we see when we can get a tight frame from a filter bank frame.

Proposition 8.4. *Let $0 < L < N$ be natural numbers and Let $\{\varphi_j\}_{j=0}^{M-1}$ be a frame for span $\{e_i\}_{i=0}^{KN+L}$, with frame operator S having eigenvectors $\{e_i\}_{i=0}^{KN+L}$ and respective eigenvalues $\{\lambda_i\}_{i=0}^{KN+L}$. Let $\varphi_{iM+j} = T_{iN}\varphi_j$ for all $0 \le j \le M-1$ and all $i \in \mathbb{Z}$. The following are equivalent:*

1. The family $\{\varphi_i\}_{i\in\mathbb{Z}}$ is a λ-tight frame for $\ell_2(\mathbb{Z})$.

2. We have

$$\lambda = \begin{cases} \sum_{j=0}^{K} \lambda_{jN+m} : & : \quad 0 \le m \le L \\ \sum_{j=0}^{K-1} \lambda_{jN+m} & : \quad L < m \le N \end{cases}$$

Proof. Let $\varphi \in \ell_2(\mathbb{Z})$ and compute

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} |\langle \varphi, \varphi_m \rangle|^2 &= \sum_{i\in\mathbb{Z}} \sum_{j=0}^{M-1} |\langle \varphi, \varphi_{iM+j} \rangle|^2 \\
&= \sum_{i\in\mathbb{Z}} \sum_{j=0}^{M-1} |\langle \varphi, T_{iN}\varphi_j \rangle|^2 \\
&= \sum_{i\in\mathbb{Z}} \sum_{n=0}^{(i+K)N+L} \lambda_{n-iN} |\varphi(n)|^2 \\
&= \sum_{i\in\mathbb{Z}} \sum_{n=0}^{KN+L} \lambda_n |\varphi(n+iN)|^2 \\
&= \sum_{m=0}^{L} \sum_{i\in\mathbb{Z}} \left(\sum_{j=0}^{K} \lambda_{jN+m} \right) |\varphi(iN+m)|^2 \\
&+ \sum_{m=L+1}^{N} \sum_{i\in Z} \left(\sum_{j=0}^{K-1} \lambda_{jN+m} \right) |\varphi(iN+m)|^2.
\end{aligned}$$

The result follows immediately from here. □

Since a filter bank frame is an equal norm frame if and only if we start with an equal norm frame before translation, the next corollary also classifies which filter bank frames are equal norm tight frames.

Corollary 8.5. *Let $\{\varphi_j\}_{j=0}^{M-1}$ be an A-tight frame for the span of $\{e_i\}_{i=0}^{KN+L}$ for $0 \le L < N$. Let $\varphi_{iM+j} = T_{iN}\varphi_j$, for all $0 \le j \le M-1$ and all $i \in \mathbb{Z}$. Then $\{\varphi_i\}_{i\in\mathbb{Z}}$ is a λ-tight frame for $\ell_2(\mathbb{Z})$ if and only if $L = 0$. Moreover, in this case $\lambda = (K+1)A$.*

Proof. Since $\{\varphi\}_{i=0}^{M-1}$ is an A-tight frame, every orthonormal basis consists of eigenvectors of the frame operator for this frame. By Proposition 8.4, $\{\varphi_i\}_{i\in\mathbb{Z}}$ is a frame with $\{e_i\}_{i\in\mathbb{Z}}$ eigenvectors for the frame operator having eigenvalues

$$\sum_{j=0}^{K} A = (K+1)A, \quad \text{for } e_{iN+m}, \quad 0 \le m \le L,$$

and

$$\sum_{j=0}^{K-1} A = KA, \quad \text{for } e_{iN+m}, \quad L+1 \leq m \leq N.$$

It follows that this is a tight frame if and only if $L = 0$. □

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