THE KADISON-SINGER PROBLEM IN MATHEMATICS AND ENGINEERING: A DETAILED ACCOUNT

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ABSTRACT. We will show that the famous, intractible 1959 Kadison-Singer problem in C^*-algebras is equivalent to fundamental unsolved problems in a dozen areas of research in pure mathematics, applied mathematics and Engineering. This gives all these areas common ground on which to interact as well as explaining why each of these areas has volumes of literature on their respective problems without a satisfactory resolution. In each of these areas we will reduce the problem to the minimum which needs to be proved to solve their version of Kadison-Singer. In some areas we will prove what we believe will be the strongest results ever available in the case that Kadison-Singer fails. Finally, we will give some directions for constructing a counter-example to Kadison-Singer.

1. Introduction

The famous 1959 Kadison-Singer Problem [61] has defied the best efforts of some of the most talented mathematicians of our time.

Kadison-Singer Problem (KS)**.** *Does every pure state on the (abelian) von Neumann algebra $\mathbb{D}$ of bounded diagonal operators on ℓ_2 have a unique extension to a (pure) state on $B(\ell_2)$, the von Neumann algebra of all bounded linear operators on the Hilbert space ℓ_2?*

A **state** of a von Neumann algebra $\mathcal{R}$ is a linear functional f on $\mathcal{R}$ for which $f(I) = 1$ and $f(T) \geq 0$ whenever $T \geq 0$ (whenever T is a positive operator). The set of states of $\mathcal{R}$ is a convex subset of the dual space of $\mathcal{R}$ which is compact in the ω^*-topology. By the Krein-Milman theorem, this convex set is the closed convex hull of its extreme points. The extremal elements in the space of states are called the **pure states** (of $\mathcal{R}$). This problem arose from the very productive collaboration of Kadison and Singer in the 1950's which culminated in their seminal work on triangular operator algebras. During this collaboration, they often discussed the fundamental work of Dirac [38] on Quantum Mechanics. In particular, they kept returning to one part of

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Dirac's work because it seemed to be problematic. Dirac wanted to find a "representation" (an orthonormal basis) for a compatible family of observables (a commutative family of self-adjoint operators). On pages 74–75 of [38] Dirac states:

> "To introduce a representation in practice
>
> (i) We look for observables which we would like to have diagonal either because we are interested in their probabilities or for reasons of mathematical simplicity;
> (ii) We must see that they all commute — a necessary condition since diagonal matrices always commute;
> (iii) We then see that they form a complete commuting set, and if not we add some more commuting observables to make them into a complete commuting set;
> (iv) We set up an orthogonal representation with this commuting set diagonal.
>
> **The representation is then completely determined** ... **by the observables that are diagonal** ..."

The emphasis above was added. Dirac then talks about finding a basis that diagonalizes a self-adjoint operator, which is troublesome since there are perfectly respectable self-adjoint operators which do not have a single eigenvector. Still, there is a *spectral resolution* of such operators. Dirac addresses this problem on pages 57-58 of [38]:

> "We have not yet considered the lengths of the basic vectors. With an orthonormal representation, the natural thing to do is to normalize the basic vectors, rather than leave their lengths arbitrary, and so introduce a further stage of simplification into the representation. However, it is possible to normalize them only if the parameters are continuous variables that can take on all values in a range, the basic vectors are eigenvectors of some observable belonging to eigenvalues in a range and are of infinite length..."

In the case of $\mathbb{D}$, the representation is $\{e_i\}_{i\in I}$, the orthonormal basis of l_2. But what happens if our observables have "ranges" (intervals) in their spectra? This led Dirac to introduce his famous δ-function — vectors of "infinite length." From a mathematical point of view, this is problematic. What we need is to replace the vectors e_i by some mathematical object that is essentially the same as the vector, when there is one, but gives us something precise and usable when there is only a δ-function. This leads to the "pure states" of

$B(\ell_2)$ and, in particular, the (vector) pure states ω_x, given by $\omega_x(T) = \langle Tx, x\rangle$, where x is a unit vector in $\mathbb{H}$. Then, $\omega_x(T)$ is the expectation value of T in the state corresponding to x. This expectation is the average of values measured in the laboratory for the "observable" T with the system in the state corresponding to x. The pure state ω_{e_i} can be shown to be completely determined by its values on $\mathbb{D}$; that is, each ω_{e_i} has a *unique* extension to $B(\ell_2)$. But there are many other pure states of $\mathbb{D}$. (The family of all pure states of $\mathbb{D}$ with the w^*-topology is $\beta(\mathbb{Z})$, the β-compactification of the integers.) Do these other pure states have unique extensions? This is the Kadison-Singer problem (KS).

By a "complete" commuting set, Dirac means what is now called a "maximal abelian self-adjoint" subalgebra of $B(\ell_2)$; $\mathbb{D}$ is one such. There are others. For example, another is generated by an observable whose"simple" spectrum is a closed interval. Dirac's claim, in mathematical form, is that each pure state of a "complete commuting set" has a unique state extension to $B(\ell_2)$. Kadison and Singer show [37] that that is *not so* for each complete commuting set other than $\mathbb{D}$. They also show that each pure state of $\mathbb{D}$ has a unique extension to the uniform closure of the algebra of linear combinations of operators T_π defined by $T_\pi e_i = e_{\pi(i)}$, where π is a permutation of $\mathbb{Z}$.

Kadison and Singer believed that KS had a negative answer. In particular, on page 397 of [61] they state: "We incline to the view that such extension is non-unique".

This paper is based on two fundamental principles.

Fundamental Principle I[Weaver, Conjecture 2.6]: The Kadison-Singer Problem is a statement about partitioning projections on finite dimensional Hilbert spaces with small diagonal into submatrices of norms $\leq 1-\epsilon$.

Fundamental Principle II[Theorem 3.5]: Every bounded operator on a finite dimensional Hilbert space is a constant times a "piece" of a projection operator from a larger Hilbert space.

Armed with these two basic principles, we will make a tour of many different areas of research. In each area we will use Fundamental Principle II (often in disguised form) to reduce their problem to a statement about (pieces of) projections. Then we will apply Fundamental Principle I to see that their problem is equivalent to the Kadison-Singer Problem.

This paper is a greatly expanded version of [31]. Let us now discuss the organization of this paper. In Sections 2-8 we will successively look at equivalents of the Kadison-Singer Problem in operator theory, frame theory, Hilbert space theory, Banach space theory, harmonic analysis, time-frequency analysis and finally in engineering. In section 9 we will address some approaches to producing a counter-example to KS. In Section 2 we will establish our first fundamental principle for showing that very general problems are equivalent to KS. In Section 3 we introduce our "universal language" of frame theory and introduce our second fundamental principle for reducing problems to KS. In Section 4, we will show that KS is equivalent to a fundamental result concerning inner products. This formulation of the problem has the advantage that it can be understood by a student one week into their first course in Hilbert spaces. In Section 5 we show that KS is equivalent to the Bourgain-Tzafriri Conjecture (and in fact, a significantly weaker form of the conjecture is equivalent to KS). This also shows that the Feichtinger Conjecture is equivalent to KS. In Section 6, we show that a fundamental problem in harmonic analysis is equivalent to KS. We also classify the uniform paving conjecture and the uniform Feichtinger Conjecture. As a consequence we will discover a surprising new identity in the area. In Section 7, we show that the Feichtinger Conjecture for frames of translates is equivalent to one of the fundamental unsolved problems in harmonic analysis. In Section 8, we look at how KS arises naturally in various problems in signal-processing, internet coding, coding theory and more.

Notation for statements of problems: Problem A (or Conjecture A) **implies** Problem B (or Conjecture B) means that a positive solution to the former implies a positive solution to the latter. They are **equivalent** if they imply each other.

Notation for Hilbert spaces: Throughout, $\ell_2(I)$ will denote a finite or infinite dimensional complex Hilbert space with a fixed orthonormal basis $\{e_i\}_{i\in I}$. If I is infinite we let $\ell_2 = \ell_2(I)$, and if $|I| = n$ write $\ell_2(I) = \ell_2^n$ with fixed orthonormal basis $\{e_i\}_{i=1}^n$. For any Hilbert space $\mathbb{H}$ we let $B(\mathbb{H})$ denote the family of bounded linear operators on $\mathbb{H}$. An n-dimensional subspace of $\ell_2(I)$ will be denoted $\mathbb{H}_n$. For an operator T on any one of our Hilbert spaces, its matrix representation with respect to our fixed orthonormal basis is the collection $(\langle Te_i, e_j\rangle)_{i,j\in I}$. If $J \subset I$, the **diagonal projection** Q_J is the matrix whose entries are all zero except for the (i,i) entries for $i \in J$ which are all one. For a matrix $A = (a_{ij})_{i,j\in I}$ let $\delta(A) = \max_{i\in I} |a_{ii}|$.

A universal language: We are going to show that the Kadison-Singer problem is equivalent to fundamental unsolved problems in a dozen different areas of research in both mathematics and engineering. But each of these areas is overrun with technical jargon which makes it difficult or even impossible for those outside the field to understand results inside the field. What we need is a *universal language* for interactions between a broad spectrum of research. For our universal language, we have chosen the language of *Hilbert space frame theory* (See Section 3) because it is simply stated and easily understood while being fundamental enough to quickly pass quite technical results between very diverse areas of research. Making it possible for researchers from a broad spectrum of research areas to understand how their problems relate to areas they may know little about will require certain redundancies. That is, we will have to reprove some results in the literature in the format of our universal language. Also, since frame theory is our universal language, we will prove some of the fundamental results in this area so that researchers will have a solid foundation for reading the rest of the paper.

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2. Kadison-Singer in Operator Theory

A significant advance on KS was made by Anderson [3] in 1979 when he reformulated KS into what is now known as the **Paving Conjecture** (See also [4, 5]). Lemma 5 of [61] shows a connection between KS and Paving.

Paving Conjecture (PC)**.** *For $\epsilon > 0$, there is a natural number r so that for every natural number n and every linear operator T on l_2^n whose matrix has zero diagonal, we can find a partition (i.e. a paving) $\{A_j\}_{j=1}^r$ of $\{1, \cdots, n\}$, such that*

$$\|Q_{A_j} T Q_{A_j}\| \le \epsilon \|T\| \quad \text{for all } j = 1, 2, \cdots, r.$$

It is important that r not depend on n in PC. We will say that an arbitrary operator T satisfies PC if $T - D(T)$ satisfies PC where $D(T)$ is the diagonal of T.

Remark 2.1. *There is a standard technique for turning finite dimensional results into infinite dimensional ones and vice-versa. We will illustrate this technique here by showing that PC is equivalent to PC for operators on ℓ_2 (which is a known result). After this, we will move freely between these cases for our later conjectures without proving that they are equivalent.*

We can use an abstract compactness argument for proving this result, but we feel that the following argument is more illuminating. We start with a limiting method for increasing sequences of partitions given in [25]. Since the proof is short we include it for completeness.

Proposition 2.2. *Fix a natural number r and assume for every natural number n there is a partition $\{A_j^n\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$. There exist natural numbers $\{k_1 < k_2 < \cdots\}$ so that if $m \in A_j^{k_m}$ for some $1 \le j \le r$ then $m \in A_j^{k_\ell}$, for all $\ell \ge m$. Hence, if $A_j = \{m \mid m \in A_j^{k_m}\}$ then*

(1) *$\{A_j\}_{j=1}^r$ is a partition of $\mathbb{N}$.*

(2) *If* $A_j = \{m_1 < m_2 < \cdots\}$*, then for all natural numbers* ℓ *we have* $\{m_1, m_2, \cdots, m_\ell\} \subset A_j^{k_{m_\ell}}$.

Proof: For each natural number n, 1 is in one of the sets $\{A_j^n\}_{j=1}^r$. Hence, there are natural numbers $n_1^1 < n_2^1 < \cdots$ and a $1 \le j \le r$ so that $1 \in A_j^{n_i^1}$ for all $i \in \mathbb{N}$. Now, for every natural number n_i^1, 2 is in one of the sets $\{A_j^{n_i^1}\}_{j=1}^r$. Hence, there is a subsequence $\{n_i^2\}$ of $\{n_i^1\}$ and a $1 \le j \le r$ so that $2 \in A_j^{n_i^2}$, for all $i \in \mathbb{N}$. Continuing by induction, for all $\ell \in \mathbb{N}$ we get a subsequence $\{n_i^{\ell+1}\}$ of $\{n_i^\ell\}$ and a $1 \le j \le r$ so that $\ell + 1 \in A_j^{n_j^{\ell+1}}$, for all $i \in \mathbb{N}$. Letting $k_i = n_i^i$ for all $i \in \mathbb{N}$ gives the conclusion of the proposition. □

Theorem 2.3. *The Paving Conjecture is equivalent to the Paving Conjecture for operators on* ℓ_2.

Proof: Assume PC holds for operators on ℓ_2^n. Let $T = (a_{ij})_{i,j=1}^\infty$ be a bounded linear operator on ℓ_2. Fix $\epsilon > 0$. By our assumption, for every natural number n there is a partition $\{A_j^n\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that if $T_n = (a_{ij})_{i,j=1}^n$ then for all $j = 1, 2, \cdots, r$

$$\|Q_{A_j^n} T_n Q_{A_j^n}\| \le \frac{\epsilon}{2}\|T_n\| \le \frac{\epsilon}{2}\|T\|.$$

Let $\{A_j\}_{j=1}^r$ be the partition of $\mathbb{N}$ given in Proposition 2.2. Fix $1 \le j \le r$, let $A_j = \{m_1 < m_2 < \cdots\}$, and for all $\ell \in \mathbb{N}$ let $Q_\ell = Q_{I_\ell}$ where $I_\ell = \{m_1, m_2, \cdots, m_\ell\}$. Fix $f \in \ell_2(\mathbb{N})$. For all large $\ell \in \mathbb{N}$ we have:

$$\begin{aligned}
\|Q_{A_j} T Q_{A_j}(f)\| &\le 2\|Q_\ell Q_{A_j} T Q_{A_j} Q_\ell(f)\| \\
&= 2\|Q_\ell Q_{A_j}^{k_{m_\ell}} T_{k_{m_\ell}} Q_{A_j}^{k_{m_\ell}} Q_\ell(f)\| \\
&\le 2\|Q_{A_j}^{k_{m_\ell}} T_{k_{m_\ell}} Q_{A_j}^{k_{m_\ell}}\|\|Q_\ell(f)\| \\
&\le 2\frac{\epsilon}{2}\|T\|\|f\| = \epsilon\|T\|\|f\|.
\end{aligned}$$

Hence, $\|Q_{A_j} T Q_{A_j}\| \le \epsilon\|T\|$.

Conversely, assume PC holds for operators on ℓ_2. We assume that PC fails for operators on ℓ_2^n and get a contradiction. If (1) fails, a little thought will yield that there must be an $\epsilon > 0$, a partition $\{I_n\}_{n=1}^\infty$ of $\mathbb{N}$ into finite subsets, operators $T_n : \ell_2(I_n) \to \ell_2(I_n)$ with $\|T_n\| = 1$ and for every partition $\{A_j^n\}_{j=1}^n$ of I_n there exists a $1 \le j \le n$ so that

$$\|Q_{A_j^n} T_n Q_{A_j^n}\| \ge \epsilon.$$

Let

$$T = \bigoplus_{n=1}^\infty T_n : \left(\bigoplus_{n=1}^\infty \ell_2(I_n)\right)_{\ell_2} \to \left(\bigoplus_{n=1}^\infty \ell_2(I_n)\right)_{\ell_2}.$$

Then, $\|T\| = \sup_n \|T_n\| = 1$. By (2), there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{N}$ so that for all $j = 1, 2, \cdots, r$

$$\|Q_{A_j} T Q_{A_j}\| \le \epsilon.$$

For every $n \in \mathbb{N}$ and every $j = 1, 2, \cdots, r$ let $A_j^n = A_j \cap I_n$. Then, $\{A_j^n\}_{j=1}^r$ is a partition of I_n. Hence, for every $j = 1, 2, \cdots, r$ we have

$$\|Q_{A_j^n} T_n Q_{A_j^n}\| = \|Q_{A_j^n} T Q_{A_j^n}\| \le \|Q_{A_j} T Q_{A_j}\| \le \epsilon.$$

If $n \ge r$, this contradicts our assumption about T_n. □

It is known [12] that the class of operators satisfying PC (the **pavable operators**) is a closed subspace of $B(\ell_2)$. The only large classes of operators which have been shown to be pavable are "diagonally dominant" matrices [10, 11, 12, 53], matrices with all entries real and positive [56] and Toeplitz operators over Riemann integrable functions (See also [57] and Section 6). Also, in [13] there is an analysis of the paving problem for certain Schatten C_p-norms. We strongly recommend that everyone read the argument of Berman, Halpern, Kaftal and Weiss [12] showing that matrices with positive entries satisfy PC. This argument is a fundamental principle concerning decompositions of matrices which has applications across the board — here, you will see it used in the proof of Theorem 8.16, and it was *vaguely* used in producing a generalization of the Rado-Horn Theorem [29] (See Theorem 8.3). We next note that in order to verify PC, it suffices to show that PC holds for any one of your favorite classes of operators.

Theorem 2.4. *The Paving Conjecture has a positive solution if any one of the following classes satisfies the Paving Conjecture:*

1. *Unitary operators.*
2. *Orthogonal projections.*
3. *Positive operators.*
4. *Self-adjoint operators.*
5. *Gram matrices* $(\langle f_i, f_j \rangle)_{i,j\in I}$ *where* $T : \ell_2(I) \to \ell_2(I)$ *is a bounded linear operator, and* $Te_i = f_i$, $\|Te_i\| = 1$ *for all* $i \in I$.
6. *Invertible operators (or invertible operators with zero diagonal).*

Proof: (1): This is immediate from the fact that every bounded operator is a multiple of a sum of three unitary operators [23].

(2): This follows from the Spectral Theorem (or see Fundamental Principle II: Theorem 3.5).

(3), (4): Since (3) or (4) immediately implies (2).

(5): We will show that (5) implies a positive solution to the Bourgain-Tzafriri Conjecture (See Section 5) and hence to PC by Theorem 5.1. Given $T : \ell_2(I) \to \ell_2(I)$ with $\|Te_i\| = 1$ for all $i \in I$, let $G = (\langle Te_i, Te_j \rangle)_{i,j\in I}$. By

(5), there is a partition $\{A_j\}_{j=1}^r$ of I which paves the Gram operator. Hence, for all $j = 1, 2, \cdots, r$ and all $f = \sum_{i \in A_j} a_i e_i$ we have

$$\begin{aligned} \|\sum_{i\in A_j} a_i T e_i\|^2 &= \langle \sum_{i\in A_j} a_i T e_i, \sum_{k\in A_j} a_k T e_k\rangle \\ &= \sum_{i\in A_j} |a_i|^2 \|Te_i\|^2 + \sum_{i\neq k\in A_j} a_i \overline{a_k} \langle Te_i, Te_j\rangle \\ &= \sum_{i\in A_j} |a_i|^2 + \langle Q_{A_j}(G - D(G))Q_{A_j} f, f\rangle \\ &\geq \sum_{i\in A_j} |a_i|^2 - \|Q_{A_j}(G-D(G))Q_{A_j}\| \|Q_{A_j} f\|^2 \\ &\geq \sum_{i\in A_j} |a_i|^2 - \epsilon \sum_{i\in A_j} |a_i|^2 \\ &= (1-\epsilon) \sum_{i\in A_j} |a_i|^2. \end{aligned}$$

Hence, the Bourgain-Tzafriri Conjecture holds (See section 5). Now we need to jump ahead to Theorem 5.3 to see that the proof of BT implies KS is done from the definition and does not need any theorems developed between here and there.

(6): Given an operator T, $(\|T\|+1)I + T$ is invertible and if it is pavable then so is T. For the second statement, given an operator T, let $S = T + (\|T\|^2+2)U$ where $U = (b_{ij})_{i,j\in I}$ is the unitary matrix given by the bilateral shift on $\mathbb{N}$ (the wrap-around shift on ℓ_2^n if $|I| = n$). Then, $S - D(S)$ is invertible and has zero diagonal. By (6), for $0 < \epsilon < 1$ there is a partition $\{A_j\}_{j=1}^r$ of I so that for all $j = 1, 2, \cdots, r$ we have

$$\|Q_{A_j}(S - D(S))Q_{A_j}\| \leq \epsilon.$$

Note that for any $i \in I$, if $i \in A_j$ then $i+1 \notin A_j$, since otherwise:

$$|(Q_{A_j}(S - D(S))Q_{A_j} e_{i+1})(i)| = |\langle Te_i, Te_{i+1}\rangle + (\|T\|^2 + 2)| \geq 1.$$

Hence, $\|Q_{A_j}(S - D(S))Q_{A_j}\| \geq 1$, which contradicts our paving of $S - D(S)$. It follows that

$$Q_{A_j}(S - D(S))Q_{A_j} = Q_{A_j}(T - D(T))Q_{A_j}, \quad \text{for all } j = 1, 2, \cdots, r.$$

So, our paving of S also paves T. □

Akemann and Anderson [1] showed that the following conjecture implies KS.

Conjecture 2.5. *There exists $0 < \epsilon, \delta < 1$ with the following property: for any orthogonal projection P on ℓ_2^n with $\delta(P) \leq \delta$, there is a diagonal projection Q such that $\|QPQ\| \leq 1 - \epsilon$ and $\|(I - Q)P(I - Q)\| \leq 1 - \epsilon$.*

It is important that ϵ, δ are independent of n in Conjecture 2.5. It is unknown if KS implies Conjecture 2.5. Weaver [82] showed that a conjectured strengthening of Conjecture 2.5 fails.

Recently, Weaver [81] provided important insight into KS by showing that a slight weakening of Conjecture 2.5 will produce a conjecture equivalent to KS. This is our first Fundamental Principle.

Conjecture 2.6 (Fundamental Principle I: Weaver)**.** *There exist universal constants* $0 < \delta, \epsilon < 1$ *and* $r \in \mathbb{N}$ *so that for all* n *and all orthogonal projections* P *on* ℓ_2^n *with* $\delta(P) \le \delta$*, there is a paving* $\{A_j\}_{j=1}^r$ *of* $\{1, 2, \cdots, n\}$ *so that* $\|Q_{A_j} P Q_{A_j}\| \le 1 - \epsilon$*, for all* $j = 1, 2, \cdots, r$*.*

This needs some explanation since there is nothing in [81] which looks anything like Conjecture 2.6. In [81], Weaver introduces what he calls "Conjecture $\mathrm{KS_r}$" (See Section 3). A careful examination of the proof of Theorem 1 of [81] reveals that Weaver shows Conjecture $\mathrm{KS_r}$ implies Conjecture 2.6 which in turn implies KS which (after the theorem is proved) is equivalent to $\mathrm{KS_r}$. We will see in Section 3 (Conjecture 3.10, Theorem 3.11) that we may assume $\|Pe_i\| = \|Pe_j\|$ for all $i, j = 1, 2, \cdots, n$ in Conjecture 2.6 with a small restriction on the $\epsilon > 0$.

3. Frame Theory: The Universal Language

A family of vectors $\{f_i\}_{i\in I}$ in a Hilbert space $\mathbb{H}$ is a **Riesz basic sequence** if there are constants $A, B > 0$ so that for all scalars $\{a_i\}_{i\in I}$ we have:

$$A \sum_{i\in I} |a_i|^2 \le \| \sum_{i\in I} a_i f_i \|^2 \le B \sum_{i\in I} |a_i|^2.$$

We call $\sqrt{A}, \sqrt{B}$ the **lower and upper Riesz basis bounds** for $\{f_i\}_{i\in I}$. If the Riesz basic sequence $\{f_i\}_{i\in I}$ spans $\mathbb{H}$ we call it a **Riesz basis** for $\mathbb{H}$. So $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathbb{H}$ means there is an orthonormal basis $\{e_i\}_{i\in I}$ so that the operator $T(e_i) = f_i$ is invertible. In particular, each Riesz basis is **bounded**. That is, $0 < \inf_{i\in I} \|f_i\| \le \sup_{i\in I} \|f_i\| < \infty$.

Hilbert space frames were introduced by Duffin and Schaeffer [42] to address some very deep problems in nonharmonic Fourier series (see [83]). A family $\{f_i\}_{i\in I}$ of elements of a (finite or infinite dimensional) Hilbert space $\mathbb{H}$ is called a **frame** for $\mathbb{H}$ if there are constants $0 < A \le B < \infty$ (called the **lower and upper frame bounds**, respectively) so that for all $f \in \mathbb{H}$

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2. \tag{3.1}$$

If we only have the right hand inequality in Equation 3.1 we call $\{f_i\}_{i\in I}$ a **Bessel sequence with Bessel bound B**. If $A = B$, we call this an A-**tight frame** and if $A = B = 1$, it is called a **Parseval frame**. If all the frame elements have the same norm, this is an **equal norm** frame and if the frame elements are of unit norm, it is a **unit norm frame**. It is immediate that $\|f_i\|^2 \le B$. If also $\inf \|f_i\| > 0$, $\{f_i\}_{i\in I}$ is a **bounded frame**. The numbers $\{\langle f, f_i\rangle\}_{i\in I}$ are the **frame coefficients** of the vector $f \in \mathbb{H}$. If $\{f_i\}_{i\in I}$ is a Bessel sequence, the **synthesis operator** for $\{f_i\}_{i\in I}$ is the bounded linear operator $T : \ell_2(I) \to \mathbb{H}$ given by $T(e_i) = f_i$ for all $i \in I$. The **analysis operator** for $\{f_i\}_{i\in I}$ is T^* and satisfies: $T^*(f) = \sum_{i\in I}\langle f, f_i\rangle e_i$. In particular,

$$\|T^* f\|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2, \quad \text{for all } f \in \mathbb{H},$$

and hence the smallest Bessel bound for $\{f_i\}_{i\in I}$ equals $\|T^*\|^2$. Comparing this to Equation 3.1 we have:

Theorem 3.1. *Let $\mathbb{H}$ be a Hilbert space and $T : \ell_2(I) \to \mathbb{H}$, $Te_i = f_i$ be a bounded linear operator. The following are equivalent:*

(1) *$\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$.*
(2) *The operator T is bounded, linear, and onto.*
(3) *The operator T^* is an (possibly into) isomorphism.*

Moreover, if $\{f_i\}_{i\in I}$ is a Riesz basis, then the Riesz basis bounds are $\sqrt{A}, \sqrt{B}$ where A, B are the frame bounds for $\{f_i\}_{i\in I}$.

It follows that a Bessel sequence is a Riesz basic sequence if and only if T^* is onto. The **frame operator** for the frame is the positive, self-adjoint invertible operator $S = TT^* : \mathbb{H} \to \mathbb{H}$. That is,

$$Sf = TT^* f = T\left(\sum_{i\in I}\langle f, f_i\rangle e_i\right) = \sum_{i\in I}\langle f, f_i\rangle Te_i = \sum_{i\in I}\langle f, f_i\rangle f_i.$$

In particular,

$$\langle Sf, f\rangle = \sum_{i\in I} |\langle f, f_i\rangle|^2.$$

It follows that $\{f_i\}_{i\in I}$ is a frame with frame bounds A, B if and only if $A \cdot I \le S \le B \cdot I$. So $\{f_i\}_{i\in I}$ is a Parseval frame if and only if $S = I$. **Reconstruction** of vectors in $\mathbb{H}$ is achieved via the formula:

$$\begin{aligned} f &= SS^{-1} f = \sum_{i\in I}\langle S^{-1} f, f_i\rangle f_i \\ &= \sum_{i\in I}\langle f, S^{-1} f_i\rangle f_i \\ &= \sum_{i\in I}\langle f, f_i\rangle S^{-1} f_i \\ &= \sum_{i\in I}\langle f, S^{-1/2} f_i\rangle S^{-1/2} f_i. \end{aligned}$$

It follows that $\{S^{-1/2}f_i\}_{i\in I}$ is a Parseval frame *equivalent* to $\{f_i\}_{i\in I}$. Two sequences $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ in a Hilbert space are *equivalent* if there is an invertible operator T between their spans with $Tf_i = g_i$ for all $i \in I$. We now show that there is a simple way to tell when two frame sequences are equivalent.

Proposition 3.2. *Let $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$ be frames for a Hilbert space $\mathbb{H}$ with analysis operators T_1 and T_2, respectively. The following are equivalent:*

(1) The frames $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are equivalent.

(2) ker T_1 = ker T_2.

Proof: (1) ⇒ (2): If $Lf_i = g_i$ is an isomorphism, then $Lf_i = LT_1e_i = g_i = T_2e_i$ quickly implies our statement about kernels.

(2) ⇒ (1): Since $T_i|_{(ker\ T_i)^\perp}$ is an isomorphism for $i = 1, 2$, if the kernels are equal, then

$$T_2\left(T_1|_{(ker\ T_2)^\perp}\right)^{-1} f_i = g_i$$

is an isomorphism. □

In the finite dimensional case, if $\{g_j\}_{j=1}^n$ is an orthonormal basis of ℓ_2^n consisting of eigenvectors for S with respective eigenvalues $\{\lambda_j\}_{j=1}^n$, then for every $1 \le j \le n$, $\sum_{i\in I} |\langle f_i, g_j\rangle|^2 = \lambda_j$. In particular, $\sum_{i\in I} \|f_i\|^2 = \text{trace S}$ ($= n$ if $\{f_i\}_{i\in I}$ is a Parseval frame). An important result is

Theorem 3.3. *If $\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$ with frame bounds A, B and P is any orthogonal projection on $\mathbb{H}$, then $\{Pf_i\}_{i\in I}$ is a frame for $P\mathbb{H}$ with frame bounds A, B.*

Proof: For any $f \in P\mathbb{H}$,

$$\sum_{i\in I} |\langle f, Pf_i\rangle|^2 = \sum_{i\in I} |\langle Pf, f_i\rangle|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2.$$

□

A fundamental result in frame theory was proved independently by Naimark and Han/Larson [35, 54]. For completeness we include its simple proof.

Theorem 3.4. *A family $\{f_i\}_{i\in I}$ is a Parseval frame for a Hilbert space $\mathbb{H}$ if and only if there is a containing Hilbert space $\mathbb{H} \subset \ell_2(I)$ with an orthonormal basis $\{e_i\}_{i\in I}$ so that the orthogonal projection P of $\ell_2(I)$ onto $\mathbb{H}$ satisfies $P(e_i) = f_i$ for all $i \in I$.*

Proof: The "only if" part is Theorem 3.3. For the "if" part, if $\{f_i\}_{i\in I}$ is a Parseval frame, then the synthesis operator $T : \ell_2(I) \to \mathbb{H}$ is a partial isometry. So T^* is an isometry and we can associate $\mathbb{H}$ with $T^*\mathbb{H}$. Now, for all $i \in I$ and all $g = T^*f \in T^*\mathbb{H}$ we have

$$\langle T^*f, Pe_i\rangle = \langle T^*f, e_i\rangle = \langle f, Te_i\rangle = \langle f, f_i\rangle = \langle T^*f, T^*f_i\rangle.$$

It follows that $Pe_i = T^* f_i$ for all $i \in I$. □

Now we can establish our Fundamental Principle II which basically states that any bounded operator on a finite dimensional Hilbert space is really just a multiple of a "piece" of a projection from a larger space.

Theorem 3.5 (Fundamental Principle II). *Let $\mathbb{H}_n$ be an n-dimensional Hilbert space with orthonormal basis $\{g_i\}_{i=1}^n$. If $T : \mathbb{H}_n \to \mathbb{H}_n$ is any bounded linear operator with $\|T\| = 1$, then there is a containing Hilbert space $\mathbb{H}_n \subset \ell_2^M$ (M=2n-1) with an orthonormal basis $\{e_i\}_{i=1}^M$ so that the orthogonal projection P from ℓ_2^M onto $\mathbb{H}_n$ satisfies:*

$$Pe_i = Tg_i, \quad \text{for all } i = 1, 2, \cdots, n.$$

Proof: Let S be the frame operator for the Bessel sequence $\{f_i\}_{i=1}^n = \{Tg_i\}_{i=1}^n$ having eigenvectors $\{x_i\}_{i=1}^n$ with respective eigenvalues $\{\lambda_i\}_{i=1}^n$ where $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For $i = 2, 3, \cdots, n$ let $h_i = \sqrt{(1-\lambda_i)}x_i$. Then, $\{f_i\}_{i=1}^n \cup \{h_i\}_{i=2}^n$ is a Parseval frame for $\mathbb{H}$ since for every $f \in \mathbb{H}$ we have

$$\begin{aligned}
\sum_{i=1}^n |\langle f, f_i\rangle|^2 + \sum_{i=2}^n |\langle f, h_i\rangle|^2 &= \langle Sf, f\rangle + \sum_{i=2}^n (1-\lambda_i)|\langle f, x_i\rangle|^2 \\
&= \sum_{i=1}^n \lambda_i |\langle f, x_i\rangle|^2 + \sum_{i=2}^n (1-\lambda_i)|\langle f, x_i\rangle|^2 \\
&= \sum_{i=1}^n |\langle f, x_i\rangle|^2 = \|f\|^2.
\end{aligned}$$

Now, by Theorem 3.4, there is a containing Hilbert space ℓ_2^{2n-1} with an orthonormal basis $\{e_i\}_{i=1}^{2n-1}$ so that the orthogonal projection P satisfies: $Pe_i = Tg_i$ for $i = 1, 2, \cdots, n$ and $Pe_i = h_i$ for $i = n+1, \cdots, 2n-1$. □

For an introduction to frame theory we refer the reader to Christensen [35].

Weaver [81] established an important relationship between frames and KS by showing that the following conjecture is equivalent to KS.

Conjecture 3.6. *There are universal constants $B \geq 4$ and $\epsilon > \sqrt{B}$ and an $r \in \mathbb{N}$ so that the following holds: Whenever $\{f_i\}_{i=1}^M$ is a unit norm B-tight frame for ℓ_2^n, there exists a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, M\}$ so that for all $j = 1, 2, \cdots, r$ and all $f \in \ell_2^n$ we have*

$$\sum_{i \in A_j} |\langle f, f_i\rangle|^2 \leq (B - \epsilon)\|f\|^2. \tag{3.2}$$

In his work on time-frequency analysis, Feichtinger [32] noted that all of the Gabor frames he was using (see Section 7) had the property that they could be divided into a finite number of subsets which were Riesz basic sequences. This led to the conjecture:

Feichtinger Conjecture (FC)**.** *Every bounded frame (or equivalently, every unit norm frame) is a finite union of Riesz basic sequences.*

There is a significant body of work on this conjecture [10, 11, 32, 53]. Yet, it remains open even for Gabor frames. In [25] it was shown that FC is equivalent to the weak BT, and hence is implied by KS (See Section 5). In [31] it was shown that FC is equivalent to KS (See Theorem 5.3). In fact, we now know that KS is equivalent to the *weak* Feichtinger Conjecture: Every unit norm Bessel sequence is a finite union of Riesz basic sequences (See Section 5). In [30] it was shown that FC is equivalent to the following conjecture.

Conjecture 3.7. *Every bounded Bessel sequence is a finite union of frame sequences.*

Let us mention two more useful equivalent formulations of KS due to Weaver [81].

Conjecture 3.8 ($\mathrm{KS_r}$)**.** *There is a natural number r and universal constants B and $\epsilon > 0$ so that the following holds. Let $\{f_i\}_{i=1}^M$ be elements of ℓ_2^n with $\|f_i\| \leq 1$ for $i = 1, 2, \cdots, M$ and suppose for every $f \in \ell_2^n$,*

$$\sum_{i=1}^{M} |\langle f, f_i \rangle|^2 \leq B\|f\|^2. \tag{3.3}$$

Then, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $f \in \ell_2^n$ and all $j = 1, 2, \cdots, r$,

$$\sum_{i \in A_j} |\langle f, f_i \rangle|^2 \leq (B - \epsilon)\|f\|^2.$$

Weaver [81] also shows that Conjecture $\mathrm{KS_r}$ is equivalent to PC if we assume equality in Equation 3.4 for all $f \in \ell_2^n$. Weaver further shows that Conjecture 3.8 is equivalent to KS even if we strengthen its assumptions so as to require that the vectors $\{f_i\}_{i=1}^M$ are of equal norm and that equality holds in 3.4, but at great cost to our $\epsilon > 0$.

Conjecture 3.9 ($\mathrm{KS_r}'$)**.** *There exists universal constants $B \geq 4$ and $\epsilon > \sqrt{B}$ so that the following holds. Let $\{f_i\}_{i=1}^M$ be elements of ℓ_2^n with $\|f_i\| \leq 1$ for $i = 1, 2, \cdots, M$ and suppose for every $f \in \ell_2^n$,*

$$\sum_{i=1}^{M} |\langle f, f_i \rangle|^2 = B\|f\|^2. \tag{3.4}$$

Then, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $f \in \ell_2^n$ and all $j = 1, 2, \cdots, r$,

$$\sum_{i \in A_j} |\langle f, f_i \rangle|^2 \leq (B - \epsilon)\|f\|^2.$$

We now strengthen the assumptions in Fundamental Principle I, Conjecture 2.6.

Conjecture 3.10. *There exist universal constants $0 < \delta, \sqrt{\delta} \le \epsilon < 1$ and $r \in \mathbb{N}$ so that for all n and all orthogonal projections P on ℓ_2^n with $\delta(P) \le \delta$ and $\|Pe_i\| = \|Pe_j\|$ for all $i, j = 1, 2, \cdots, n$, there is a paving $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that $\|Q_{A_j} P Q_{A_j}\| \le 1 - \epsilon$, for all $j = 1, 2, \cdots, r$.*

Using Conjecture 3.9 we can see that KS is equivalent to Conjecture 3.10.

Theorem 3.11. *KS is equivalent to Conjecture 3.10.*

Proof: It is clear that Conjecture 2.6 (which is equivalent to KS) implies Conjecture 3.10. So we assume that Conjecture 3.10 holds and we will show that Conjecture 3.9 holds. Let $\{f_i\}_{i=1}^M$ be elements of $\mathbb{H}_n$ with $\|f_i\| = 1$ for $i = 1, 2, \cdots, M$ and suppose for every $f \in \mathbb{H}_n$,

$$\sum_{i=1}^{M} |\langle f, f_i \rangle|^2 = B\|f\|^2, \tag{3.5}$$

where $\frac{1}{B} \le \delta$. It follows from Equation 3.5 that $\{\frac{1}{\sqrt{B}} f_i\}_{i=1}^M$ is an equal norm Parseval frame and so there is a larger Hilbert space ℓ_2^M and a projection $P : \ell_2^M \to \mathbb{H}_n$ so that $Pe_i = f_i$ for all $i = 1, 2, \cdots, M$. Now, $\|Pe_i\|^2 = \langle Pe_i, e_i \rangle = \frac{1}{B} \le \delta$ for all $i = 1, 2, \cdots, M$. So by Conjecture 3.10, there is a paving $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, M\}$ so that $\|Q_{A_j} P Q_{A_j}\| \le 1 - \epsilon$, for all $j = 1, 2, \cdots, r$. Now, for all $1 \le j \le r$ and all $f \in \ell_2^n$ we have:

$$\begin{aligned}
\|Q_{A_j} P f\|^2 &= \sum_{i=1}^{M} |\langle Q_{A_j} P f, e_i \rangle|^2 = \sum_{i=1}^{M} |\langle f, P Q_{A_j} e_i \rangle|^2 \\
&= \frac{1}{B} \sum_{i \in A_j} |\langle f, f_i \rangle|^2 \\
&\le \|Q_{A_j} P\|^2 \|f\|^2 \\
&= \|Q_{A_j} P Q_{A_j}\| \|f\|^2 \le (1 - \epsilon)\|f\|^2.
\end{aligned}$$

It follows that for all $f \in \mathbb{H}_n$ we have

$$\sum_{i \in A_j} |\langle f, f_i \rangle|^2 \le (B - \epsilon B)\|f\|^2.$$

Since $\epsilon B > \sqrt{B}$, we have verified Conjecture 3.9. □

We give one final formulation of KS in Hilbert space frame theory.

Theorem 3.12. *The following are equivalent:*

(1) The Paving Conjecture.

(2) For every unit norm B-Bessel sequence $\{f_i\}_{i=1}^M$ *in* $\mathbb{H}_n$ *and every* $\epsilon > 0$, *there exists* $r = f(B, \epsilon)$ *and a partition* $\{A_j\}_{j=1}^r$ *of* $\{1, 2, \cdots, M\}$ *so that for every* $j = 1, 2, \cdots, r$ *and all scalars* $\{a_i\}_{i \in A_j}$ *we have*

$$\sum_{n \in A_j} |\langle f_n, \sum_{n \neq m \in A_j} a_m f_m \rangle|^2 \leq \epsilon \| \sum_{m \in A_j} a_m f_m \|^2.$$

Proof: (1) $\Rightarrow$ (2): Let G be the Gram operator for $\{f_i\}_{i=1}^M$. By PC, we can partition $\{1, 2, \cdots, M\}$ into $\{A_j\}_{j=1}^r$ so that for all $j = 1, 2, \cdots, r$ we have

$$\|P_{A_j}(G - D(G))Q_{A_j}\| \leq \epsilon.$$

Now, for any $j = 1, 2, \cdots, r$ and any scalars $\{a_m\}_{m \in A_j}$ we have

$$\begin{aligned} \sum_{n \in A_j} |\langle f_n, \sum_{n \neq m \in A_j} a_m f_m \rangle|^2 &= \|Q_{A_j}(G - D(G))Q_{A_j}(\sum_{m \in A_j} a_m f_m)\|^2 \\ &\leq \epsilon \sum_{n \in A_j} |a_n|^2 \\ &\leq \frac{\epsilon}{1 - \epsilon} \| \sum_{n \in A_j} a_n f_n \|^2, \end{aligned}$$

where the last inequality follows from the R_ϵ-Conjecture (actually, its proof using PC, see section 4).

(2) $\Rightarrow$ (1): Given (2), we have

$$\begin{aligned} \| \sum_{n \in A_j} a_n f_n \|^2 &= \sum_{n \in A_j} |a_n|^2 + \sum_{n \neq m \in A_j} a_n \overline{a_m} \langle f_n, f_m \rangle \\ &= \sum_{n \in A_j} |a_n|^2 + \sum_{n \in A_j} a_n \langle f_n, \sum_{n \neq m \in A_j} a_m f_m \rangle. \end{aligned}$$

Using (2) we now compute:

$$\begin{aligned} \left| \sum_{n \in A_j} a_n \langle f_n, \sum_{n \neq m \in A_j} a_m f_m \rangle \right|^2 &\leq \left(\sum_{n \in A_j} |a_n|^2 \right) \sum_{n \in A_j} |\langle f_n, \sum_{n \neq m \in A_j} a_m f_m \rangle|^2 \\ &\leq \left(\sum_{n \in A_j} |a_n|^2 \right) \cdot \epsilon \| \sum_{m \in A_j} a_m f_m \|^2 \\ &\leq \left(\sum_{n \in A_j} |a_n|^2 \right) \cdot \epsilon \cdot B \sum_{n \in A_j} |a_n|^2. \end{aligned}$$

This is enough to verify the R_ϵ-Conjecture (See Section 4). □

An important open problem in frame theory is:

Problem 3.13. *Classify the equal norm Parseval frames with special properties.*

The *special properties* here could be *translation invariance* (Section 7), *reconstruction after erasures* (Section 8), *frames which decompose into good frame sequences* (see Section 8), etc. The idea here is to build up a "bookshelf" of equal norm frames with special properties which can be used for applications such as we have for wavelets. As we will see, this problem shows up in many formulations of KS.

4. Kadison-Singer in Hilbert space theory

In this section we will see that KS is actually a fundamental result concerning inner products. Recall that a family of vectors $\{f_i\}_{i\in I}$ is a **Riesz basic sequence** in a Hilbert space $\mathbb{H}$ if there are constants $A, B > 0$ so that for all scalars $\{a_i\}_{i\in I}$ we have:

$$A\sum_{i\in I}|a_i|^2 \le \|\sum_{i\in I} a_i f_i\|^2 \le B\sum_{i\in I}|a_i|^2.$$

We call $\sqrt{A}, \sqrt{B}$ the **lower and upper Riesz basis bounds** for $\{f_i\}_{i\in I}$. If $\epsilon > 0$ and $A = 1-\epsilon, B = 1+\epsilon$ we call $\{f_i\}_{i\in I}$ an ϵ**-Riesz basic sequence**. If $\|f_i\| = 1$ for all $i \in I$ this is a **unit norm** Riesz basic sequence. A natural question is whether we can improve the Riesz basis bounds for a unit norm Riesz basic sequence by partitioning the sequence into subsets. Formally:

Conjecture 4.1 (R_ϵ-Conjecture)**.** *For every $\epsilon > 0$, every unit norm Riesz basic sequence is a finite union of ϵ-Riesz basic sequences.*

The R_ϵ-Conjecture was first stated in [32] where it was shown that KS implies this conjecture. It was recently shown in [31] that KS is equivalent to the R_ϵ-Conjecture. We include this argument here since it demonstrates a fundamental principle we will employ throughout this paper.

Theorem 4.2. *The following are equivalent:*
(1) The Paving Conjecture.
(2) If $T : \ell_2 \to \ell_2$ is a bounded linear operator with $\|Te_i\| = 1$ for all $i \in I$, then for every $\epsilon > 0$, $\{Te_i\}_{i\in I}$ is a finite union of ϵ-Riesz basic sequences.
(3) The R_ϵ-Conjecture.

Proof: (1) $\Rightarrow$ (2): Fix $\epsilon > 0$. Given T as in (2), let $S = T^*T$. Since S has ones on its diagonal, the (infinite form of the) Paving Conjecture gives $r = r(\epsilon, \|T\|)$ and a partition $\{A_j\}_{j=1}^r$ of I so that for every $j = 1, 2, \cdots, r$ we have

$$\|Q_{A_j}(I-S)Q_{A_j}\| \le \delta\|I-S\|$$

where $\delta = \epsilon/(\|S\|+1)$. Now, for all $f = \sum_{i\in I} a_i e_i$ we have

$$\begin{aligned}
\|\sum_{i\in A_j} a_i T e_i\|^2 &= \|TQ_{A_j} f\|^2 \\
&= \langle TQ_{A_j} f, TQ_{A_j} f\rangle \\
&= \langle T^*TQ_{A_j} f, Q_{A_j} f\rangle \\
&= \langle Q_{A_j} f, Q_{A_j} f\rangle - \langle Q_{A_j}(I-S)Q_{A_j} f, Q_{A_j} f\rangle \\
&\geq \|Q_{A_j} f\|^2 - \delta\|I-S\|\|Q_{A_j} f\|^2 \\
&\geq (1-\epsilon)\|Q_{A_j} f\|^2 = (1-\epsilon)\sum_{i\in A_j} |a_i|^2.
\end{aligned}$$

Similarly, $\|\sum_{i\in A_j} a_i T e_i\|^2 \leq (1+\epsilon)\sum_{i\in A_j}|a_i|^2$.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (1): Let $T \in B(\ell_2)$ with $Te_i = f_i$ and $\|f_i\| = 1$ for all $i \in I$. We need to show that the Gram operator $\mathbf{G}$ of $\{f_i\}_{i\in I}$ is pavable. Fix $0 < \delta < 1$ and let $\epsilon > 0$. Let $g_i = \sqrt{1-\delta^2} f_i \oplus \delta e_i \in \ell_2 \oplus \ell_2$. Then ,$\|g_i\| = 1$ for all $i \in I$ and for all scalars $\{a_i\}_{i\in I}$

$$\begin{aligned}
\delta\sum_{i\in I}|a_i|^2 &\leq \|\sum_{i\in I} a_i g_i\|^2 = (1-\delta^2)\|\sum_{i\in I} a_i T e_i\|^2 + \delta^2 \sum_{i\in I}|a_i|^2 \\
&\leq \left[(1-\delta^2)\|T\|^2 + \delta^2\right]\sum_{i\in I}|a_i|^2.
\end{aligned}$$

So $\{g_i\}_{i\in I}$ is a unit norm Riesz basic sequence and $\langle g_i, g_k\rangle = (1-\delta^2)\langle f_i, f_k\rangle$ for all $i \neq k \in I$. By the R_ϵ-Conjecture, there is a partition $\{A_j\}_{j=1}^r$ so that for all $j = 1, 2, \cdots, r$ and all $f = \sum_{i\in I} a_i e_i$,

$$\begin{aligned}
(1-\epsilon)\sum_{i\in A_j}|a_i|^2 &\leq \|\sum_{i\in A_j} a_i g_i\|^2 = \langle \sum_{i\in A_j} a_i g_i, \sum_{k\in A_j} a_k g_k\rangle \\
&= \sum_{i\in A_j}|a_i|^2\|g_i\|^2 + \sum_{i\neq k\in A_j} a_i\overline{a_k}\langle g_i, g_k\rangle \\
&= \sum_{i\in A_j}|a_i|^2 + (1-\delta^2)\sum_{i\neq k\in A_j} a_i\overline{a_k}\langle f_i, f_k\rangle \\
&= \sum_{i\in A_j}|a_i|^2 + (1-\delta^2)\langle Q_{A_j}(G - D(G))Q_{A_j} f, f\rangle \\
&\leq (1+\epsilon)\sum_{i\in A_j}|a_i|^2.
\end{aligned}$$

Subtracting $\sum_{i\in A_j}|a_i|^2$ through the inequality yields,

$$-\epsilon\sum_{i\in A_j}|a_i|^2 \le (1-\delta^2)\langle Q_{A_j}(G-D(G))Q_{A_j}f,f\rangle \le \epsilon\sum_{i\in A_j}|a_i|^2.$$

That is,

$$(1-\delta^2)|\langle Q_{A_j}(G-D(G))Q_{A_j}f,f\rangle| \le \epsilon\|f\|^2.$$

Since $Q_{A_j}(G-D(G))Q_{A_j}$ is a self-adjoint operator, we have $(1-\delta^2)\|Q_{A_j}(G-D(G))Q_{A_j}\| \le \epsilon$. That is, $(1-\delta^2)G$ (and hence G) is pavable. □

Remark 4.3. *The proof of* (3) $\Rightarrow$ (1) *of Theorem 4.2 illustrates a standard method for turning conjectures about unit norm Riesz basic sequences* $\{g_i\}_{i\in I}$ *into conjectures about unit norm Bessel sequences* $\{f_i\}_{i\in I}$. *Namely, given* $\{f_i\}_{i\in I}$ *and* $0<\delta<1$, *let* $g_i=\sqrt{1-\delta^2}f_i\oplus\delta e_i\in\ell_2(I)\oplus\ell_2(I)$. *Then,* $\{g_i\}_{i\in I}$ *is a unit norm Riesz basic sequence and for* δ *small enough,* g_i *is close enough to* f_i *to pass inequalities from* $\{g_i\}_{i\in I}$ *to* $\{f_i\}_{i\in I}$.

It follows from Remark 2.1 that we can finite-dimensionalize the result in Theorem 4.2.

Conjecture 4.4. *For every* $\epsilon>0$ *and every* $T\in B(\ell_2^n)$ *with* $\|Te_i\|=1$ *for* $i=1,2,\cdots,n$ *there is an* $r=r(\epsilon,\|T\|)$ *and a partition* $\{A_j\}_{j=1}^r$ *of* $\{1,2,\cdots,n\}$ *so that for all* $j=1,2,\cdots,r$ *and all scalars* $\{a_i\}_{i\in A_j}$ *we have*

$$(1-\epsilon)\sum_{i\in A_j}|a_i|^2 \le \|\sum_{i\in A_j}a_iTe_i\|^2 \le (1+\epsilon)\sum_{i\in A_j}|a_i|^2.$$

By Remark 4.3, we can reformulate Conjecture 4.4 into a statement about unit norm Riesz basic sequences.

One advantage of the R_ϵ-Conjecture is that it can be shown to students right at the beginning of a course in Hilbert spaces. We note that this conjecture fails for equivalent norms on a Hilbert space. For example, if we renorm ℓ_2 by letting $|\{a_i\}|=\|a_i\|_{\ell_2}+\sup_i|a_i|$, then the R_ϵ-Conjecture fails for this equivalent norm. To see this, let $f_i=(e_{2i}+e_{2i+1})/(\sqrt{2}+1)$ where $\{e_i\}_{i\in\mathbb{N}}$ is the unit vector basis of ℓ_2. This is now a unit norm Riesz basic sequence, but no infinite subset satisfies the R_ϵ-Conjecture. To check this, let $J\subset\mathbb{N}$ with $|J|=n$ and $a_i=1/\sqrt{n}$ for $i\in J$. Then,

$$|\sum_{i\in J}a_if_i| = \frac{1}{\sqrt{2}+1}\left(\sqrt{2}+\frac{1}{\sqrt{n}}\right).$$

Since the norm above is bounded away from one for $n\ge 2$, we cannot satisfy the requirements of the R_ϵ-Conjecture. It follows that a positive solution to KS would imply a fundamental new result concerning "inner products", not just norms. Actually, the R_ϵ-Conjecture is way too strong for proving KS. As we will see, having either the upper inequality or the lower inequality hold is a sufficient enough assumption to prove KS - and for each of these we just need a universal constant to work instead of $1-\epsilon$ or $1+\epsilon$.

Using Conjecture 3.6 we can show that the following conjecture is equivalent to KS:

Conjecture 4.5. *There is a universal constant $1 \le D$ so that for all $T \in B(\ell_2^n)$ with $\|Te_i\| = 1$ for all $i = 1, 2, \cdots, n$, there is an $r = r(\|T\|)$ and a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $j = 1, 2, \cdots, r$ and all scalars $\{a_i\}_{i\in A_j}$*

$$\|\sum_{i\in A_j} a_i Te_i\|^2 \le D \sum_{i\in A_j} |a_i|^2.$$

Theorem 4.6. *Conjecture 4.5 is equivalent to KS.*

Proof: Since Conjecture 4.4 clearly implies Conjecture 4.5, we just need to show that Conjecture 4.5 implies Conjecture 3.6. So, choose D as in Conjecture 4.5 and choose $B \ge 4$ and $\epsilon > \sqrt{B}$ so that $D \le B - \epsilon$. Let $\{f_i\}_{i\in I}$ be a unit norm B tight frame for ℓ_2^n. If $Te_i = f_i$ is the synthesis operator for this frame, then $\|T\|^2 = \|T^*\|^2 = B$. So by Conjecture 4.5, there is an $r = r(\|B\|)$ and a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $j = 1, 2, \cdots, r$ and all scalars $\{a_i\}_{i\in A_j}$

$$\|\sum_{i\in A_j} a_i Te_i\|^2 = \|\sum_{i\in A_j} a_i f_i\|^2 \le D \sum_{i\in A_j} |a_i|^2 \le (B-\epsilon) \sum_{i\in A_j} |a_i|^2.$$

So $\|TQ_{A_j}\|^2 \le B - \epsilon$ and for all $f \in \ell_2^n$

$$\sum_{i\in A_j} |\langle f, f_i\rangle|^2 = \|(Q_{A_j}T)^* f\|^2 \le \|TQ_{A_j}\|^2 \|f\|^2 \le (B-\epsilon)\|f\|^2.$$

This verifies that Conjecture 3.6 holds and so KS holds. □

Remark 4.3 and Conjecture 4.5 show that we only need any universal upper bound in the R_ϵ-Conjecture to hold to get KS.

5. Kadison-Singer in Banach space theory

In this section we state a fundamental theorem of Bourgain and Tzafriri called the *restricted invertibility principle*. This theorem led to the *(strong and weak) Bourgain-Tzafriri Conjectures*. We will see that these conjectures are equivalent to KS.

In 1987, Bourgain and Tzafriri [16] proved a fundamental result in Banach space theory known as the **restricted invertibility principle**.

Theorem 5.1 (Bourgain-Tzafriri)**.** *There are universal constants $A, c > 0$ so that whenever $T : \ell_2^n \to \ell_2^n$ is a linear operator for which $\|Te_i\| = 1$, for $1 \le i \le n$, then there exists a subset $\sigma \subset \{1, 2, \cdots, n\}$ of cardinality*

$|\sigma| \geq cn/\|T\|^2$ so that for all $j = 1, 2, \cdots, n$ and for all choices of scalars $\{a_j\}_{j\in\sigma}$,

$$\|\sum_{j\in\sigma} a_j Te_j\|^2 \geq A \sum_{j\in\sigma} |a_j|^2.$$

Theorem 5.1 gave rise to a problem in the area which has received a great deal of attention [17, 31, 32].

Bourgain-Tzafriri Conjecture (BT)**.** *There is a universal constant $A > 0$ so that for every $B > 1$ there is a natural number $r = r(B)$ satisfying: For any natural number n, if $T : \ell_2^n \to \ell_2^n$ is a linear operator with $\|T\| \leq B$ and $\|Te_i\| = 1$ for all $i = 1, 2, \cdots, n$, then there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $j = 1, 2, \cdots, r$ and all choices of scalars $\{a_i\}_{i\in A_j}$ we have:*

$$\|\sum_{i\in A_j} a_i Te_i\|^2 \geq A \sum_{i\in A_j} |a_i|^2.$$

It had been "folklore" for years that KS and BT must be equivalent. But no one was quite able to actually give a proof of this fact. Recently, Casazza and Vershynin [32] gave a formal proof of the equivalence of KS and BT. Sometimes BT is called **strong BT** since there is a weakening of it called **weak BT**. In weak BT we allow A to depend upon the norm of the operator T. A significant amount of effort has been invested in trying to show that strong and weak BT are equivalent [10, 25, 32]. Recently, Casazza and Tremain [31] proved this equivalence by showing that these results are all equivalent to yet another conjecture.

Conjecture 5.2. *There exists a constant $A > 0$ and a natural number r so that for all natural numbers n, if $T : \ell_2^n \to \ell_2^n$ with $\|Te_i\| = 1$ for all $i = 1, 2, \cdots, n$ and $\|T\| \leq 2$, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $j = 1, 2, \cdots, r$ and all scalars $\{a_i\}_{i\in A_j}$ we have*

$$\|\sum_{i\in A_j} a_i Te_i\|^2 \geq A \sum_{i\in A_j} |a_i|^2.$$

The proof of the following theorem from [31] demonstrates how we will use our two Fundamental Principles.

Theorem 5.3. *The following are equivalent:*

(1) *The Kadison-Singer Problem.*
(2) *The (strong) BT.*
(3) *The (weak) BT.*
(4) *Conjecture 5.2*
(5) *The Feichtinger Conjecture.*

Proof: (1) $\Rightarrow$ (2): By the R_ϵ-Conjecture, KS implies (strong) BT.

It is clear that (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1): It suffices to show that Conjecture 5.2 implies Conjecture 2.6. Let r, A satisfy Conjecture 5.2. Fix $0 < \delta \leq 3/4$ and let P be an orthogonal projection on ℓ_2^n with $\delta(P) \leq \delta$ (notation from Section 1). Now, $\langle Pe_i, e_i\rangle = \|Pe_i\|^2 \leq \delta$ implies $\|(I-P)e_i\|^2 \geq 1-\delta \geq \frac{1}{4}$. Define $T : \ell_2^n \to \ell_2^n$ by $Te_i = (I-P)e_i/\|(I-P)e_i\|$. For any scalars $\{a_i\}_{i=1}^n$ we have

$$\begin{aligned}
\|\sum_{i=1}^n a_i Te_i\|^2 &= \|\sum_{i=1}^n \frac{a_i}{\|(I-P)e_i\|}(I-P)e_i\|^2 \\
&\leq \sum_{i=1}^n \left|\frac{a_i}{\|(I-P)e_i\|}\right|^2 \\
&\leq 4\sum_{i=1}^n |a_i|^2.
\end{aligned}$$

So $\|Te_i\| = 1$ and $\|T\| \leq 2$. By Conjecture 5.2, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $j = 1, 2, \cdots, r$ and all scalars $\{a_i\}_{i\in A_j}$ we have

$$\|\sum_{i\in A_J} a_i Te_i\|^2 \geq A\sum_{i\in A_j} |a_i|^2.$$

Hence,

$$\begin{aligned}
\|\sum_{i\in A_j} a_i(I-P)e_i\|^2 &= \|\sum_{i\in A_j} a_i\|(I-P)e_i\|Te_i\|^2 \\
&\geq A\sum_{i\in A_j} |a_i|^2\|(I-P)e_i\|^2 \\
&\geq \frac{A}{4}\sum_{i\in A_j} |a_i|^2.
\end{aligned}$$

It follows that for all scalars $\{a_i\}_{i\in A_j}$,

$$\begin{aligned}
\sum_{i\in A_j} |a_i|^2 &= \|\sum_{i\in A_j} a_i Pe_i\|^2 + \|\sum_{i\in A_j} a_i(I-P)e_i\|^2 \\
&\geq \|\sum_{i\in A_j} a_i Pe_i\|^2 + \frac{A}{4}\sum_{i\in A_j} |a_i|^2.
\end{aligned}$$

Now, for all $f = \sum_{i=1}^n a_i e_i$

$$\|PQ_{A_j} f\|^2 = \|\sum_{i\in A_j} a_i Pe_i\|^2 \leq (1 - \frac{A}{4})\sum_{i\in A_j} |a_i|^2.$$

Thus,

$$\|Q_{A_j}PQ_{A_j}\| = \|PQ_{A_j}\|^2 \leq 1 - \frac{A}{4}.$$

So Conjecture 2.6 holds.

(1) $\Rightarrow$ (5): Since every unit norm frame $\{f_i\}_{i\in I}$ has the property that the operator $T(e_i) = f_i$ is bounded where $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathbb{H}$, it follows that from Theorem 4.2 that PC implies FC.

(5) $\Rightarrow$ (4): This arguement comes from [25]. We will prove the contrapositive. So we assume that (4) fails. Then for every $M \in \mathbb{N}$ and for every $A > 0$ there is an $n = n(M, A) \in \mathbb{N}$, a finite dimensional Hilbert space H and a Bessel sequence $\{f_i\}_{i=1}^n$ in H with Bessel constant 2 and $\|f_i\| = 1$, for all $1 \leq i \leq n$, and whenever we partition $\{1, 2, \cdots, n\}$ into sets $\{I_j\}_{j=1}^M$, then there exists some $1 \leq \ell \leq M$ and a set of scalars $\{a_i\}_{i\in I_\ell}$ with

$$\|\sum_{i\in I_\ell} a_i f_i\|^2 \leq A \sum_{i\in I_\ell} |a_i|^2.$$

Now, for each $k \in \mathbb{N}$, we can choose a finite dimensional Hilbert space H_k of dimension, say m_k, and letting $M = k$ and $A = 1/k$ above we can choose $n_k = n(k, 1/k)$ and $\{f_i^k\}_{i=1}^{n_k}$ satisfying the above conditions. Let $H = (\sum \oplus H_k)_{\ell_2}$ and consider $\{f_i^k\}_{i=1,k=1}^{n_k\ ,\infty}$ as elements of H. For each $k \in \mathbb{N}$, let $\{e_i^k\}_{i=1}^{m_k}$ be an orthonormal basis for H_k and consider $\{e_i^k\}_{i=1,k=1}^{m_k\ ,\infty}$ as elements of H. Since $\{e_i^k\}_{i=1,k=1}^{m_k\ ,\infty}$ is an orthonormal bais for H, the family $\{f_i^k\}_{i=1,k=1}^{n_k\ ,\infty} \cup \{e_i^k\}_{i=1,k=1}^{m_k\ ,\infty}$ is a family of norm one vectors in H with Bessel bound 3 and lower frame bound ≥ 1 and hence forms a frame for H. Fix $M, A > 0$ and assume we can partition this frame into M sets of Riesz basic sequences each with lower Riesz basis bound A. In particular, we can partition $\{f_i^k\}_{i=1,k=1}^{n_k\ ,\infty}$ into M sets of Riesz basic sequences each with lower Riesz basis bound A. But, for all k with $k \geq M$ and $1/k \leq A$, $\{f_i^k\}_{i=1}^{n_k}$ cannot be partitioned into M sets each with lower Riesz basis bound $\geq A$, and hence $\{f_i^k\}_{i=1,k=1}^{n_k\ ,\infty}$ cannot be partitioned this way. This shows that (5) fails. □

Finally, let us note that Remark 4.3 and BT imply that KS is equivalent to just the lower inequality in the R_ϵ-Conjecture and even without the lower constant having to be close to one.

6. Kadison-Singer in harmonic analysis

In this section, we present a detailed study of the Paving Conjecture for Toeplitz operators, reducing this problem to an old and fundamental problem in Harmonic Analysis. Given $\phi \in L^\infty([0,1])$, the corresponding **Toeplitz operator** is $T_\phi : L^2[0,1] \to L^2[0,1]$, $T_\phi(f) = f \cdot \phi$. In the 1980's, much effort was put into showing that the class of Toeplitz operators satisfies the Paving Conjecture (see Berman, Halpern, Kaftal and Weiss [12, 56, 55, 57]) during which time the uniformly pavable operators were classified and it was shown that T_ϕ is pavable if ϕ is Riemann integrable [56]. In Section 6.1 we will reduce the conjecture to a fundamental question in harmonic analysis and find the weakest conditions which need to be established to verify PC. In Section 6.2 we give harmonic analysis classifications of the Uniform Paving Property for Toeplitz operators and for the Uniform Feichtinger Conjecture. As a consequence, we will discover a surprising universal identity for all functions $f \in L^2[0,1]$. Throughout this section we will use the following notation.

Notation: If $I \subset \mathbb{Z}$, we let $S(I)$ denote the $L^2([0,1])$-closure of the span of the exponential functions with frequencies taken from I:

$$S(I) = \mathrm{cl}(\mathrm{span}\{\mathrm{e}^{2\pi \mathrm{i}nt}\}_{n\in I}).$$

6.1. **The Paving Conjecture for Toeplitz operators.** A deep and fundamental question in Harmonic Analysis is to understand the distribution of the norm of a function $f \in S(I)$. It is known (Proposition 6.5) if that if $[a,b] \subset [0,1]$ and $\epsilon > 0$, then there is a partition of $\mathbb{Z}$ into arithmetic progressions $A_j = \{nr+j\}_{n\in\mathbb{Z}}$, $0 \le j \le r-1$ so that for all $f \in S(A_j)$ we have

$$(1-\epsilon)(b-a)\|f\|^2 \le \|f \cdot \chi_{[a,b]}\|^2 \le (1+\epsilon)(b-a)\|f\|^2.$$

What this says is that the functions in $S(A_j)$ have their norms nearly uniformly distributed across $[a,b]$ and $[0,1]\setminus[a,b]$. The central question is whether such a result is true for arbitrary measurable subsets of $[0,1]$ (but it is known that the partitions can no longer be arithmetic progressions [18, 56, 57]). If E is a measurable subset of $[0,1]$, let P_E denote the orthogonal projection of $L^2[0,1]$ onto $L^2(E)$, that is, $P_E(f) = f\cdot\chi_E$. The fundamental question here is then

Conjecture 6.1. *If $E \subset [0,1]$ is measurable and $\epsilon > 0$ is given, there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that for all $j = 1,2,\cdots,r$ and all $f \in S(A_j)$*

$$(1-\epsilon)|E|\|f\|^2 \le \|P_E(f)\|^2 \le (1+\epsilon)|E|\|f\|^2. \tag{6.1}$$

Despite the many deep results in the field of Harmonic Analysis, almost nothing is known about the distribution of the norms of functions coming from the span of a finite subset of the characters, except that this question has deep connections to Number Theory [18] (Also, see Theorem 7.11). Very little progress has ever been made on Conjecture 6.1 except for a specialized result of Bourgain and Tzafriri [17]. Any advance on this problem would have broad applications throughout the field.

To this day, the Paving Conjecture for Toeplitz operators remains a deep mystery. The next theorem (from [31]) helps explain why so little progress has been made on KS for Toeplitz operators — this problem is in fact equivalent to Conjecture 6.1. Because this result is fundamental for the rest of this section, we will give the proof from [31]. To prove the theorem we will first look at the decomposition of Toeplitz operators of the form P_E.

Proposition 6.2. *If $E \subset [0,1]$ and $A \subset \mathbb{Z}$ then for every $f \in L^2[0,1]$ we have*

$$\|P_E Q_A f\|^2 = |E|\|Q_A f\|^2 + \langle Q_A(P_E - D(P_E))Q_A f, f\rangle,$$

where Q_A is the orthogonal projection of $L^2[0,1]$ onto $S(A)$.

Proof: For any $f = \sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{2\pi int} \in L^2[0,1]$ we have

$$\begin{aligned}
\|P_E Q_A f\|^2 &= \langle P_E Q_A f, P_E Q_A f\rangle = \langle \sum_{n\in A} a_n P_E(\mathrm{e}^{2\pi \mathrm{i}nt}), \sum_{m\in A} a_m P_E(\mathrm{e}^{2\pi \mathrm{i}mt})\rangle \\
&= \sum_{n\in A} |a_n|^2 \|\chi_E \cdot \mathrm{e}^{2\pi int}\|^2 + \sum_{n\neq m\in A} a_n \overline{a_m} \langle P_E \mathrm{e}^{2\pi \mathrm{i}nt}, \mathrm{e}^{2\pi \mathrm{i}mt}\rangle \\
&= |E| \sum_{n\in A} |a_n|^2 + \langle (P_E - D(P_E)) \sum_{n\in A} a_n \mathrm{e}^{2\pi int}, \sum_{n\in A} a_n \mathrm{e}^{2\pi \mathrm{i}nt}\rangle \\
&= |E|\|Q_A f\|^2 + \langle Q_A(P_E - D(P_E))Q_A f, f\rangle.
\end{aligned}$$

□

Now we are ready for the theorem from [31].

Theorem 6.3. *The following are equivalent:*
(1) Conjecture 6.1.
(2) For every measurable $E \subset [0,1]$, the Toeplitz operator P_E satisfies PC.
(3) All Toeplitz operators satisfy PC.

Proof: (2) ⇔ (3): This follows from the fact that the class of pavable operators is closed and the class of Toeplitz operators are contained in the closed linear span of the Toeplitz operators of the form P_E. That is, an arbitrary bounded measurable function on $[0,1]$ may be essentially uniformly approximated by simple functions.

(1) ⇔ (2): By Proposition 6.2, Conjecture 6.1 holds if and only if for all $\epsilon > 0$, there exists a partition $\{A_j\}_{j=1}^r$ such that

$$\begin{aligned}
(1-\epsilon)|E|\|Q_{A_j} f\|^2 &\leq |E|\|Q_{A_j} f\|^2 + \langle Q_{A_j}(P_E - D(P_E)Q_{A_j} f, f\rangle \\
&\leq (1+\epsilon)|E|\|Q_{A_j} f\|^2
\end{aligned}$$

for all $j = 1, 2, \cdots, r$ and all $f \in L^2[0,1]$.

Subtracting like terms through the inequality yields that this inequality is equivalent to

$$|\langle Q_{A_j}(P_E - D(P_E)Q_{A_j} f, f\rangle| \leq \epsilon |E|\|Q_{A_j} f\|^2. \tag{6.2}$$

Since $Q_{A_j}(P_E - D(P_E)Q_{A_j}$ is a self-adjoint, Equation 6.2 is equivalent to $\|Q_{A_j}(P_E - D(P_E)Q_{A_j}\| \leq \epsilon|E|$, and so P_E is pavable. □

Next we state a useful result from [68].

Proposition 6.4. *Suppose that $\lambda_1, \lambda_2 \cdots, \lambda_N$ are distinct real numbers, and suppose that $\delta > 0$ is chosen so that $|\lambda_m - \lambda_n| \geq \delta$ whenever $n \neq m$. Then for any coefficients $\{a_n\}_{n=1}^N$, and any $T > 0$ we have*

$$\int_0^T |\sum_{n=1}^N a_n e^{2\pi i \lambda_n t}|^2 dt = \left(T + \frac{\theta}{\delta}\right) \sum_{n=1}^N |a_n|^2,$$

for some θ with $-1 \leq \theta \leq 1$.

As an immediate consequence of 6.4 we have

Proposition 6.5. *Conjecture 6.1 holds for intervals. Moreover, we can use a partition of $\mathbb{Z}$ made up of arithmetic progressions.*

We next show that a significantly weaker conjecture than Conjecture 6.1 is equivalent to PC for Toeplitz operators. It is clear that this is the weakest inequality we can have and still get PC.

Conjecture 6.6. *There is a universal constant $0 < K$ so that for any measurable set $E \subset [0, 1]$ there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that for every $f \in S(A_j)$ we have $\|P_E f\|^2 \leq K|E|\|f\|^2$.*

Now we will see that Conjecture 6.6 is equivalent to PC for Toeplitz operators.

Proposition 6.7. *Conjecture 6.6 is equivalent to Conjecture 6.1.*

Proof: Conjecture 6.1 clearly implies Conjecture 6.6. Assuming Conjecture 6.6 holds, fix $\epsilon > 0$ and a measurable set $E \subset [0, 1]$. Choose intervals $\{[a_k, b_k]\}_{k=1}^\infty$ so that

$$E \subset \bigcup_{k=1}^\infty [a_k, b_k],$$

and

$$\left| \sum_{k=1}^\infty (b_k - a_k) - |E| \right| < \frac{\epsilon |E|}{3K}.$$

Next, choose N so that

$$\sum_{k=N+1}^\infty (b_k - a_k) < \frac{\epsilon |E|}{3K},$$

and let

$$F = \bigcup_{k=1}^N [a_k, b_k], \quad E_1 = E \cap F, \quad E_2 = E \setminus E_1.$$

Note that

$$|F \setminus E_1| \leq \left|(F \setminus E_1) \cup \left(\bigcup_{k=N+1}^{\infty} [a_k, b_k] \setminus E_2 \right)\right| \leq \left|\bigcup_{k=1}^{\infty}[a_k, b_k] \setminus E\right| < \frac{\epsilon|E|}{3}.$$

By Conjecture 6.6, we can partition $\mathbb{Z}$ into $\{A_j\}_{j=1}^{r}$ so that for each j and all $f \in S(A_j)$ we have

$$\|P_{E_2} f\|^2 \leq K|E_2|\|f\|^2,$$

and

$$\|P_{F\setminus E_1} f\|^2 \leq K|F \setminus E_1|\|f\|^2.$$

Since $E_2 \subset \bigcup_{k=N+1}^{\infty} [a_k, b_k]$, for every $f \in S(A_j)$ we have

$$\|P_{E_2} f\|^2 \leq K\frac{\epsilon|E|}{3K}\|f\|^2 = \frac{\epsilon|E|}{3}\|f\|^2.$$

Fix $1 \leq j \leq r$. By Proposition 6.5, there is a partition $\{B_k\}_{k=1}^{M_j}$ of A_j so that for every $f \in S(B_k)$ we have

$$(|F| - \frac{\epsilon|E|}{3})\|f\|^2 \leq \|P_F f\|^2 \leq (|F| + \frac{\epsilon|E|}{3})\|f\|^2.$$

Now, for every $f \in S(B_k)$ we have

$$\begin{aligned}
\|P_E f\|^2 &\leq \|P_{E_1} f\|^2 + \|P_{E_2} f\|^2 \\
&\leq \|P_F f\|^2 + \frac{\epsilon|E|}{3}\|f\|^2 \\
&\leq (|F| + \frac{\epsilon|E|}{3})\|f\|^2 + \frac{\epsilon|E|}{3}\|f\|^2 \\
&\leq \sum_{k=1}^{N}(b_k - a_k)\|f\|^2 + \frac{2\epsilon|E|}{3}\|f\|^2 \\
&\leq (|E| + \frac{\epsilon|E|}{3K})\|f\|^2 + \frac{2\epsilon|E|}{3}\|f\|^2 \\
&= (1+\epsilon)|E|\|f\|^2,
\end{aligned}$$

where, without loss of generality, we have assumed $K > 1$. For the other direction, we note that $P_F f = P_{E_1} f + P_{F\setminus E_1} f$ and $P_{E_1} f \perp P_{F\setminus E_1} f$ and so

$$\begin{aligned}\|P_E f\|^2 \geq \|P_{E_1} f\|^2 &= \|P_F f\|^2 - \|P_{F\setminus E_1} f\|^2 \\ &\geq (|F| - \frac{\epsilon|E|}{3})\|f\|^2 - \frac{\epsilon|E|}{3}\|f\|^2 \\ &\geq (|E| - \frac{\epsilon|E|}{3} - \frac{2\epsilon|E|}{3})\|f\|^2 \\ &= (1-\epsilon)|E|\|f\|^2.\end{aligned}$$

□

We next present several equivalent formulations of a slightly weaker conjecture.

Conjecture 6.8. *Suppose $E \subset [0,1]$ with $0 < |E|$. There is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that for all $j = 1, 2, \cdots, r$, P_E is an isomorphism of $S(A_j)$ onto its range.*

Definition 6.9. *We say the Toeplitz operator T_ϕ satisfies the Feichtinger Conjecture if there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that $\{T_\phi e^{2\pi int}\}_{n\in A_j}$ is a Riesz basic sequence for every $j = 1, 2, \cdots, r$.*

Conjecture 6.8 is equivalent to all Toeplitz operators satisfying the Feichtinger Conjecture. That is, for any Toeplitz operator T_ϕ, $\{T_\phi e^{2\pi int}\}_{n\in\mathbb{Z}}$ is a finite union of Riesz basic sequences.

Theorem 6.10. *The following are equivalent:*

(1) The Feichtinger Conjecture for Toeplitz operators.

(2) The Feichtinger Conjecture for P_E for every measurable set $E \subset [0,1]$ with $0 < |E|$.

(3) Conjecture 6.8.

Proof: (1) ⇒ (2): This is obvious.

(2) ⇒ (3): If E is a measurable subset of $[0,1]$ with $0 < |E|$ and we assume (2), then there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ and a constant $0 < A$ so that $\{P_E e^{2\pi int}\}_{n\in A_j}$ is a Riesz basic sequence for all $j = 1, 2, \cdots, r$ with lower Riesz basis bound A. Hence, for every $j = 1, 2, \cdots, r$ and every $f = \sum_{n\in A_j} a_n e^{2\pi int}$ we have

$$\|P_E f\|^2 = \|\sum_{n\in A_j} a_n P_E e^{2\pi int}\|^2 \geq A^2 \sum_{n\in A_j} |a_n|^2 = A^2\|f\|^2.$$

(3) ⇒ (1): Let T_ϕ be a non-zero Toeplitz operator on $L^2[0,1]$. Choose $\epsilon > 0$ and a measurable set $E \subset [0,1]$ with $|E| > 0$ so that $|\phi(t)| \geq \epsilon$ for all $t \in E$. By our assumption (3), there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that P_E is an

isomorphism of $S(A_j)$ onto its range with, say, lower isomorphism bound A. That is, for all $j = 1, 2, \cdots, r$ and all $\{a_n\}_{n\in A_j}$ we have

$$\begin{aligned} \|\sum_{n\in A_j} a_n T_\phi \mathrm{e}^{2\pi int}\|^2 &= \int_0^1 |\sum_{n\in A_j} a_n \mathrm{e}^{2\pi int}|^2 |\phi(t)|^2 \, dt \\ &\geq \int_0^1 |\sum_{n\in A_j} a_n \mathrm{e}^{2\pi int}|^2 \epsilon^2 |\chi_E|^2 \, dt \\ &\geq \epsilon^2 \|P_E \sum_{n\in A_j} a_n \mathrm{e}^{2\pi int}\|^2 \\ &\geq \epsilon^2 A^2 \sum |a_n|^2. \end{aligned}$$

□

At this time we do not know if the Feichtinger Conjecture for Toeplitz operators is equivalent to PC for Toeplitz operators. The main problem here is that we do not have an equivalent form of Fundamental Principle I (Conjecture 2.6) for Toeplitz operators.

There is some evidence for believing that Conjecture 6.8 might be true since a weaker version of it holds as we see in the next result.

Proposition 6.11. *Assume $E \subset [0, 1]$ with $0 < |E|$. Then, there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that for every $j = 1, 2, \cdots, r$ if $f \in S(A_j)$ and $f|_E = 0$ then $f = 0$.*

Proof: We actually prove a stronger result, namely that there is a single partition that works for all measurable sets E. In particular, let $A_1 = \mathbb{N} \cup \{0\}$ and $A_2 = \mathbb{Z} \setminus A_1$. Now, if $f \in S(A_1)$ then $\log |f| \in L^1[0, 1]$ (See Duran [43], Page 16) and so $f \neq 0$ on any set of positive measure. A similar argument (applied to the complex conjugate of f) applies in the case where $f \in S(A_2)$. □

We end this section with one more equivalence of PC for Toeplitz operators.

Proposition 6.12. *For a Toeplitz operator T_g the following are equivalent:*

(1) T_g satisfies PC.

(2) For every $\epsilon > 0$ there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that for every $j = 1, 2, \cdots, r$ and for every $f \in S(A_j)$ we have:

$$\|f\|(\|g\| - \epsilon) \leq \|f \cdot g\| \leq \|f\|(\|g\| + \epsilon).$$

Proof: (1) ⇒ (2): If T_g has the Paving Property, then Conjecture 6.6 (and hence Conjecture 6.1) holds. Fix $\epsilon > 0$. Choose a simple function $h = \sum_{k=1}^M a_k \chi_{E_k}$ such that $|g - h| < \epsilon$ almost everywhere, where, without loss of generality, the sets E_k are mutually disjoint. By Conjecture 6.1, there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that for all $j = 1, 2, \cdots, r$ and for all $f \in S(A_j)$ we have

$$\|f\|^2 \left(|E_k| - \frac{\delta}{\sum_{k=1}^M |a_k|^2} \right) \le \|f \cdot \chi_{E_k}\|^2 \le \|f\|^2 \left(|E_k| + \frac{\delta}{\sum_{k=1}^M |a_k|^2} \right)$$

for all $k = 1, \ldots, M$. Now,

$$\begin{aligned}
\|f \cdot h\|^2 &= \|\sum_{k=1}^M a_k f \cdot \chi_{E_k}\|^2 \\
&= \sum_{k=1}^M |a_k|^2 \|f \cdot \chi_{E_k}\|^2 \\
&\le \sum_{k=1}^M |a_k|^2 \|f\|^2 \left(|E_k| + \frac{\delta}{\sum_{k=1}^M |a_k|^2} \right) \\
&\le \|f\|^2 \left(\sum_{k=1}^M |a_k|^2 |E_k| + \delta \right) \\
&= \|f\|^2 (\|h\|^2 + \delta).
\end{aligned}$$

Similarly,

$$\|f \cdot h\|^2 \ge \|f\|^2 (\|h\|^2 - \delta).$$

Hence, for $\delta > 0$ small enough we have

$$\begin{aligned}
\|f \cdot g\| &\le \|f \cdot h\| + \|f \cdot (g - h)\| \\
&\le \|f\| \sqrt{\|h\|^2 + \delta} + \|f\| \delta \\
&\le \|f\| \left(\sqrt{(\|g\| + \delta)^2 + \delta} + \delta \right) \\
&\le \|f\| \sqrt{\|g\|^2 + \epsilon}.
\end{aligned}$$

Thus,

$$\|f \cdot g\|^2 \le \|f\|^2 (\|g\|^2 + \epsilon).$$

Similarly,

$$\|f \cdot g\|^2 \ge \|f\|^2 (\|g\|^2 - \epsilon).$$

(2) $\Rightarrow$ (1): This is immediate from Conjecture 6.1, and Theorem 6.3. $\square$

6.2. **The Uniform Paving Property.** In this section we will classify the Toeplitz operators which have the uniform paving property and the uniform Feichtinger property.

Definition 6.13. *A Toeplitz operator T_g has the uniform paving property if for every $\epsilon > 0$, there is a $K \in \mathbb{N}$ so that if $A_k = \{nK+k\}_{n\in\mathbb{Z}}$ for $0 \le k \le K-1$ then*

$$\|P_{A_k}(T_g - D(T_g))P_{A_k}\| < \epsilon.$$

Definition 6.14. *A Toeplitz operator T_g has the uniform Feichtinger property if there is a $K \in \mathbb{N}$ so that if $A_k = \{nK+k\}_{n\in\mathbb{Z}}$ for $0 \le k \le K-1$, then $\{T_g e^{2\pi i n t}\}_{n\in A_k}$ is a Riesz basic sequence for all $k = 0, 1, \cdots, K-1$.*

Halpern, Kaftal and Weiss [56] made a detailed study of the uniform paving property and in particular they showed that T_ϕ is uniformly pavable if ϕ is a Riemann integrable function. As we saw in Section 6.1, the uniform paving property is really a fundamental question in harmonic analysis. In this section we will give classifications of the Toeplitz operators having the uniform paving property and those having the uniform Feichtinger property. Our approach will be of a harmonic analysis flavor and will lead to a new identity which holds for all $f \in L^2[0,1]$.

Notation 6.15. *For all $g \in L^2[0,1]$, $K \in \mathbb{N}$ and any $0 \le k \le K-1$ we let*

$$g_k^K(t) = \sum_{n\in\mathbb{Z}} \langle g, e^{2\pi i(nK+k)t}\rangle e^{2\pi i(nK+k)t}.$$

Also,

$$g_K(t) = \frac{1}{K}\sum_{k=0}^{K-1} |g(t-\frac{k}{K})|^2.$$

The main theorems of this section are:

Theorem 6.16. *Let $g \in L^\infty[0,1]$ and T_g the Toeplitz operator of multiplication by g. The following are equivalent:*
(1) T_g has uniform PC.
(2) There is an increasing sequence of natural numbers $\{K_n\}$ so that

$$\lim_{n\to\infty} \frac{1}{K_n}\sum_{k=0}^{K_n-1} |g(t-\frac{k}{K_n})|^2 = \|g\|^2 \quad a.e.$$

uniformly over t. That is, for every $\epsilon > 0$ there is a $K \in \mathbb{N}$ so that

$$|\frac{1}{K}\sum_{k=0}^{K-1} |g(t-\frac{k}{K})|^2 - \|g\|^2| < \epsilon \quad a.e.$$

Theorem 6.17. *Let $g \in L^\infty([0,1])$ and T_g the Toeplitz operator of multiplication by g. The following are equivalent:*
(1) T_g has the uniform Feichtinger property.

(2) There is a natural number $K \in \mathbb{N}$ and an $\epsilon > 0$ so that

$$\frac{1}{K}\sum_{k=0}^{K-1} |g(t-\frac{k}{K})|^2 \geq \epsilon \quad a.e.$$

(3) There is an $\epsilon > 0$ and K measurable sets

$$E_k \subset [\frac{k}{K}, \frac{k+1}{K}], \quad \text{for all } 0 \leq k \leq K-1,$$

satisfying:

(a) The sets $\{E_k - \frac{k}{K}\}_{k=0}^{K-1}$ are disjoint in $[0, \frac{1}{K}]$.

(b) $\bigcup_{k=0}^{K-1}(E_k - k) = [0, \frac{1}{K}]$.

(c) $|g(t)| \geq \epsilon$ on $\bigcup_{k=0}^{K-1} E_k$.

With a little effort we can recover the Halpern, Kaftal and Weiss result [56].

Corollary 6.18. *For all Riemann integrable functions ϕ, the Toeplitz operator T_ϕ satisfies the uniform paving property. If $|g| \geq \epsilon > 0$ on an interval then g has the uniform Feichtinger property. Hence, if g is continuous at one point and is not zero at that point, then g has the uniform Feichtinger property.*

Examples of Toeplitz operators failing uniform pavability were given in [18, 56].

Example 6.19. *An example of a Toeplitz operator which fails the uniform Feichtinger property.*

Proof: Choose $0 < a_n$ so that $\sum na_n < 1$. For each n, choose

$$F_n \subset [0, \frac{1}{n}], \quad \text{with} \quad |F_n| = a_n.$$

Let

$$E_n = \bigcup_{k=0}^{n-1}(F_n + \frac{k}{n}), \quad \text{and} \quad E = \bigcup_{n=0}^{\infty} E_n.$$

Now,

$$|E| \leq \sum_n |E_n| = \sum na_n < 1.$$

It is easily seen that E^c contains no intervals. If $g = \chi_{E^c}$, then for all $K \in \mathbb{Z}$ we have

$$\sum_{k=0}^{K-1} |g(t-\frac{k}{K})|^2 = 0, \quad \text{on} \quad E_K.$$

So uniform Feichtinger fails. □

Example 6.20. *There is an open set $F \subset [0,1]$ so that for $g = \chi_F$ the Toeplitz operator T_g fails the uniform paving property.*

Proof: Let $g = \chi_F$ where F is the set given in Example 6.19. Let F_n be the sets given in that example also. Then F is an open set with $|F| < 1$ and for all $K \in \mathbb{N}$ we have

$$\frac{1}{K}\sum_{k=0}^{K-1} |g(t-\frac{k}{K})|^2 = 1, \quad \text{for all } t \in F_n.$$

Hence, T_g fails uniform paving by Corollary 4.5. □

Example 6.21. *There is a $g \in L^2[0,1]$ so that the Toeplitz operator T_g has the uniform Feichtinger property but fails the uniform paving property.*

Proof: Choose a measurable set $F \subset [0,1/2]$ with χ_F failing uniform paving and let $E = F \cup [1/2,1]$. By Corollary 6.18 P_E has the uniform Feichtinger property but still fails uniform paving. □

In order to prove Theorems 6.16 and 6.17, we will need to do some preliminary work. Parts (1), (2) of this proposition were done originally by Halpern, Kaftal and Weiss [56], Lemma 3.4.

Proposition 6.22. *For any $g \in L^2([0,1])$ and any positive integer K,*

(1) $g_k^K(t) = \frac{1}{K}\sum_{j=0}^{K-1} g(t-\frac{j}{K})\mathrm{e}^{\frac{2\pi \mathrm{i} jk}{K}}$ *for all* $0 \le k \le K-1$,

(2) *For all $k, \ell \in \mathbb{Z}$ we have:*

$$g_k^K(t-\frac{\ell}{K}) = \mathrm{e}^{-\frac{2\pi \mathrm{i} k\ell}{K}} g_k^K(t).$$

(3) *If now $f = f_k^K$, then*

$$(f\cdot g)_\ell^K(t) = f(t) g_{\ell-k}^K(t).$$

Proof: (1): For any $k = 0, \ldots, K-1$, consider $h \in L^2([0,1])$,

$$h(t) = \frac{1}{K}\sum_{j=0}^{K-1} g(t-\frac{j}{K}).$$

We want to show $h = g_k^K$, namely,

$$h(t) = \sum_{n\in\mathbb{Z}} \langle g, \mathrm{e}^{2\pi \mathrm{i}(nK+k)t}\rangle \mathrm{e}^{2\pi \mathrm{i}(nK+k)t}.$$

To do this, note

$$
\begin{aligned}
\widehat{h}(n) &= \int_0^1 h(t)e^{-2\pi int}dt \\
&= \frac{1}{K}\sum_{j=0}^{K-1} \mathrm{e}^{\frac{2\pi \mathrm{i} jk}{K}} \int_0^1 g(t-\frac{j}{K})e^{-2\pi int}dt, \quad \text{let } s = t - \tfrac{j}{K} \\
&= \frac{1}{K}\sum_{j=0}^{K-1} \mathrm{e}^{\frac{2\pi \mathrm{i} jk}{K}} \int_0^1 g(s)e^{-2\pi in(s+\frac{j}{K})}ds \\
&= \frac{1}{K}\sum_{j=0}^{K-1} \mathrm{e}^{\frac{2\pi \mathrm{i} jk}{K}} \mathrm{e}^{-\frac{2\pi \mathrm{i} nj}{K}} \int_0^1 g(s)e^{-2\pi ins}ds \\
&= \frac{1}{K}\sum_{j=0}^{K-1} \left[\mathrm{e}^{-\frac{2\pi \mathrm{i}(n-k)}{K}}\right]^j \hat{g}(n) \\
&= \begin{cases} \hat{g}(n) & \mathrm{e}^{-\frac{2\pi \mathrm{i}(n-k)}{K}} = 1 \\ 0 & \mathrm{e}^{-\frac{2\pi \mathrm{i}(n-k)}{K}} \neq 1 \end{cases} \\
&= \begin{cases} \hat{g}(n) & n \equiv k \mod K \\ 0 & \text{else} \end{cases} .
\end{aligned}
$$

Thus, h and g_k^K have the same Fourier coefficients, that is, $h = g_k^K$.

(2): We have:

$$
\begin{aligned}
g_k(t-\frac{\ell}{K}) &= \frac{1}{K}\sum_{j=0}^{K-1} g(t-\frac{\ell}{K}-\frac{j}{K})\mathrm{e}^{\frac{2\pi \mathrm{i} jk}{K}} \\
&= \frac{1}{K}\sum_{j=0}^{K-1} g(t-\frac{\ell+j}{K})\mathrm{e}^{\frac{2\pi \mathrm{i} jk}{K}} \\
&= \frac{1}{K}\sum_{j=0}^{K-1} g(t-\frac{\ell+j}{K})\mathrm{e}^{\frac{2\pi \mathrm{i}(\ell+j)}{K}} \mathrm{e}^{-\frac{2\pi \mathrm{i}\ell k}{K}} \\
&= \mathrm{e}^{-\frac{2\pi \mathrm{i}\ell k}{K}} g_k(t).
\end{aligned}
$$

(3): We have:

$$\begin{aligned}(f\cdot g)_\ell(t) &= \frac{1}{K}\sum_{k=0}^{K} f(t-\frac{j}{K})g(t-\frac{j}{K})\mathrm{e}^{\frac{2\pi\mathrm{i}j\ell}{K}}\\ &= \frac{1}{K}\sum_{j=0}^{K-1} f_k(t-\frac{j}{K})g(t-\frac{j}{K})\mathrm{e}^{\frac{2\pi\mathrm{i}j\ell}{K}}\\ &= \frac{1}{K}\sum_{j=0}^{K-1} \mathrm{e}^{-\frac{2\pi\mathrm{i}jk}{K}} f_k(t)g(t-\frac{j}{K})\mathrm{e}^{\frac{2\pi\mathrm{i}j\ell}{K}}\\ &= f_k(t)\frac{1}{K}\sum_{j=0}^{K-1} g(t-\frac{j}{K})\mathrm{e}^{\frac{2\pi\mathrm{i}j(\ell-k)}{K}} = f(t)g_{\ell-k}(t).\end{aligned}$$

□

Now we can establish an important relationship between g_k^K and g.

Theorem 6.23. *For all $g\in L^2[0,1]$ we have*

$$\sum_{k=0}^{K-1}|g_k^K(t)|^2 = \frac{1}{K}\sum_{k=0}^{K-1}|g(t-\frac{k}{K})|^2.$$

Proof: With

$$g_k(t) = \frac{1}{K}\sum_{j=0}^{K-1} g(t-\frac{j}{K})\mathrm{e}^{\frac{2\pi\mathrm{i}jk}{K}},$$

and for $t\in[0,1]$, let $h_t\in\ell(\mathbb{Z}_K)$ be given by: $h_t(j)=g(t-\frac{j}{K})$. Then

$$g_k(t) = \frac{1}{K}\sum_{j=0}^{K-1} h_t(j)\mathrm{e}^{-\frac{2\pi\mathrm{i}jk}{K}} = \frac{1}{\sqrt{K}}\widehat{h}_t(-k).$$

Now,

$$\begin{aligned}\sum_{k=0}^{K-1}|g_k^K(t)|^2 &= \sum_{k=0}^{K-1}|g_k(t)|^2 = \sum_{k=0}^{K-1}|\frac{1}{\sqrt{K}}\widehat{h}_t(-k)|^2\\ &= \frac{1}{K}\sum_{k=0}^{K-1}|\widehat{h}_t(-k)|^2\\ &= \frac{1}{K}\|\widehat{h}_t\|^2 = \frac{1}{K}\|h_t\|^2\\ &= \frac{1}{K}\sum_{j=0}^{K-1}|h_t(j)|^2 = \frac{1}{K}\sum_{j=0}^{K-1}|g(t-\frac{j}{K})|^2.\end{aligned}$$

Combined with our lemmas, this proves Theorem 6.23. □

The next theorem gives an identity which holds for all $f \in L^2[0,1]$. It says that *pointwise*, any Fourier series can be divided into its subseries of arithmetic progressions so that the square sums of the functions given by the subseries spreads the norm nearly equally over the interval $[0,1]$ (Compare this to the discussion at the beginning of Section 6.1).

Theorem 6.24. *For any $g \in L^\infty([0,1])$ there is an increasing sequence of natural numbers $\{K_n\}_{n=1}^\infty$ so that*

$$\lim_{n\to\infty}\left(\sum_{k=0}^{K_n-1}|g_k^{K_n}(t)|^2\right)^{1/2} = \|g\|\chi_{[0,1]} \quad a.e.$$

Proof: By Theorem 6.23, we can work with the functions g_K. Also, we observe that it suffices to prove that this sum converges in measure to $\|g\|^2\chi_{[0,1]}$. We will do the proof in steps.

Step 1: The result holds for $g = \chi_{[S,T]}$. And in fact, the convergence is uniform for these functions.

Proof of Step 1: It suffices to assume $S = 0$ and $T < 1$. Fix K and choose $1 \le k \le K$ so that

$$\frac{k-1}{K} \le T < \frac{k}{K}.$$

So, $k - 1 \le TK \le k$. Now,

$$\frac{1}{K}\sum_{j=0}^{K-1}|g(t-\frac{j}{K})|^2 \le \frac{k}{K} = \frac{k-1}{K} + \frac{1}{K} \le T + \frac{1}{K}.$$

Similarly,

$$T - \frac{1}{K} \le \frac{1}{K}\sum_{j=0}^{K-1}|g(t-\frac{j}{K})|^2.$$

This completes Step 1.

Step 2: If $\{g_j\}_{j=1}^{\ell}$ are disjointly supported functions on $[0,1]$, $g = \sum_{j=1}^{\ell} g_j$ and all the g_j satisfy the theorem (respectively, satisfy the theorem with uniform convergence), then g satisfies the theorem (respectively, with uniform convergence).

Proof of Step 2: Since the g_j are disjointly supported,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{K_n}\sum_{k=0}^{K_n-1}|g(t-\frac{k}{K_n})|^2 &= \lim_{n\to\infty}\frac{1}{K_n}\sum_{k=0}^{K_n-1}\sum_{j=1}^{\ell}|g_j(t-\frac{k}{K_n})|^2\\ &= \sum_{j=1}^{\ell}\lim_{n\to\infty}\frac{1}{K_n}|g_j(t-\frac{k}{K_n})|^2\\ &= \sum_{j=1}^{\ell}\|g_j\|^2\chi_{[0,1]}\\ &= \|g\|^2\chi_{[0,1]},\quad \text{a.e. t (respectively, uniformly).}\end{aligned}$$

This completes the proof of Step 2.

Step 3: Let E be a measurable subset of $[0,1]$, $\epsilon>0$, $K\in\mathbb{N}$ and set

$$F=\{t\in[0,1]|\frac{1}{K}\sum_{k=0}^{K-1}|\chi_E(t-\frac{k}{K})|^2\geq\epsilon\}.$$

Let $F_1=F\cap[0,\frac{1}{K}]$. If $|F|\geq\epsilon$, then $|E|\geq\epsilon^2$.

Proof of Step 3: Let $E_k=E\cap[\frac{k}{K},\frac{k+1}{K}]$ and compute

$$\begin{aligned}\epsilon^2\leq\epsilon|F|=\epsilon K|F_1| &\leq \int_0^{\frac{1}{K}}|\chi_E(t-\frac{k}{K})|^2dt\\ &= \int_0^{\frac{1}{K}}\sum_{k=0}^{K-1}|\chi_E(t+\frac{k}{K})|dt\\ &= \int_0^{\frac{1}{K}}\sum_{k=0}^{K-1}|\chi_{E_k}(t+\frac{k}{K})|dt\\ &= \sum_{k=0}^{K-1}\int_0^{\frac{1}{K}}\chi_{E_k}(t+\frac{k}{K})dt\\ &= \sum_{k=0}^{K-1}\int_{\frac{k}{K}}^{\frac{k+1}{K}}\chi_{E_k}(t)dt\\ &= \sum_{k=0}^{K-1}|\chi_{E_k}|=|E|.\end{aligned}$$

This completes the proof of Step 3.

Step 4: If E is a measurable set which is a countable union of intervals, then the theorem holds.

Proof of Step 4: Fix $\epsilon > 0$ arbitrary and choose $\delta > 0$ to be specified later. Assume that

$$E = \cup_{n=1}^{\infty}(a_n, b_n),$$

and choose a natural number N so that

$$\sum_{n=N+1}^{\infty}(b_n - a_n) < \delta^2.$$

Also let

$$G = \cup_{n=N+1}^{\infty}(a_n, b_n), \text{so that } \ |G| < \delta^2.$$

Let

$$H = \cup_{n=1}^{N}(a_n, b_n),$$

and note that we may as well assume that $\{(a_n, b_n)\}_{n=1}^N$ are disjoint. Finally, let

$$F = \{t \in [0,1] | \frac{1}{K}\sum_{k=0}^{K-1}|\chi_G(t - \frac{k}{K})|^2 \geq \delta\}, \quad \text{so that } \ |F| < \delta.$$

By Step 1,

$$\lim_{K\to\infty}\frac{1}{K}\sum_{k=0}^{K-1}|\chi_G(t - \frac{k}{K})|^2 = |G|\chi_{[0,1]},$$

uniformly. Choose a natural number K_0 so that for all $K \geq K_0$ we have

$$\frac{1}{K}\sum_{k=0}^{K-1}|\chi_G(t - \frac{k}{K})|^2 \leq (|G| + \delta)\chi_{[0,1]}.$$

Now we compute,

$$\begin{aligned}
\sqrt{\frac{1}{K}\sum_{k=0}^{K-1}|\chi_E(t - \frac{k}{K})|^2} &\leq \sqrt{\frac{1}{K}\sum_{k=0}^{K-1}|\chi_G(t - \frac{k}{K}) + \chi_H(t - \frac{k}{K})|^2} \\
&\leq \sqrt{\frac{1}{K}\sum_{k=0}^{K-1}|\chi_G(t - \frac{k}{K})|^2} + \sqrt{\frac{1}{K}\sum_{k=0}^{K-1}|\chi_H(t - \frac{k}{K})|^2} \\
&\leq \sqrt{|H| + \delta}\chi_{[0,1]} + \sqrt{\frac{1}{K}\sum_{k=0}^{K-1}|\chi_G(t - \frac{k}{K})|^2} \\
&\leq \sqrt{|H| + \delta}\chi_{[0,1]} + \delta\chi_{[0,1]},
\end{aligned}$$

off of the set F where $|F| < \delta < \epsilon$ and δ is chosen so that

$$\sqrt{|H| + \delta}\chi_{[0,1]} + \delta\chi_{[0,1]} \leq \sqrt{|H| + \epsilon}\chi_{[0,1]}.$$

Similarly, for $K \geq K_0$ we have

$$\sqrt{\frac{1}{K}\sum_{k=0}^{K-1}|\chi_E(t-\frac{k}{K})|^2} \geq \left(\sqrt{|G|+\delta}-\delta\right)\chi_{[0,1]} \geq \sqrt{|G|+\epsilon}\chi_{[0,1]}.$$

This completes the proof of Step 4.

Step 5: The Theorem holds for χ_E for every measurable set E.

Proof of Step 5: Given a measurable set E in $[0,1]$ and an $\epsilon > 0$, choose intervals $\{(a_n,b_n)\}_{n=1}^{\infty}$ so that

$$E \subset \cup_{n=1}^{\infty}(a_n,b_n) =: F,$$

and

$$||E| - \sum_{n=1}^{\infty}(b_n-a_n)| < \frac{\epsilon}{3}.$$

Then, there is a measurable set G with $|G| < \epsilon/3$ and a natural number K_0 so that for every $K \geq K_0$ and for all $t \notin G$ we have

$$\begin{aligned}\frac{1}{K}\sum_{k=0}^{K-1}|\chi_E(t-\frac{k}{K})|^2 &\leq \frac{1}{K}\sum_{k=0}^{K-1}|\chi_F(t-\frac{k}{K})|^2 \\ &\leq (|F|+\frac{\epsilon}{3})\chi_{[0,1]} \leq (|E|+\frac{2\epsilon}{3})\chi_{[0,1]}.\end{aligned}$$

Similarly, there is a measurable set G_1 with $|G_1| < \epsilon/3$ and a natural number K_1 so that for all $K \geq K_1$ and all $t \notin G_1$ we have

$$\frac{1}{K}\sum_{k=0}^{K-1}|\chi_{E^c}(t-\frac{k}{K})|^2 \leq (|E^c|+\epsilon)\chi_{[0,1]}.$$

We next note that

$$\begin{aligned}\chi_{[0,1]}(t) &= \frac{1}{K}\sum_{k=0}^{K-1}\left(|\chi_E(t-\frac{k}{K})|^2 + |\chi_{E^c}(t-\frac{k}{K-1})|^2\right) \\ &= \frac{1}{K}\sum_{k=0}^{K-1}|\chi_E(t-\frac{k}{K})|^2 + \frac{1}{K}\sum_{k=0}^{K-1}|\chi_{E^c}(t-\frac{k}{K-1})|^2.\end{aligned}$$

Now, for all $t \notin G_1$ we have

$$\frac{1}{K}\sum_{k=0}^{K-1}|\chi_E(t-\frac{k}{K})|^2 \geq 1 - \frac{1}{K}\sum_{k=0}^{K-1}|\chi_{E^c}(t-\frac{k}{K})|^2 \geq 1-(|E^c|+\epsilon) = |E|-\epsilon.$$

Hence, $|G \cup G_1| < \epsilon$ and for all $t \notin G \cup G_1$ we have

$$|\frac{1}{K}\sum_{k=0}^{K-1}|\chi_E(t-\frac{k}{K})|^2 - |E|| < \epsilon.$$

Hence,

$$\{\frac{1}{K}\sum_{k=0}^{K-1}|\chi_E(t-\frac{k}{K})|^2\}$$

converges to $|E|\chi_{[0,1]}$ in measure. This completes the proof of Step 5.
Step 6: The general case for the theorem.

Proof: If $g \in L^2[0,1]$ and $\epsilon > 0$ is given, fix a $\delta > 0$ (to be chosen later) and choose a simple function

$$h = \sum_{j=1}^{M} a_j \chi_{E_j},$$

so that $|g - h| < \delta$ a.e. Then,

$$\begin{aligned}\sqrt{\frac{1}{K}\sum_{k=0}^{K-1}|g(t-\frac{k}{K})|^2} &\le \sqrt{\frac{1}{K}\sum_{k=0}^{K-1}|h(t-\frac{k}{K})|^2} + \sqrt{\frac{1}{K}\sum_{k=0}^{K-1}(\frac{\delta}{2})^2} \\ &\le \sqrt{\frac{1}{K}\sum_{k=0}^{K-1}|h(t-\frac{k}{K})|^2} + \frac{\delta}{2}.\end{aligned}$$

Similarly,

$$\sqrt{\frac{1}{K}\sum_{k=0}^{K-1}|g(t-\frac{k}{K})|^2} \ge \sqrt{\frac{1}{K}\sum_{k=0}^{K-1}|h(t-\frac{k}{K})|^2} - \frac{\delta}{2}.$$

Now, there is a measurable set $|G| < \delta$ and a natural number K_0 so that for all $K \ge K_0$ and all $t \notin G$ we have

$$\begin{aligned}\frac{1}{K}\sum_{k=0}^{K-1}|h(t-\frac{k}{K})|^2 &= \frac{1}{K}\sum_{k=0}^{K-1}\sum_{j=1}^{M}|a_j|^2|\chi_{E_j}(t-\frac{k}{K})|^2 \\ &= \sum_{j=1}^{M}|a_j|^2\frac{1}{K}\sum_{k=0}^{K-1}|\chi_{E_j}(t-\frac{k}{K})|^2 \\ &\le \sum_{j=1}^{M}|a_j|^2\left(|E_j| + \frac{\delta}{2M\sum|a_j|^2}\right) \\ &= \sum_{j=1}^{M}|a_j|^2|E_j| + \frac{\delta}{2} = \|h\|^2 + \frac{\delta}{2}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{K}\sum_{k=0}^{K-1}|g(t-\frac{k}{K})|^2 &\leq (\|h\|^2+\frac{\delta}{2})+\frac{\delta}{2}\\ &\leq \|h\|^2+\delta \leq (\|g\|+\frac{\delta}{2})^2+\delta\\ &\leq \|g\|^2+\frac{\delta}{2}\|g\|+\frac{\delta^2}{4}+\delta\\ &\leq \|g\|^2+\epsilon,\end{aligned}$$

for an appropriately chosen $\delta > 0$ and all $t \notin G$. Similarly,

$$\frac{1}{K}\sum_{k=0}^{K-1}|g(t-\frac{k}{K})|^2 \geq \|g\|^2-\epsilon,$$

for all $t \notin G$. This completes the proof of Step 6 and hence of the theorem. □

Now we proceed to the proof of the main theorems of this section. For the proofs we will need a proposition.

Proposition 6.25. *Fix $g \in L^\infty([0,1])$ and $K \in \mathbb{N}$. For $0 \leq k \leq K-1$ let*

$$g_k(t) = \sum_{n\in\mathbb{Z}} \langle g, \mathrm{e}^{2\pi \mathrm{i}(nK+k)t}\rangle \mathrm{e}^{2\pi \mathrm{i}(nK+k)t}.$$

Then, for every $f \in \mathrm{cl}(\mathrm{span}\{\mathrm{e}^{2\pi \mathrm{i}(nK+k)t}\}_{n\in\mathbb{Z}})$ we have:

$$\|f\cdot g\|^2 = \|f\cdot\left(\sum_{k=0}^{K-1}|g_k|^2\right)^{1/2}\|.$$

Proof: We will do the case $k = 0$; the others require only notational changes. So, we compute:

$$\begin{aligned}
\|f \cdot g\|^2 &= \sum_{n\in\mathbb{Z}} |\langle f \cdot g, \mathrm{e}^{2\pi int}\rangle|^2 \\
&= \sum_{n\in\mathbb{Z}} |\langle f, \mathrm{e}^{2\pi int}\overline{g}\rangle|^2 \\
&= \sum_{k=0}^{K-1}\sum_{n\in\mathbb{Z}} |\langle f, \mathrm{e}^{2\pi \mathrm{i}(nK+k)t}\overline{g}\rangle|^2 \\
&= \sum_{k=0}^{K-1}\sum_{n\in\mathbb{Z}} |\langle e^{-2\pi ikt} f, \mathrm{e}^{2\pi \mathrm{i}nKt}\overline{g}\rangle|^2 \\
&= \sum_{k=0}^{K-1}\sum_{n\in\mathbb{Z}} |\langle e^{-2\pi inkt} f, \mathrm{e}^{2\pi \mathrm{i}nKt}\overline{g_k}\rangle|^2 \\
&= \sum_{k=0}^{K-1}\sum_{n\in\mathbb{Z}} |\langle e^{-2\pi ikt} f, \mathrm{e}^{2\pi \mathrm{i}nt}\overline{g_k}\rangle|^2 \\
&= \sum_{k=0}^{K-1}\sum_{n\in\mathbb{Z}} |\langle e^{-2\pi ikt} f \cdot g_k, \mathrm{e}^{2\pi \mathrm{i}nt}\rangle|^2 \\
&= \sum_{k=0}^{K-1} \|e^{-2\pi ikt} f \cdot g_k\|^2 = \sum_{k=0}^{K-1} \|f \cdot g_k\|^2 \\
&= \sum_{k=0}^{K-1} \int_0^1 |f(t)|^2 |g_k(t)|^2 dt \\
&= \int_0^1 |f(t)|^2 \sum_{k=0}^{K-1} |g_k(t)|^2 dt \\
&= \|f \cdot \left(\sum_{k=0}^{K-1} |g_k|^2\right)^{1/2} \|^2.
\end{aligned}$$

□

Proof of Theorem 6.16: By Corollary 4.5, T_g has the uniform Kadison-Singer Property if and only if for every $\epsilon > 0$ there is a natural number K so that for all $f \in \mathrm{cl}(\mathrm{span}\{\mathrm{e}^{2\pi \mathrm{i}(nK+k)t}\}_{n\in\mathbb{Z}})$ we have:

$$\|f\|^2(\|g\|^2 - \epsilon) \le \|f \cdot (\sum_{k=0}^{K-1} |g_k^K(t)|^2)^{1/2}\|^2$$

$$= \int_0^1 |f(t)|^2 \frac{1}{K} \sum_{k=0}^{K-1} |g(t - \frac{k}{K})|^2 dt \leq \|f\|^2(\|g\|^2 + \epsilon).$$

The proof of (2) ⇒ (1) is immediate from here.

(1) ⇒ (2): If this implication fails, there is an $\epsilon > 0$ so that for all $K \in \mathbb{N}$ there is a measurable set E_0 so that either

$$\frac{1}{K} \sum_{k=0}^{K-1} |g(t - \frac{k}{K})|^2 \geq \|g\|^2 + \epsilon,$$

or

$$\frac{1}{K} \sum_{k=0}^{K-1} |g(t - \frac{k}{K})|^2 \leq \|g\|^2 - \epsilon.$$

We will do the first case since the second is similar. Choose $\|h\| = \frac{1}{\sqrt{K}}$ so that $h = h \cdot \chi_{E_0}$. Let $E = \cup_{k=0}^{K-1}(E_0 + k)$ and let

$$f = \frac{1}{K} \sum_{j=0}^{K-1} h(t - \frac{j}{K}) \mathrm{e}^{\frac{2\pi \mathrm{i} jk}{K}}.$$

Then, $g \in \mathrm{cl}(\mathrm{span}\{\mathrm{e}^{2\pi \mathrm{i}(nK+k)t}\}_{n\in\mathbb{Z}})$ and

$$\begin{aligned} \|f \cdot g\|^2 &= \int_0^1 |f(t)|^2 \frac{1}{K} \sum_{k=0}^{K-1} |g(t - \frac{k}{K})|^2 dt \\ &ge \int_E |f(t)|^2 dt \left(\|g\|^2 + \epsilon\right) \\ &= \|g\|^2 + \epsilon. \end{aligned}$$

Similarly, for the other case we have

$$\|f \cdot g\|^2 = \int_0^1 |f(t)|^2 \frac{1}{K} \sum_{k=0}^{K-1} |g(t - \frac{k}{K})|^2 dt \leq \|g\|^2 - \epsilon.$$

So the Paving Conjecture fails for T_g by Corollary 4.5.

Proof of Theorem 6.17: This is similar to the above. The Toeplitz operator T_g has the uniform Feichtinger property if and only if there exists an $\epsilon > 0$ and a natural number K so that for all $f \in \mathrm{cl}(\mathrm{span}\{\mathrm{e}^{2\pi \mathrm{i}(nK+k)t}\}_{n\in\mathbb{Z}})$ we have

$$\|f \cdot g\|^2 = \int_0^1 |f(t)|^2 \frac{1}{K} \sum_{k=0}^{K} g(t - \frac{k}{K})|^2 dt \geq \epsilon.$$

As in the proof of Theorem 6.16, this holds if and only if there exists an $\epsilon > 0$ and a K so that

$$\frac{1}{K}\sum_{k=0}^{K}|g(t-\frac{k}{K})|^2 \geq \epsilon \;\; a.e.$$

This shows that (1) $\Leftrightarrow$ (2).

(3) $\Rightarrow$ (2): By (3) we have that

$$\frac{1}{K}\sum_{k=0}^{K}|g(t-\frac{k}{K})|^2 \geq \epsilon \;\; a.e.$$

(2) $\Rightarrow$ (3): If

$$\frac{1}{K}\sum_{k=0}^{K}|g(t-\frac{k}{K})|^2 \geq \epsilon \;\; a.e.,$$

then for all $0 \leq k \leq K-1$ let

$$F_k = \{t \in [\frac{k}{K}, \frac{k+1}{K}] \mid |g(t)| \geq \epsilon\}.$$

Now,

$$\cup_{k=0}^{K}(F_k - k) = [0, \frac{1}{K}].$$

Letting $E_0 = F_0$ and

$$E_{k+1} = F_k \setminus \cup_{j=0}^{k-1}(F_j - j),$$

produces the desired sets. This completes the proof of Theorem 6.17. □

7. Kadison-Singer in time-frequency analysis

Although the Fourier transform has been a major tool in analysis for over a century, it has a serious lacking for signal analysis in that it hides in its phases information concerning the moment of emission and duration of a signal. What was needed was a localized time-frequency representation which has this information encoded in it. In 1946 Gabor [45] filled this gap and formulated a fundamental approach to signal decomposition in terms of elementary signals. Gabor's method has become the paradigm for signal analysis in Engineering as well as its mathematical counterpart: Time-Frequency Analysis.

To build our elementary signals, we choose a **window function** $g \in L^2(\mathbb{R})$. For $x, y \in \mathbb{R}$ we define **modulation by x** and **translation by y** of g by:

$$M_x g(t) = \mathrm{e}^{2\pi \mathrm{i} xt} g(t), \;\; T_y g(t) = g(t-y).$$

If $\Lambda \subset \mathbb{R} \times \mathbb{R}$ and $\{E_x T_y g\}_{(x,y)\in\Lambda}$ forms a frame for $L^2(\mathbb{R})$ we call this an (irregular) **Gabor frame**. Standard Gabor frames are the case where Λ is a lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ where $a, b > 0$ and $ab \leq 1$. For an introduction to time-frequency analysis we recommend the excellent book of Grochenig [52].

It was in his work on time-frequency analysis that Feichtinger observed that all the Gabor frames he was working with could be decomposed into a finite union of Riesz basic sequences. This led him to formulate the Feichtinger Conjecture - which we now know is equivalent to KS. There is a significant amount of literature on the Feichtinger Conjecture for Gabor frames as well as wavelet frames and frames of translates [10, 11, 18]. It is known that Gabor frames over rational lattices [25] and Gabor frames whose window function is "localized" satisfy the Feichtinger Conjecture [10, 11, 53]. But the general case has defied solution.

Translates of a single function play a fundamental role in frame theory, time-frequency analysis, sampling theory and more [2, 18]. If $g \in L^2(\mathbb{R})$, $\lambda_n \in \mathbb{R}$ for $n \in \mathbb{Z}$ and $\{T_{\lambda_n} g\}_{n\in\mathbb{Z}}$ is a frame for its closed linear span, we call this a **frame of translates**. Although considerable effort has been invested in the Feichtinger Conjecture for frames of translates, little progress has been made. One exception is a surprising result from [26].

Theorem 7.1. *Let $I \subset \mathbb{Z}$ be bounded below, $a > 0$ and $g \in L^2(\mathbb{R})$. Then $\{T_{na} g\}_{n\in I}$ is a frame sequence if and only if it is a Riesz basic sequence.*

Our next theorem will explain why the Feichtinger Conjecture for frames of translates, wavelet frames and Gabor frames has proven to be so difficult. This is due to the fact that this problem is equivalent to a deep problem in harmonic analysis, namely Conjecture 6.8, which in turn is equivalent to having all Toeplitz operators satisfy the Feichtinger Conjecture (Theorem 6.10).

The proof our of our theorem is complicated, and requires some preliminary work. The main idea is to apply the Fourier transform to turn this into a problem concerning functions of the form $\{e^{2\pi i\lambda_n t}\phi\}_{n\in\Lambda}$ with $\phi \in L^2(\mathbb{R})$. Then we want to use perturbation theory to reduce this problem into one with evenly spaced exponentials $\{e^{2\pi int}\phi\}_{n\in\Lambda}$. There are two technical problems with this. The first is that our functions ϕ are no longer in $L^\infty[0,1]$ which causes technicalities. Second, perturbation theory fails miserably in the frame setting if we perturb a frame by a sequence from outside the space - as we have to do here (see Example 7.2 below). What makes this all eventually work is that perturbation theory *does work from outside the space* for Riesz basic sequences and we are just trying to divide our family of vectors into a finite number of Riesz basic sequences.

Example 7.2. *Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for ℓ_2. For all $i \in \mathbb{N}$, define $g_{2i} = g_{2i+1} = e_{2i}$, $f_{2i+1} = e_{2i}$ and*

$$f_{2i} = e_{2i} + \frac{\epsilon}{2^{i+1}} e_{2i+1}.$$

Then $\{g_i\}_{i=1}^\infty$ *is clearly a 2-tight frame for the span of* $\{e_{2i}\}_{i=1}^\infty$. *Also, for any finitely non-zero sequence of scalars* $\{a_i\}_{i=1}^\infty$ *we have*

$$\| \sum_{i=1}^\infty a_i(g_i - f_i)\|^2 = \| \sum_{i=1}^\infty \frac{\epsilon}{2^{i+1}} a_{i+1}e_{i+1}\|^2 \le \epsilon \sum_{i=1}^\infty |a_i|^2 \le \epsilon \sum_{i=1}^\infty |a_i|^2.$$

So $\{f_i\}_{i=1}^\infty$ *is a small perturbation of* $\{g_i\}_{i=1}^\infty$ *but* $\{f_i\}_{i=1}^\infty$ *is not a frame for its span since for any* $j \in \mathbb{N}$ *we have*

$$\sum_{i=1}^\infty |\langle e_{2j+1}, f_i\rangle|^2 = \frac{\epsilon}{2^{i+1}}.$$

We will state the main theorems here, then develop some theory for solving them and give the proofs at the end.

Theorem 7.3. *The following are equivalent:*

(1) Conjecture 6.10.

(2) For every $0 \ne \phi \in L^2(\mathbb{R})$ *and* $\lambda_n \in \mathbb{R}$ *for* $n \in \Lambda$, *if* $\{T_{\lambda_n}\phi\}_{n\in\Lambda}$ *is a Bessel sequence, then it is a finite union of Riesz basic sequences.*

(3) For every $\Lambda \subset \mathbb{Z}$ *and every* $0 \ne \phi \in L^2(\mathbb{R})$, *if* $\{T_n\phi\}_{n\in\Lambda}$ *is a Bessel sequence, then it is a finite union of Riesz basic sequences.*

(4) For every $0 \ne \phi \in L^2(\mathbb{R})$ *and* $\lambda_n \in \mathbb{R}$ *for* $n \in \Lambda$, *if* $\{T_{\lambda_n}\phi\}_{n\in\Lambda}$ *is a frame sequence, then it is a finite union of Riesz basic sequences.*

(5) For every $\Lambda \subset \mathbb{Z}$ *and every* $0 \ne \phi \in L^2(\mathbb{R})$, *if* $\{T_n\phi\}_{n\in\Lambda}$ *is a Bessel sequence, then it is a finite union of frame sequences.*

Instead of proving Theorem 7.3, we will take the Fourier transform of all this and prove the equivalent formulation given in the next theorem.

Theorem 7.4. *The following are equivalent:*

(1) Conjecture 6.10.

(2) For every $\phi \in L^2(\mathbb{R})$ *and every* $\{\lambda_n\}_{n\in\Lambda}$, *if* $\{e^{2\pi i\lambda_n t}\phi\}_{n\in\Lambda}$ *is Bessel in* $L^2(\mathbb{R})$, *then it is a finite union of Riesz basic sequences.*

(3) For every $0 \ne \phi \in L^2[0,1]$ *and every* $\Lambda \subset \mathbb{Z}$, *if* $\{e^{2\pi i n t}\phi\}_{n\in\Lambda}$ *is a Bessel sequence then it is a finite union of Riesz basic sequences.*

(4) For every $\phi \in L^2(\mathbb{R})$ *and every* $\{\lambda_n\}_{n\in\Lambda}$, *if* $\{e^{2\pi i\lambda_n t}\phi\}_{n\in\Lambda}$ *is a frame sequence in* $L^2(\mathbb{R})$, *then it is a finite union of Riesz basic sequences.*

(5) For every $\Lambda \subset \mathbb{Z}$ *and every* $0 \ne \phi \in L^2[0,1]$, *if* $\{e^{2\pi i n t}\phi\}_{n\in\Lambda}$ *is a Bessel sequence, then it is a finite union of frame sequences.*

The first thing we will do is derive the perturbation theorem we need for proving our results. We start with a theorem due to Christensen [33] which is a generalization of the Paley-Wiener theorem [64] (We state a slightly stronger conclusion at the end which easily follows from the proof of [33]).

Theorem 7.5. *Let $\mathbb{H}$ be a Hilbert space and $\{f_i\}_{i\in I}$ a frame for $\mathbb{H}$ with frame bounds A, B. Let $\{g_i\}_{i\in I}$ be a sequence in $\mathbb{H}$. Assume there exists a $\lambda, \mu > 0$ with $\lambda + \frac{\mu}{\sqrt{A}} < 1$ and an increasing sequence of subsets $I_1 \subset I_2 \subset \cdots \subset I$ with $\cup_{n=1}^{\infty} I_n = I$ so that for all $n = 1, 2, \cdots$ and all families of scalars $\{a_i\}_{i\in I_n}$ we have*

$$\|\sum_{i\in I_n} a_i(f_i - g_i)\| \le \lambda \|\sum_{i\in I_n} a_i f_i\| + \mu \left(\sum_{i\in I_n} |a_i|^2\right)^{1/2}.$$

Then $\{g_i\}_{i\in I}$ is a frame for $\mathbb{H}$ with frame bounds

$$A(1-\lambda-\frac{\mu}{\sqrt{A}})^2, \qquad B(1+\lambda+\frac{\mu}{\sqrt{B}})^2.$$

Moreover, if $\{f_i\}_{i\in I}$ is a Riesz basic sequence, then $\{g_i\}_{i\in I}$ is also a Riesz basic sequence.

We will need a variation of a result proved independently by Balan [6] and Christensen [34]. Since this is a straightforward generalization where we just insert a function into the calculations of Balan [6], we will outline the proof.

Theorem 7.6. *Let $\phi \in L^2[-\gamma, \gamma]$ and assume $\{e^{i\lambda_n t}\phi\}_{n\in\mathbb{Z}}$ is a Bessel sequence with Bessel bound B. Set*

$$L(\gamma) = \frac{\pi}{4\gamma} - \frac{1}{\gamma} \arcsin\left(\frac{1}{\sqrt{2}}\left(1 - \sqrt{\frac{A}{B}}\right)\right).$$

Suppose $\mu_n \in \mathbb{R}$ and $\sup_n |\mu_n - \lambda_n| = \delta < 1/4$. Then $\{e^{i\mu_n t}\phi\}_{n\in I}$ is a Bessel sequence with Bessel bound $B(2 - \cos\ \gamma\delta + \sin\ \gamma\delta)^2$. Moreover, if $\{e^{i\lambda_n t}\phi\}_{n\in I}$ is a frame sequence with frame bounds A, B and $\delta < L(\gamma)$ then $\{e^{i\mu_n t}\phi\}_{n\in I}$ is a frame sequence with frame bounds

$$A\left(1 - \sqrt{\frac{A}{B}}(1 - \cos\ \gamma\delta + \sin\ \gamma\delta)\right)^2, \quad B(2 - \cos\ \gamma\delta + \sin\ \gamma\delta)^2.$$

Proof: By a change of scale ($\hat{\lambda}_n = \frac{\gamma}{\pi}\lambda_n$) we may assume $\gamma = \pi$ and we need to show:

$$L(\pi) = \frac{1}{4} - \frac{1}{\pi} \arcsin\left[\frac{1}{\sqrt{2}}\left(1 - \sqrt{\frac{A}{B}}\right)\right].$$

To prove this result, we rely on Kadec's classical estimations for computing the Paley-Wiener constant [62]. Let $\{a_n\}_{n\in I}$ be scalars, $I_N \subset I$ with $|I_N| < \infty$ and let $\delta_n = \mu_n - \lambda_n$. We compute:
(7.1)

$$U =: \|\sum_{n\in I_N} a_n \left(\frac{1}{\sqrt{2\pi}} e^{i\lambda_n t} - \frac{1}{\sqrt{2\pi}} e^{i\mu_n t}\right)\phi(t)\| = \frac{1}{\sqrt{2\pi}} \|\sum_{n\in I_n} a_n e^{i\lambda_n t}(1 - e^{i\delta_n t})\phi(t)\|.$$

By expanding $1-e^{i\delta_n t}$ into a Fourier series relative to the orthogonal system $\{1, \cos\ \nu t, \sin(\nu-\frac{1}{2})t\}$, $\nu=1,2,\cdots$ we have

$$1-e^{i\delta_n t} = \left(1-\frac{\sin\ \pi\delta_n}{\pi\delta_n}\right)+\sum_{\nu=1}^{\infty}\frac{(-1)^\nu 2\delta_n \sin\ \pi\delta_n}{\pi(\nu^2-\delta_n^2)}\cos\ (\nu t) \tag{7.2}$$

$$+\ i\sum_{\nu=1}^{\infty}\frac{(-1)^\nu 2\delta_n \cos\ \pi\delta_n}{\pi((\nu-\frac{1}{2})^2-\delta_n^2)}\sin\left(\left(\nu-\frac{1}{2}\right)t\right). \tag{7.3}$$

We next insert (7.2) into (7.1), change the order of summation, apply the triangle inequality and use the bounds $\|(\cos\ \nu t)f(t)\| \le \|f\|$ and $\|(\sin(\nu-\frac{1}{2})t)f(t)\| \le \|f\|$ to arrive at

$$\begin{aligned} U \ \le\ & \Big\|\sum_{n\in I_N}\left(1-\frac{\sin\ \pi\delta_n}{\pi\delta_n}\right)a_n e^{i\lambda_n t}\phi(t)\Big\| + \sum_{\nu=1}^{\infty}\Big\|\sum_{n\in I_N}\frac{2\delta_n\ \sin\ \pi\delta_n}{\pi(\nu^2-\delta_n^2)}a_n e^{i\lambda_n t}\phi(t)\Big\| \\ +\ & \sum_{\nu=1}^{\infty}\Big\|\sum_{n\in I_N}\frac{2\delta_n\ \cos\ \pi\delta_n}{\pi((\nu-\frac{1}{2})^2-\delta_n^2)}a_n e^{i\lambda_n t}\phi(t)\Big\|. \end{aligned}$$

Now we use the fact that $\{e^{i\lambda_n t}\phi(t)\}_{n\in I}$ is a B-Bessel sequence. Therefore, each norm above can be bounded as:

$$\Big\|\sum_{n\in I_N} c_n a_n e^{i\lambda_n t}\phi(t)\Big\| \le \sqrt{B}\|\{c_n a_n\}_{n\in I_N}\| \le \sqrt{B}\sum_{n\in I_N}|c_n|\|\{a_n\}_{n\in I_N}\|.$$

Also,

$$\left|1-\frac{\sin\ \pi\delta_n}{\pi\delta_n}\right| \le 1-\frac{\sin\ \pi\delta}{\pi\delta},$$

$$\left|\frac{2\delta_n \sin\ \pi\delta_n}{\pi(\nu^2-\delta_n^2)}\right| \le \frac{2\delta \sin\ \pi\delta}{\pi(\nu^2-\delta^2)},$$

$$\left|\frac{2\delta_n \cos\ \pi\delta_n}{\pi((\nu-\frac{1}{2})^2-\delta_n^2)}\right| \le \frac{2\delta \cos\ \pi\delta}{\pi((\nu-\frac{1}{2})^2-\delta^2)},$$

where the last inequality holds because $\delta<\frac{1}{4}$. Thus,

$$U \le \sqrt{B}(Re(1-e^{i\pi\delta})-Im(1-e^{i\pi\delta}))\left(\sum_{n\in I_N}|a_n|^2\right)^{1/2}.$$

That is,

$$U \le \sqrt{B}(1-\cos\ \pi\delta+\sin\ \pi\delta)\left(\sum_{n\in I_n}|a_n|^2\right)^{1/2}.$$

Now we apply Theorem 7.5 with $\lambda = 0$ and $\mu = \sqrt{B}(1-\cos\ \pi\delta + \sin\ \pi\delta)$. The condition of that theorem becomes $\mu < \sqrt{A}$ or $1 - \cos\ \pi\delta + \sin\ \pi\delta < \sqrt{\frac{A}{B}}$. Standard trigonometry yields

$$\delta < L = \frac{1}{4} - \frac{1}{\pi} \arcsin\ \left(\frac{1}{\sqrt{2}} \left(1 - \sqrt{\frac{A}{B}} \right) \right).$$

The frame bounds come from $A(1 - \frac{\mu}{\sqrt{A}})^2$ and $B(1 + \frac{\mu}{\sqrt{B}})^2$. This completes the proof. □

We also need a simple observation.

Lemma 7.7. *Suppose $\{\phi_n\}_{n\in\Lambda}$ is a Riesz basic sequence in $L^2(I)$, $I \subset \mathbb{R}$ with Riesz basis bounds A, B. If $|\phi| = 1$ a.e. then $\{\phi_n \phi\}_{n\in\Lambda}$ is a Riesz basic sequence with Riesz basis bounds A, B.*

Proof: For any sequence of scalars $\{a_n\}_{n\in\Lambda}$ we have

$$\begin{aligned} \|\sum_{n\in\Lambda} a_n \phi_n \phi\|^2 &= \int_I |\sum_{n\in\Lambda} a_n \phi_n(t)\phi(t)|^2 dt \\ &= \int_I |\sum_{n\in\Lambda} a_n \phi_n(t)|^2 dt \\ &= \|\sum_{n\in\Lambda} a_n \phi_n\|^2. \end{aligned}$$

□

Corollary 7.8. *Suppose $\phi \in L^2(\mathbb{R})$, $\{e^{2\pi i n t}\phi\}_{n\in\Lambda}$ is a Riesz basic sequence in $L^2[0,1]$ and $\lambda_n \in \mathbb{R}$ with $|n - \lambda_n| < 1$ for every $n \in \Lambda$. Then $\{e^{2\pi i \lambda_n t}\phi\}_{n\in\Lambda}$ is a finite union of Riesz basic sequences.*

Proof: Suppose $\{e^{2\pi i n t}\phi\}_{n\in\Lambda}$ has Riesz basis bounds A, B. Choose N so that $1/(2\pi N) < 1/4$ and

$$\sqrt{\frac{B}{A}}(1 - \cos\ \frac{1}{N} + \sin\ \frac{1}{N}) < 1.$$

For $j = 0, 1, \cdots, N-1$ let

$$\Lambda_j = \{n \in \Lambda | n + \frac{j}{N} \le \lambda_n \le n + \frac{j+1}{N}\},$$

and

$$\mu_n = n + \frac{j}{N}, \quad \text{for } n \in \Lambda_n.$$

Then

$$\{e^{2\pi i \mu_n t}\phi\}_{n\in\Lambda_j} = \{e^{2\pi i n t}\phi e^{2\pi i \frac{j}{N} t}\}_{n\in\Lambda_j},$$

is a Riesz basic sequence with Riesz basis bounds A, B, by Lemma 7.7. Since $|\mu_n - \lambda_n| < \frac{1}{N}$ by Theorem 7.6 (rescaled to this setting) we have

$$\|\sum a_k(\mathrm{e}^{2\pi \mathrm{i}\mu_n t} - \mathrm{e}^{2\pi \mathrm{i}\lambda_n t})\phi\| \leq \sqrt{B}(1 - \cos\ \frac{1}{N} + \sin\ \frac{1}{N}),$$

and

$$\sqrt{\frac{B}{A}}(1 - \cos\ \frac{1}{N} + \sin\ \frac{1}{N}) < 1.$$

So by Corollary 7.8,

$$\{\mathrm{e}^{2\pi \mathrm{i}\lambda_n t}\phi\}_{n\in\Lambda_j}$$

is a Riesz basic sequence for $j = 0, 1, \cdots, N-1$. □

We will need a little more notation. If $\phi \in L^2(\mathbb{R})$ we define $\Phi_b : [0,1] \to \mathbb{R}$ by

$$\Phi_b(t) = \sum_{n\in\mathbb{Z}} \left| \hat{\phi}\left(\frac{t+n}{b}\right)\right|^2.$$

If $\Lambda \subset \mathbb{Z}$ we let $S(\Lambda)$ be the closed subspace of $L^2([0,1])$ generated by the characters $\mathrm{e}^{2\pi int}$ for $n \in \Lambda$. We let E_Λ be the closed subspace of $S(\Lambda)$ of all f such that $\Phi_b(t)f(t) = 0$ a.e. If $f \in S(\Lambda)$ we denote by $d(f, E_\Lambda)$ the distance of f from the subspace E_Λ. We denote T_x the translation operator on $L^2(\mathbb{R})$ by x. Now we can state the result from [26].

Theorem 7.9. *Suppose $\phi \in L^2(\mathbb{R})$ and $b > 0$. If $\Lambda \subset \mathbb{Z}$ then $\{T_{nb}\phi\}_{n\in\Lambda}$ is a frame sequence with frame bounds A, B if and only if for every $f \in S(\Lambda)$ we have*

$$Ad(f, E_\Lambda)^2 \leq \frac{1}{b}\int_0^1 |f(t)|^2 \Phi_b(t)dt \leq B\|f\|^2,$$

or equivalently, for all $f \in S(\Lambda) \cap E_\Lambda^\perp$,

$$A\|f\|^2 \leq \frac{1}{b}\int_0^1 |f(t)|^2 \Phi_b(t)dt \leq B\|f\|^2.$$

Furthermore, if this condition is satisfied, $\{T_{nb}\phi\}_{n\in\Lambda}$ is a Riesz basic sequence with the same frame bounds if and only if $E_\Lambda = \{0\}$.

Now we are ready to prove our main theorem.

Proof of Theorem 7.4:

(3) ⇒ (2): We first note the existence of a natural number N so that any interval of length one in $\mathbb{R}$ contains at most N of the λ_n's. If not, then for every natural number N there is a set $I \subset \Lambda$ with $|I| = N$ and

$$|\langle \mathrm{e}^{2\pi \mathrm{i}\lambda_n t}\phi, \mathrm{e}^{2\pi \mathrm{i}\mu_n t}\phi\rangle|^2 \geq \frac{1}{2}\|\phi\|^2,$$

for all $\lambda_n, \mu_n \in I$. Now, for $\lambda_n \in I$ fixed we have

$$\sum_{\mu_n \in I} |\langle \mathrm{e}^{2\pi \mathrm{i}\lambda_n t}\phi, \mathrm{e}^{2\pi \mathrm{i}\mu_n t}\phi\rangle|^2 \geq \frac{N}{2}\|\phi\|^2.$$

So $\{\mathrm{e}^{2\pi \mathrm{i}\lambda_n t}\}_{n\in\Lambda}$ is not Bessel which contradicts our assumption. It follows that we can write Λ as a finite union of sets so that $|\lambda_n - \lambda_m| \geq 1$ for all $n \neq m$. So we may just assume that Λ has this property. By reindexing, we may assume there is some $\Lambda \subset \mathbb{Z}$ so that for $n \in \Lambda$ we have $|\lambda_n - n| < 1$. Let $\{\Lambda_j\}_{j=0}^4$ be:

$$\Lambda_i = \{n \in \Lambda | n + \frac{i}{5} \leq \lambda_n < n + \frac{i+1}{5}\}.$$

Fix $0 \leq j \leq 4$. Since $\{\mathrm{e}^{2\pi \mathrm{i}\lambda_n t}\phi\}_{n\in\Lambda_j}$ is Bessel with Bessel bound B, by Theorem 7.6 we have that $\{\mathrm{e}^{2\pi int}\phi\}_{n\in\Lambda_j}$ is Bessel with Bessel bound

$$B_1 = B[2 - \cos\ \frac{\pi}{5} + \sin\frac{\pi}{5}].$$

By our assumption (3), we can partition Λ_j into a finite number of sets $\{\Lambda_{jk}\}_{k=1}^{M_j}$ so that for every $k = 1, 2, \cdots, M_j$, the family $\{\mathrm{e}^{2\pi int}\phi\}_{n\in\Lambda_{jk}}$ is a Riesz basic sequence with some lower Riesz basis bound $A > 0$. By Corollary 7.8, we have that $\{\mathrm{e}^{2\pi \mathrm{i}\lambda_n t}\phi\}_{n\in\Lambda_{jk}}$ is a finite union of Riesz basic sequences.

(2) $\Rightarrow$ (4): This is obvious.

(4) $\Rightarrow$ (1): Since $\{\mathrm{e}^{2\pi int}\chi_E\}_{n\in\mathbb{Z}}$ is a Parseval frame, by (4) there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that $\{\mathrm{e}^{2\pi int}\phi\}_{n\in A_j}$ is a Riesz basic sequence (with lower Riesz basis bound $A > 0$) for all $j = 1, 2, \cdots, M$. Hence, for any $f = \sum_{n\in A_j} a_n \mathrm{e}^{2\pi int}$, we have that $\|f\|^2 = \sum_{n\in A_j} |a_n|^2$ and

$$\|P_E f\|^2 = \|\sum_{n\in A_j} a_n \mathrm{e}^{2\pi int}\chi_E\|^2 \geq A^2 \sum_{n\in A_j} |a_n|^2 = A^2\|f\|^2.$$

That is, P_E is an isomorphism onto its range.

(1) $\Rightarrow$ (3): Suppose $\{\mathrm{e}^{2\pi int}\phi\}_{n\in\Lambda}$ is Bessel in $L^2(\mathbb{R})$. So there exists a $B > 0$ so that for all $f \in H_\Lambda$ we have

$$\int_0^1 |f(t)|^2 \Phi(t) dt \leq B\|f\|^2.$$

Since $\phi \neq 0$, $\Phi \neq 0$. So there is a measurable set $E \subset [0, 1]$ with $0 < |E|$ and an $\epsilon > 0$ so that $|\Phi(t)| \geq \epsilon$ for all $t \in E$. By the above, $\{\mathrm{e}^{2\pi int}\Phi\}_{n\in\Lambda}$ is a bounded Bessel sequence in $L^2[0, 1]$. By (1), there is a partition of $\mathbb{Z}\cap\Lambda$ of the form $\{A_j\}_{j=1}^M$ so that P_E is an isomorphism on $S(A_j)$, for every $j = 1, 2, \cdots, M$

with lower isomorphism bound $A > 0$. Hence, for every $\{a_n\}_{n\in A_j}$ we have

$$\| \sum_{n\in A_j} a_n e^{2\pi int}\Phi\|_{L^2[0,1]} \geq \| \sum_{n\in A_j} a_n e^{2\pi int}\chi_E\|_{L^2[0,1]} \geq A \left(\sum_{n\in A_j} |a_n|^2 \right)^{1/2} .$$

So $\{e^{2\pi int}\Phi\}_{n\in A_j}$ is a Riesz basic sequence for all $j = 1, 2, \cdots, M$.

(5) ⇒ (3): Assume $\{e^{2\pi int}\phi\}_{n\in\Lambda}$ is Bessel. Let

$$\Lambda^+ = \{n \in \Lambda | 0 \leq n\}, \quad \Lambda^- = \{n \in \Lambda | n < 0\}.$$

Now, $\{e^{2\pi int}\phi\}_{n\in\Lambda^+}$ is Bessel with the same Bessel bound. So by (5), we can write it as a finite union of frame sequences. By Proposition 6.11 and by Theorem 7.9, these frame sequences are all Riesz basic sequences.

(3) ⇒ (5): This is obvious.

This completes the proof of the theorem. □

We end this section with a result of Bownik and Speegle [18] which makes a connection between number theory and PC for Toeplitz operators. This is related to a possible generalization of van der Waerden's theorem [67, 68].

Definition 7.10. *Let $g : \mathbb{R}^2 \to \mathbb{R}$. We say that $I \subset \mathbb{Z}$ satisfies the $g(\ell, N)$-arithmetic progression condition if for every $\delta > 0$ there exists $M, N, \ell \in \mathbb{Z}$ such that*

(1) $g(\ell, N) < \delta$, and

(2) $\{M, M + \ell, \cdots, M + N\ell\} \subset I$.

Taking the Fourier transform through theorem 4.1.2 in [18] yields:

Theorem 7.11. *A positive solution to the Feichtinger Conjecture for Toeplitz operators implies there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that each A_j fails the $g(\ell, N) = \ell N^{-1/2} log^3 N$ arithmetic progression condition.*

In [18], it is observed then if we randomly assign each integer to one of r subsets $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$, then with probability one, for each j and r there will exist an M_j such that

$$\{M_j, M_j + 1, \cdots, M_j + L\} \subset A_j.$$

Now, if $\mathbb{Z}$ is partitioned as $\{A_j\}_{j=1}^r$, the probability that $\{T_n\phi\}_{n\in A_j}$ (and hence $\{e^{2\pi int}\hat{\phi}\}_{n\in A_j}$) is a Riesz basic sequence is zero.

8. Kadison-Singer in Engineering

Frames have traditionally been used in signal processing because of their resilience to additive noise, resilience to quantization, numerical stability of reconstruction and the fact that they give greater freedom to capture important signal characteristics [43, 49]. Recently, Goyal, Kovačević and Vetterli [49] (see also [46, 47, 50, 51]) proposed using the redundancy of frames to mitigate the losses in packet based transmission systems such as the internet. These systems transport packets of data from a "source" to a "recipient". These packets are sequences of information bits of a certain length surrounded by error-control, addressing and timing information that assure that the packet is delivered without errors. It accomplishes this by not delivering the packet if it contains errors. Failures here are due primarily to buffer overflows at intermediate nodes in the network. So to most users, the behavior of a packet network is not characterized by random loss but rather by *unpredictable transport time*. This is due to a protocol, invisible to the user, that retransmits lost or damaged packets. Retransmission of packets takes much longer than the original transmission and in many applications retransmission of lost packets is not feasible. If a lost packet is independent of the other transmitted data, then the information is truly lost. But if there are dependencies between transmitted packets, one could have partial or complete recovery despite losses. This leads us to consider using frames for encoding. But which frames? In this setting, when frame coefficients are lost we call them **erasures**. It was shown in [48] that an equal norm frame minimizes mean-squared error in reconstruction with erasures if and only if it is tight. So a fundamental question is to identify the optimal classes of equal norm Parseval frames for doing reconstruction with erasures. Since the lower frame bound of a family of vectors determines the computational complexity of reconstruction, it is this constant we need to control. Formally, this is a max/min problem which looks like:

Problem 8.1. *Given natural numbers k, K find the class of equal norm Parseval frames $\{f_i\}_{i=1}^{Kn}$ in ℓ_2^n which maximize the minimum below:*

$$\textit{min}\ \{A_J : J \subset \{1, 2, \cdots n\},\ |J| = k,\ A_J \text{ the lower frame bound of } \{f_i\}_{i \in J^c}\}.$$

This problem has proved to be very difficult. We only have a complete solution to the problem for two erasures [14, 27, 58]. Recently, Bodemann and Paulsen [14] have given sharp error bounds for an arbitrary number of erasures and, more importantly, have characterized when we have equality in these bounds. In some settings, this proves that equal norm tight frames are optimal. Vershynin [80] shows that for any n-dimensional frame, any source can be linearly reconstructed from only n log n randomly chosen frame coefficients, with a small error and with high probability. Thus every frame expansion withstands random erasures better (for worst case sources) than the orthogonal basis expansion, for which the n log n bound is attained. It was hoped that some special cases of the problem would be more tractable and serve as a starting point for the classification since the frames we are looking for are contained in this class.

Conjecture 8.2. *There exists an $\epsilon > 0$ so that for large K, for all n and all equal norm Parseval frames $\{f_i\}_{i=1}^{Kn}$ for ℓ_2^n, there is a $J \subset \{1, 2, \cdots, Kn\}$ so* *that both* $\{f_i\}_{i\in J}$ *and* $\{f_i\}_{i\in J^c}$ *have lower frame bounds which are greater than* ϵ.

The ideal situation would be for Conjecture 8.2 to hold for all $K \geq 2$. In order for $\{f_i\}_{i\in J}$ and $\{f_i\}_{i\in J^c}$ to both be frames for ℓ_2^n, they at least have to span ℓ_2^n. So the first question is whether we can partition our frame into spanning sets. This will follow from the Rado-Horn theorem [59, 71]. For a generalization of the theorem see [29].

Theorem 8.3 (Rado-Horn). *Let* I *be a finite or countable index set and let* $\{f_i\}_{i\in I}$ *be a collection of vectors in a vector space. There is a partition* $\{A_j\}_{j=1}^r$ *such that for each* $j = 1, 2, \cdots, r$, $\{f_i\}_{i\in A_j}$ *is linearly independent if and only if for all finite* $J \subset I$

$$\frac{|J|}{\dim \operatorname{span}\{f_i\}_{i\in J}} \leq r. \tag{8.1}$$

The terminology "Rado-Horn Theorem" was introduced, to our knowledge, in the paper [15]. This theorem has had several interesting applications in analysis, for one, a characterization of Sidon sets in $\Pi_{k=1}^{\infty}\mathbb{Z}_p$ due to Bourgain and Pisier [15, 70]. Another application is a proof that the Feichtinger Conjecture is equivalent Conjecture 3.7 [29]. In [31] it was shown that the Rado-Horn Theorem will decompose our frames for us.

Proposition 8.4. *Every equal norm Parseval frame* $\{f_i\}_{i=1}^{Kn}$ *for* ℓ_2^n *can be partitioned into* K *linearly independent spanning sets.*

Proof: If $J \subset \{1, 2, \cdots, Kn\}$, let P_J be the orthogonal projection of ℓ_2^n onto $\operatorname{span}\{f_i\}_{i\in J}$. Since $\{f_i\}_{i=1}^{Kn}$ is an equal norm Parseval frame (see Section 3) $\sum_{i=1}^{Kn}\|f_i\|^2 = Kn\|f_1\|^2 = n$. Now,

$$dim(\operatorname{span}\{f_i\}_{i\in J}) = \sum_{i=1}^{Kn}\|P_J f_i\|^2 \geq \sum_{i\in J}\|P_J f_i\|^2 = \sum_{i\in J}\|f_i\|^2 = \frac{|J|}{K}.$$

So the Rado-Horn conditions hold with constant $r = K$. If we divide our family of Kn vectors into K linearly independent sets, since each set cannot contain more than n-elements, it follows that each has exactly n-elements. □

If we are going to be able to erase arbitrary k-element subsets of our frame, then the frame must be a union of erasure sets. So a generalization of Conjecture 8.2 which is a class containing the class given in Problem 8.1 is

Conjecture 8.5. *There exists* $\epsilon > 0$ *and a natural number* r *so that for all large* K *and all equal norm Parseval frames* $\{f_i\}_{i=1}^{Kn}$ *in* ℓ_2^n *there is a partition* $\{A_j\}_{j=1}^r$ *of* $\{1, 2, \cdots, Kn\}$ *so that for all* $j = 1, 2, \cdots, r$ *the Bessel bound of* $\{f_i\}_{i\in A_j}$ *is* $\leq 1 - \epsilon$.

Little progress has been made on this list of problems. But before we discuss why, let us turn to another setting where these problems arise. For many years engineers have believed that it should be possible to do signal reconstruction without phase. Recently, Balan, Casazza and Edidin [7] verified this longstanding conjecture of the signal processing community by constructing new classes of equal norm Parseval frames. This problem comes from a fundamental problem in speech recognition technology called the "cocktail party problem".

Cocktail Party Problem. *We have a tape recording of a group of people talking at a cocktail party. Can we recover each individual voice with all of its voice characteristics?*

As we will see, the main problem here is "signal reconstruction with noisy phase". One standard technique for removing noise from a *signal* $f \in L^2(\mathbb{R})$ is to digitalize f by sending it through the **fast Fourier transform** [7]. This proceedure just computes the frame coefficients of f with respect to a Gabor frame (see Section 7), say $\{\langle f, f_i\rangle\}_{i\in I}$. Next, we take the absolute values of the frame coefficients to be **processed** and **store** the phases

$$X_i(f) = \frac{\langle f, f_i\rangle}{|\langle f, f_i\rangle|}.$$

There are countless methods for processing a signal. One of the simplest is **thresholding**. This is a process of deleting any frame coefficients whose moduli fall outside of a "threshold interval," say $[A, B]$, where $0 < A < B$. The idea is that if our frame is chosen carefully enough then the deleted coefficients will represent the "noise" in the signal. Now it is time to reconstruct a clear signal. This is done by passing our signal back through the inverse fast Fourier transform (that is, we are inverting the frame operator). But to do this we need phases for our coefficients. So we take our stored $X_i(f)$ and put them back on the processed frame coefficients which are at this time all non-negative real numbers. This is where the problem arises. If the noise in the signal is actually in the phases (which occurs in speech recognition), then we just put the noise back into the signal. The way to avoid this is to construct frames for which reconstruction can be done directly from the absolute value of the frame coefficients and not needing the phases. This was done in [7].

Theorem 8.6. *For a generic real frame on ℓ_2^n with at least $(2n-1)$-elements, the mapping $\pm f \to \{|\langle f, f_i\rangle|\}_{i\in I}$ is one-to-one.*

For a generic complex frame on ℓ_2^n with at least $(4n-2)$-elements, the mapping $cf \to \{|\langle f, f_i\rangle|\}_{i\in I}$, $|c| = 1$, is one-to-one.

"Generic" here means that the set of frames with this property is dense in the class of all frames in the Zariski topology on the Grassman manifold [7].

In the process of looking for algorithms for doing reconstruction directly from the absolute value of the frame coefficients, it was discovered in the real case (the complex case is much more complicated) that the standard algorithms failed when the vector was getting approximately half of its norm from the positive frame coefficients and half from the negative coefficients [8]. The algorithms behave as if one of these sets has been "erased". The necessary conditions for reconstruction without phase in [7] help explain why. These conditions imply that every vector in the space must be reconstructable from either the positive frame coefficients or the negative ones. It is also shown in [8] that signal reconstruction without phase is equivalent to a (P_0) problem with additional constraints (See equation 8.2 below). So once again we have bumped into Problem 8.1 and Conjectures 8.2 and 8.5.

The next theorem (from [31]) helps to explain why all of these reconstruction problems have proved to be so difficult. Namely, because KS has come into play again.

Theorem 8.7. *(1) Conjecture 8.2 implies Conjecture 8.5.*
(2) Conjecture 8.5 is equivalent to KS.

Proof: (1): Fix $\epsilon > 0$, r, K as in Conjecture 8.2. Let $\{f_i\}_{i=1}^{Kn}$ be an equal norm Parseval frame for an n-dimensional Hilbert space $\mathbb{H}_n$. By Theorem 3.4 there is an orthogonal projection P on ℓ_2^{Kn} with $Pe_i = f_i$ for all $i = 1, 2, \cdots, Kn$. By Conjecture 8.2, there is a $J \subset \{1, 2, \cdots, Kn\}$ so that $\{Pe_i\}_{i \in J}$ and $\{Pe_i\}_{i \in J^c}$ both have a lower frame bound of $\epsilon > 0$. Hence, for $f \in \mathbb{H}_n = P(\ell_2^{Kn})$,

$$\begin{aligned} \|f\|^2 &= \sum_{i=1}^{n} |\langle f, Pe_i \rangle|^2 = \sum_{i \in J} |\langle f, Pe_i \rangle|^2 + \sum_{i \in J^c} |\langle f, Pe_i \rangle|^2 \\ &\geq \sum_{i \in J} |\langle f, Pe_i \rangle|^2 + \epsilon \|f\|^2. \end{aligned}$$

That is, $\sum_{i \in J} |\langle f, Pe_i \rangle|^2 \leq (1-\epsilon)\|f\|^2$. So the upper frame bound of $\{Pe_i\}_{i \in J}$ (which is the norm of the analysis operator $(PQ_J)^*$ for this frame) is $\leq 1 - \epsilon$. Since PQ_J is the synthesis operator for this frame, we have that $\|Q_J P Q_J\| = \|PQ_J\|^2 = \|(PQ_J)^*\|^2 \leq 1 - \epsilon$. Similarly, $\|Q_{J^c} P Q_{J^c}\| \leq 1 - \epsilon$. So Conjecture 8.5 holds for $r = 2$.

(2): We will show that Conjecture 8.5 implies Conjecture 3.6. Choose an integer K and an $r, \epsilon > 0$ with $\frac{1}{\sqrt{K}} < \epsilon$. Let $\{f_i\}_{i=1}^{M}$ be a unit norm K-tight frame for an n-dimensional Hilbert space $\mathbb{H}_n$. Then (see Section 3) $M = \sum_{i=1}^{M} \|f_i\|^2 = Kn$. Since $\{\frac{1}{\sqrt{K}} f_i\}_{i=1}^{M}$ is an equal norm Parseval frame, by Theorem 3.4, there is an orthogonal projection P on ℓ_2^M with $Pe_i = \frac{1}{\sqrt{K}} f_i$, for $i = 1, 2, \cdots, M$. By Conjecture 8.5, we have universal $r, \epsilon > 0$ and a partition

$\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, M\}$ so that the Bessel bound $\|(PQ_{A_j})^*\|^2$ for each family $\{f_i\}_{i\in A_j}$ is $\leq 1-\epsilon$. So for $j = 1, 2, \cdots, r$ and any $f \in \ell_2^n$ we have

$$\begin{aligned}\sum_{i\in A_j} |\langle f, \frac{1}{\sqrt{K}} f_i\rangle|^2 &= \sum_{i\in A_j} |\langle f, PQ_{A_j} e_i\rangle|^2 = \sum_{i\in A_j} |\langle Q_{A_j} Pf, e_i\rangle|^2 \leq \|Q_{A_j} Pf\|^2 \\ &\leq \|Q_{A_j} P\|^2 \|f\|^2 = \|(PQ_{A_j})^*\|^2 \|f\|^2 \leq (1-\epsilon)\|f\|^2.\end{aligned}$$

Hence,

$$\sum_{i\in A_j} |\langle f, f_i\rangle|^2 \leq K(1-\epsilon)\|f\|^2 = (K - K\epsilon)\|f\|^2.$$

Since $K\epsilon > \sqrt{K}$, we have verified Conjecture 3.6.

For the converse, choose r, δ, ϵ satisfying Conjecture 2.6. If $\{f_i\}_{i=1}^{Kn}$ is an equal norm Parseval frame for an n-dimensional Hilbert space $\mathbb{H}_n$ with $\frac{1}{K} \leq \delta$, by Theorem 3.4 we have an orthogonal projection P on ℓ_2^{Kn} with $Pe_i = f_i$ for $i = 1, 2, \cdots, Kn$. Since $\delta(P) = \|f_i\|^2 \leq \frac{1}{K} \leq \delta$ (see the proof of Proposition 8.4), by Conjecture 2.6 there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, Kn\}$ so that for all $j = 1, 2, \cdots, r$,

$$\|Q_{A_j} PQ_{A_j}\| = \|PQ_{A_j}\|^2 = \|(PQ_{A_j})^*\|^2 \leq 1-\epsilon.$$

Since $\|(PQ_{A_j})^*\|^2$ is the Bessel bound for $\{Pe_i\}_{i\in A_j} = \{f_i\}_{i\in A_j}$, we have that Conjecture 8.5 holds. □

Theorem 8.7 yields yet another equivalent form of KS. That is, KS is equivalent to finding a quantitative version of the Rado-Horn Theorem.

We end this section with a class of Conjectures which were thought to be equivalent to KS. But, we will show that these conjectures are just weak enough to have a positive solution. There is currently a flury of activity surrounding sparse solutions to vastly underdetermined systems of linear equations. This has applications to problems in signal processing (recovering signals from highly incomplete measurements), coding theory (recovering an input vector from corrupted measurements) and much more. If A is an $n \times m$ matrix with $n < m$, the sparsest solution to $Af = g$ is

$$(P_0) \quad \min_{f\in\mathbb{R}^m} \|f\|_{\ell_0} \quad \text{subject to } Af = g, \tag{8.2}$$

where $\|f\|_{\ell_0} = |\{i : f(i) \neq 0\}|$. The problem with (P_0) is that it is NP hard in general [39, 69]. This has led researchers to consider the ℓ_1 version of the problem known as *basis pursuit*.

$$(P_1) \quad \min_{f\in\mathbb{R}^m} \|f\|_{\ell_1} \quad \text{subject to } Af = g,$$

where $\|f\|_{\ell_1} = \sum_{i=1}^m |f(i)|$. Building on the groundbreaking work of Donoho and Huo [40], it has now been shown [19, 22, 20, 21, 39, 41, 44, 78] that there are classes of matrices for which the problems (P_0) and (P_1) have the same unique solutions. Since (P_1) is a convex program, it can be solved by its classical reformulation as a linear program. A recent approach to these problems involves *restricted isometry constants* [20]. If A is a matrix with column vectors $\{v_j\}_{j\in J}$, for all $1 \leq S \leq |J|$ we define the *S-restricted isometry constant* δ_S to be the smallest constant so that for all $T \subset J$ with $|T| \leq S$ and for all $\{a_j\}_{j\in T}$,

$$(1-\delta_S)\sum_{j\in T}|a_j|^2 \leq \|\sum_{j\in T} a_j v_j\|^2 \leq (1+\delta_S)\sum_{j\in T}|a_j|^2.$$

The fundamental principle here is the construction of (nearly) unit norm frames for which subsets of a fixed size are (nearly) Parseval (or better, nearly orthogonal). The classification of these frames is out of our grasp at this time. But this did lead to a natural conjecture.

Conjecture 8.8. *For every $S \in \mathbb{N}$, for every $0 < \delta < 1$ and for every unit norm tight frame $\{f_i\}_{i=1}^{\infty}$, there is a partition $\{A_j\}_{j=1}^{r}$ of $\mathbb{N}$ so that for all $j = 1, 2, \cdots, r$, $\{f_i\}_{i\in A_j}$ is a frame sequence with S-restricted isometry constant $\delta_S \leq \delta$.*

There is also a finite dimensional version of Conjecture 8.8.

Conjecture 8.9. *For every $S \in \mathbb{N}$ and B and every $0 < \delta < 1$, there is a natural number $r = r(\delta, S, B)$ so that for every n and every unit norm B-tight frame $\{f_i\}_{i=1}^{M}$ for ℓ_2^n there is a partition $\{A_j\}_{j=1}^{r}$ of $\{1, 2, \cdots, M\}$ so that for all $j = 1, 2, \cdots, r$, $\{f_i\}_{i\in A_j}$ is a frame sequence with S-restricted isometry constant $\delta_S \leq \delta$.*

We would like to invoke Remark 2.1 here to see that Conjectures 8.8 and 8.9 are equivalent. It is easily seen by that remark that Conjecture 8.8 implies Conjecture 8.9. Unfortunately, this approach does not directly work for the converse since we are working with unit norm tight frames and if we take finite "parts" of these, say $\{f_i\}_{i=1}^{M}$, then these are not tight frames. We will not prove in detail that these are equivalent but instead just point out that combined with the following result of Balan, Casazza, Edidin and Kutyniok [9], Remark 2.1 will work.

Theorem 8.10. *If $\{f_i\}_{i\in I}$ is a unit norm Bessel sequence in $\mathbb{H}_n$ with Bessel bound B, then there is a unit norm family $\{g_j\}_{j\in J}$ so that $\{f_i\}_{i\in I} \cup \{g_j\}_{j\in J}$ is a unit norm tight frame for $\mathbb{H}_n$ with tight frame bound $\lambda \leq B + 2$.*

Remark 8.11. *By Theorem 8.10 and Remark 4.3, Conjectures 8.8 and 8.9 are equivalent to the same conjectures with "unit norm tight frame" replaced by "unit norm frame" or replaced by "unit norm Bessel sequence".*

A particularly interesting place to look for frames with good restricted isometry constants is in $L^2[0, 1]$.

Conjecture 8.12. *For every measurable set $E \subset [0,1]$ with $0 < |E|$, for every natural number S and for every $0 < \delta < 1$ there is a a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that for every $j = 1, 2, \cdots, r$ the family*

$$\{\frac{1}{\sqrt{|E|}} \mathrm{e}^{2\pi i n t} \chi_E\}_{n \in A_j},$$

is a frame sequence with S-restricted isometry constant $\delta_S \le \delta$.

These conjectures deal directly with the frame. If we want to deal with the columns of the frame vectors we have the following conjecture.

Conjecture 8.13. *Let $\{f_i\}_{i=1}^M$ be a frame for ℓ_2^n with frame bounds A, B. Let $\{e_k\}_{k=1}^n$ be the unit vector basis of ℓ_2^n and assume the column vectors $\{v_k\}_{k=1}^n$ are norm one. That is, assume $\|v_k\|^2 = \sum_{i=1}^M |\langle f_i, e_k\rangle|^2 = 1$ for every $1 \le k \le n$. For every $\delta > 0$ and for every $S \le n$ there exists a natural number $r = r(\delta, S, B)$ and a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $j = 1, 2, \cdots, r$, the family $\{v_k\}_{k \in A_j}$ has S-restricted isometry constant $\delta_S \le \delta$.*

We do not need to assume the column vectors are norm one in Conjecture 8.13 but rather that they are within ϵ of being one. The following result of Casazza, Kutyniok and Lammers [28] yields that Conjecture 3.10 is equivalent to Conjecture 8.13.

Theorem 8.14. *A family $\{f_i\}_{i=1}^M$ is a frame for ℓ_2^n with frame bounds A, B if and only if the column vectors of the frame vectors form a Riesz basic sequence in ℓ_2^M with Riesz basis bounds $\sqrt{A}, \sqrt{B}$.*

It is immediate from the R_ϵ-Conjecture (See section 4) that a positive solution to KS would imply a positive solution to Conjectures 8.8, 8.9, 8.12, and 8.13. Actually, all these conjectures are true as we will now see. For this we need to recall a result of Berman, Halpern, Kaftal and Weiss [12].

Theorem 8.15. *There is a natural number $r = r(B)$ satisfying the following. Let $(a_{ij})_{i,j=1}^n$ be a self-adjoint matrix with non-negative entries and with zero diagonal so that*

$$\sum_{m=1}^n a_{im} \le B, \quad \textit{for all } i = 1, 2, \cdots, n.$$

Then for every $r \in \mathbb{N}$ there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for every $j = 1, 2, \cdots, r$,

$$\sum_{m \in A_j} a_{im} \le \frac{1}{r} \sum_{m \in A_\ell} a_{im}, \quad \textit{for every } i \in A_j \textit{ and } \ell \ne j. \tag{8.3}$$

Now we can prove our conjectures hold true.

Theorem 8.16. *Conjectures 8.8, 8.9, 8.12, and 8.13 have a positive solution.*

Proof: We will prove Conjecture 8.9 for unit norm Bessel sequences. Let $\{f_i\}_{i=1}^M$ be a unit norm B-Bessel sequence in ℓ_2^n. Let H be the matrix

$$H = (\langle f_i, f_m\rangle)_{i,m=1}^M .$$

For each $1 \le i \le M$,

$$\sum_{m=1}^M |\langle f_i, f_m\rangle|^2 \le B.$$

Fix a $k \in \mathbb{N}$ with $\sqrt{\frac{BS}{k}} \le \delta$ and fix S as in Conjecture 8.9. By Theorem 8.15, there is a natural number $r = r(B,S,k) \in \mathbb{N}$ and a partition $\{A_j\}_{j=1}^r$ so that $H - D(H)$ satisfies Equation 8.3. Fix $1 \le j \le r$, let $T \subset A_j$ with $|T| \le S$ and let $(a_i)_{i\in T}$ be scalars. Then,

$$\begin{aligned}
\|\sum_{i\in T} a_i f_i\|^2 &= \sum_{i\in T} |a_i|^2 \|f_i\|^2 + \sum_{i\ne m \in T} a_i \overline{a_m} \langle f_i, f_m\rangle \\
&\le \sum_{i\in T} |a_i|^2 + \left(\sum_{i\ne m\in T} |a_i|^2 |a_m|^2\right)^{1/2} \left(\sum_{i\ne m\in T} |\langle f_i, f_m\rangle|^2\right)^{1/2} \\
&\le \sum_{i\in T} |a_i|^2 + \left[\left(\sum_{i\in T} |a_i|^2\right)^2\right]^{1/2} \left(\sum_{i\in T} \frac{B}{k}\right)^{1/2} \\
&\le \sum_{i\in T} |a_i|^2 + \left(\sum_{i\in T} |a_i|^2\right)\sqrt{\frac{BS}{k}} \\
&\le \sum_{i\in T} |a_i|^2 + \delta \sum_{i\in T} |a_i|^2 \\
&= (1+\delta) \sum_{i\in T} |a_i|^2.
\end{aligned}$$

Similarly we have

$$\|\sum_{i\in T} a_i f_i\|^2 \ge (1-\delta) \sum_{i\in T} |a_i|^2.$$

It follows that $\{f_i\}_{i\in A_j}$ has S-restricted isometry constant $\delta_S \le \delta$. □

What this section is trying to tell us is the following. In applied mathematics and engineering problems we are generally looking for the best examples we can find to use in practice. However, if we instead ask the question: *Let's classify all objects which satisfy our requirements*, then we have entered the world of the deepest unsolved problems in pure mathematics.

9. Towards a counter-example to Kadison-Singer

In this section we will give some more equivalents of Kadison-Singer which lend themselves to viable approaches for constructing a counterexample to KS. Throughout this section we will use the notation:

Notation: If $E \subset I$ we let P_E denote the orthogonal projection of $\ell_2(I)$ onto $\ell_2(E)$. Also, recall that we write $\{e_i\}_{i\in I}$ for the standard orthonormal basis for $\ell_2(I)$.

For results on frames, see Section 3.

Definition 9.1. *A subspace $\mathbb{H}$ of $\ell_2(I)$ is A-large for $A>0$ if it is closed and for each $i \in I$, there is a vector $f_i \in \mathbb{H}$ so that $\|f_i\| = 1$ and $|f_i(i)| \geq A$. The space $\mathbb{H}$ is large if it is A-large for some $A > 0$.*

We are going to classify PC in terms of A-large subspaces of $\ell_2(I)$. To do this we need some preliminary results.

Lemma 9.2. *Let $T^* : \mathbb{H} \to \ell_2(I)$ be the analysis operator for a frame $\{f_i\}_{i\in I}$ for $\mathbb{H}$ and let P be the orthogonal projection of $\ell_2(I)$ onto $\mathbb{H}$. Then $\{Pe_i\}_{i\in I}$ is a Parseval frame for $T^*(\mathbb{H})$ which is equivalent to $\{f_i\}_{i\in I}$.*

Proof: Note that $\{Pe_i\}_{i\in I}$ is a Parseval frame (Theorem 3.4) with synthesis operator P and analysis operator T_1^* satisfying $T_1^*(\mathbb{H}) = P(\ell_2(I)) = T^*(\mathbb{H})$. By Proposition 3.2, $\{Pe_i\}_{i\in I}$ is equivalent to $\{f_i\}_{i\in I}$. □

Proposition 9.3. *Let $\mathbb{H}$ be a subspace of $\ell_2(I)$. The following are equivalent:*

(1) The subspace $\mathbb{H}$ is large.

(2) If P is the orthogonal projection of $\ell_2(I)$ onto $\mathbb{H}$ then there is an $A > 0$ so that $\|Pe_i\| \geq A$, for all $i \in I$.

(3) The subspace $\mathbb{H}$ is the range of the analysis operator of some bounded frame $\{f_i\}_{i\in I}$.

Proof: (1) ⇒ (2): Suppose $\mathbb{H}$ is large. So, there exists an $A > 0$ such that for each $i \in I$, there exists a vector $f_i \in \mathbb{H}$ with $\|f_i\| = 1$ and $|f_i(i)| \geq A$. Given the projection P of (2) we have

$$A \leq |f_i(i)| = |\langle e_i, f_i\rangle| = |\langle Pe_i, f_i\rangle| \leq \|Pe_i\|\|f_i\| = \|Pe_i\|.$$

(2) ⇒ (3): By (2), $\{Pe_i\}_{i\in I}$ is a bounded sequence which is a Parseval frame by Theorem 3.4 and having $\mathbb{H}$ as the range of its analysis operator.

(3) ⇒ (1): Assume $\{f_i\}_{i\in I}$ is a bounded frame for a Hilbert space $\mathbb{K}$ with analysis operator T^* and $T^*(\mathbb{K}) = \mathbb{H}$. Now, $\{Pe_i\}_{i\in I}$ is a Parseval frame for $\mathbb{H}$ which is the range of its own analysis operator. Hence, $\{f_i\}_{i\in I}$ is equivalent to $\{Pe_i\}_{i\in I}$ by Proposition 3.2. Since $\{f_i\}_{i\in I}$ is bounded, so is $\{Pe_i\}_{i\in I}$. Choose $A > 0$ so that $A \leq \|Pe_i\| \leq 1$, for all $i \in I$. Then

$$A \leq |\langle Pe_i, Pe_i\rangle| = |\langle Pe_i, e_i\rangle| = |Pe_i(i)|.$$

So $\mathbb{H}$ is a large subspace. □

Now we need to learn how to decompose the range of the analysis operator of our frames.

Definition 9.4. *A closed subspace $\mathbb{H}$ of $\ell_2(I)$ is r-decomposable if for some natural number r there exists a partition $\{E_j\}_{j=1}^r$ of I so that $P_{E_j}(\mathbb{H}) = \ell_2(E_j)$, for all $j = 1, 2, \cdots, r$. The subspace $\mathbb{H}$ is finitely decomposable if it is r-decomposable for some r.*

For the next proposition we need a small observation.

Lemma 9.5. *Let $\{f_i\}_{i\in I}$ be a Bessel sequence in $\mathbb{H}$ having synthesis operator T and analysis operator T^*, let $E \subset I$, and let $\{f_i\}_{i\in E}$ have analysis operator $(T|_E)^*$. Then*

$$P_E T^* = (T|_E)^*.$$

Proof: For all $f \in \mathbb{H}$,

$$P_E T^*(f) = P_E\left(\sum_{i\in I}\langle f, f_i\rangle e_i\right) = \sum_{i\in E}\langle f, f_i\rangle e_i = (T|_E)^*(f).$$

□

We now have

Proposition 9.6. *A frame $\{f_i\}_{i\in I}$ for $\mathbb{K}$ satisfies the Feichtinger Conjecture if and only if $\mathbb{H} = T^*(\mathbb{K})$ is finitely decomposable.*

Proof: We can partition I into $\{E_j\}_{j=1}^r$ so that each $\{f_i\}_{i\in E_j}$ is a Riesz basic sequence if and only if (see the discussion after Theorem 3.1) $(T|_{E_j})^*$ is onto for every $j = 1, 2, \cdots, r$ if and only if (by Lemma 9.5) $P_{E_j}T^*$ is onto for all $j = 1, 2, \cdots, r$. □

Now we can put this altogether.

Theorem 9.7. *The following are equivalent:*

(1) The Kadison-Singer Problem.

(2) Every large subspace of $\ell_2(I)$ is finitely decomposable.

(3) For every $0 < A < 1$ there is a natural number $r = r(A)$ so that every A-large subspace of $\ell_2(I)$ is r-decomposable.

Proof: (1) ⇔ (2): This is immediate from Propositions 9.3 and 9.6.

(2) ⇒ (3): We prove the contrapositive. If (3) fails, then there is an $0 < A < 1$ and a sequence of subspaces $\mathbb{H}_j$ $j = 1, 2, \cdots$ of $\ell_2(I)$ so that each $\mathbb{H}_j$ is A-large but not j-decomposable. But now, $(\oplus_{j\in\mathbb{N}}\mathbb{H}_j)_{\ell_2}$ is an A-large subspace of $(\oplus_{j\in\mathbb{N}}\ell_2(I))_{\ell_2}$ which fails to be decomposable.

(3) ⇒ (2): This is obvious. □

Now we want to give quite explicit information about the existence of certain families of vectors in every large subspace of $\ell_2(I)$. We will see that this gives us an approach to producing a counterexample to KS.

Proposition 9.8. *Let $E \subset I$, and assume for every $i \in E$ there are vectors*

$$f_i = e_i + g_i \in \ell_2(I),$$

where each $g_i \in \ell_2(E^c)$, and the collection $\{g_i\}_{i\in E}$ is a Bessel sequence. Then, $\{f_i\}_{i\in E}$ is a Riesz basic sequence. Moreover, if $\mathbb{K}$ is the closed span of $\{f_i\}_{i\in E}$, then $P_E\mathbb{K} = \ell_2(E)$.

Proof: That $\{f_i\}$ is a Bessel sequence is obvious, and so $\{f_i\}$ possesses an upper basis bound.

We establish a lower basis bound. For all sequences of scalars $\{a_i\}_{i\in E}$ we have:

$$\begin{aligned} \|\sum_{i\in E} a_i f_i\| &= \|\sum_{i\in E} a_i e_i + \sum_{i\in E} a_i g_i\| \\ &\geq \|\sum_{i\in E} a_i e_i\| \qquad (9.1) \\ &= \left(\sum_{i\in E} |a_i|^2\right)^{1/2}, \end{aligned}$$

where the estimate in (9.1) follows by virtue of the orthogonality of $\sum_{i\in E} a_i e_i$ and $\sum_{i\in E} a_i g_i$.

If $\sum_{i\in E} a_i e_i \in \ell_2(E)$, then $\sum_{i\in E} a_i f_i \in \mathbb{K}$ and, since the g_i's are supported outside of E,

$$P_E\left(\sum_{i\in E} a_i f_i\right) = \sum_{i\in E} a_i e_i.$$

□

The following is a converse to Proposition 9.8.

Theorem 9.9. *Let $\mathbb{H}$ be a closed subspace of $\ell_2(I)$. The following are equivalent:*

(1) $\mathbb{H}$ is finitely decomposable.

(2) We can partition I into subsets $\{E_j\}_{j=1}^r$ so that for every $j = 1, 2, \cdots, r$ and all $i \in E_j$ we can find vectors

$$f_{ji} = e_i + g_{ji} \in \mathbb{H},$$

so that $g_{ji} \in \text{span}_{k\notin E_j} e_k$ and $\{g_{ji}\}_{i\in E_j}$ is Bessel.

Proof: (1) $\Rightarrow$ (2): Assume $\mathbb{H}$ is finitely decomposable. Let $\{E_j\}_{j=1}^r$ be a partition of I which satisfies Definition 9.4. Fix $1 \leq j \leq r$. Since $P_{E_j} : \mathbb{H} \to \ell_2(E_j)$ is bounded, linear and onto, it follows that $P_{E_j}^*$ is an (into) isomorphism. Therefore, $\{P_{E_j}^* e_i\}_{i\in E_j}$ is a Riesz basis for its span. Let $\{f_{ji}\}_{i\in E_j}$ be the dual functionals for this Riesz basis. Now, for all $i, \ell \in E_j$ we have

$$\delta_{\ell i} = \langle P_{E_j}^* e_\ell, f_{ji}\rangle = \langle e_\ell, P_{E_j} f_{ji}\rangle.$$

It follows that $f_{ji}(\ell) = 0$ if $i \in E_j$ and $i \neq \ell$, and $f_{ji}(i) = 1$. Hence, $f_{ji} = e_i + g_{ji}$ where $g_{ji} \in \text{span}_{i \notin E_j} e_i$. Finally, since $\{f_{ji}\}_{i \in E_j}$ is a Riesz basis, it follows that $\{g_{ji}\}_{i \in E_j}$ is Bessel.

(2) ⇒ (1): This is immediate from Proposition 9.8. □

Remark 9.10. *The vectors which arise in Theorem 9.9 are unique. That is, if*

$$\hat{f}_{ki} = e_i + \hat{g}_{ki} \in \mathbb{H},$$

(even without any assumption that the $\{\hat{g}_{ki}\}_{i \in E_j}$ are Bessel), then $\hat{f}_{ki} = f_{ki}$, for all $i \in E_j$. This follows from the fact that P_{E_j} is invertible on the range of $P^*_{E_j}$.

We will now discuss why we believe that Theorem 9.9 gives a viable approach to constructing a counterexample to KS. Basically, we want to construct a sequence of vectors f_i in $\ell_2(\mathbb{N})$ each having at least one big coordinate but so that whenever we partition $\mathbb{N}$ into a finite number of sets, one of these sets has sufficient density to guarantee that the vectors f_{ki} cannot be Bessel. As we have seen, these vectors are unique. To get the vectors f_{ki} we have to "row reduce" the $\{f_i\}_{i \in E_j}$ accross the coefficients of E_j. If the $\{f_i\}$ are chosen appropriately, we believe that this row reduction process will leave us with g_{ki} which are no longer Bessel.

This may seem esoteric, but all of this was built on existing deep constructions in the Banach space approximation property due to Szankowski [73, 74, 75] (see [24, 66]). A look at [73] shows that Szankowski constructs vectors with 6 ones in each vector (and this is their only support) in such a way that when these vectors sit in ℓ_p, $p \neq 2$ in a careful way, they span a sublattice failing the approximation property. Of course, Hilbert spaces have the approximation property. But our above propositions show that the Kadison-Singer Problem is asking for a specialized class of operators to give the required approximation. That is, Kadison-Singer is a *restricted* approximation property for ℓ_2. What we need to do is add a bounded set of vectors onto the Szankowski construction so that the set is Bessel, but when we do the required row reduction to get the vectors in Theorem 9.9, we end up with a non-Bessel sequence $\{g_{ki}\}_{i \in E_j}$ for one of the $j = 1, 2, \cdots, r$.

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