CHARACTER SUMS WITH PIATETSKI-SHAPIRO SEQUENCES

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Abstract

For \( c > 1, c \not\in \mathbb{Z} \) and \( \chi \) a primitive character \((\text{mod } q)\), we estimate the sum

\[
\sum_{x < n \leq x+y} \chi([nc]).
\]

Our results complement earlier work of Banks and Shparlinski. We also give a new bound for the least \( n \) for which \([nc]\) is a quadratic non-residue \((\text{mod } q)\), in the case of prime \( q \), for \( 1 < c < \frac{371}{309} \).

1. Introduction

Let \( c > 1, c \not\in \mathbb{Z} \) and let \( \chi \) be a primitive Dirichlet character \((\text{mod } q)\). We estimate the sum

\[
S_{c,\chi}(x,y) = \sum_{x < n \leq x+y} \chi([nc]).
\]

See [4] for background information about the Piatetski-Shapiro sequences \([nc] : n = 1, 2, \ldots \).

Although Garaev [10] gave a bound for the least quadratic non-residue \((\text{mod } q)\) of the form \([nc]\) for a short range of \( c > 1 \), he did not use sums \( S_{c,\chi}(x,y) \). Later his result was improved by Banks et al. [6]. In Section 5, we shall give the following further improvement. (Again, sums \( S_{c,\chi}(x,y) \) are not used in the proof.)

\begin{align*}
\text{Theorem 1.1} \quad &\text{Let } q \text{ be prime and } 1 < c < \frac{371}{309} = 1.2006 \ldots \text{ Then} \\
&\left( \frac{[nc]}{q} \right) = -1 \quad \text{for some } n \ll q^{1/4\sqrt{e} + \varepsilon}. \quad (1.1)
\end{align*}

Here and subsequently, \( \varepsilon \) is an arbitrary positive number, which we may take to be small. Constants implied by ‘\( \ll \)’ and ‘\( O \)’ may depend on \( c \) and \( \varepsilon \). We write \( \mathcal{L} \) for \( \log x \).

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The sums $S_{c,\chi}(x, y)$ are considered, again with $q$ restricted to prime values, by Banks and Shparlinski [5]. They show that for $c < \frac{4}{3}$ and
\[ x^{c+\varepsilon} < q < x^{4-2c-\varepsilon} \] (1.2)
a non-trivial estimate holds of the form
\[ S_{c,\chi}(0, x) \ll x^{1-\delta(\varepsilon)}, \quad \delta(\varepsilon) > 0. \] (1.3)

In the present paper, we give the bound (1.3), without any arithmetic restriction on the modulus, but for a range of $q$ of the form
\[ 3 \leq q < x^{\theta(c)-\varepsilon}. \] (1.4)

**Theorem 1.2** Let
\[
\theta(c) = \begin{cases}
6 - 4c & \left(1 < c \leq \frac{5}{4}\right), \\
1 & \left(\frac{5}{4} < c \leq \frac{9}{7}\right), \\
31 - 14c & \left(\frac{9}{7} < c \leq \frac{16}{11}\right), \\
\frac{13}{6 - 3c} & \left(\frac{16}{13} < c \leq \frac{14}{9}\right), \\
\frac{46 - 24c}{13} & \left(\frac{14}{9} < c \leq \frac{87}{55}\right), \\
\frac{11 - 5c}{5} & \left(\frac{87}{55} < c \leq \frac{9}{5}\right), \\
1 - \frac{c}{3} & \left(\frac{9}{5} < c < 3, c \neq 2\right).
\end{cases}
\]

An estimate (1.3) holds provided that $q$ satisfies (1.4). The bound (1.3) also holds for $c > 2$, $c \notin \mathbb{Z}$ with $\theta(c) = b/c^2$ for a certain positive constant $b$.

Our range (1.4) overlaps with (1.2) only for $c < \frac{6}{5}$.

We shall deduce Theorem 1.2 from several rather more general results.

**Theorem 1.3** Let $q \geq 3$ and let $q_0$ be a divisor of $q$. Then for $1 < c < \frac{3}{2}$, $q \leq x^{6-4c-\varepsilon}$, $1 \leq y \leq x$, we have
\[
S_{c,\chi}(x, y) \ll x^y \left\{ y^{1/2} x^{c/2-1/2} \left( q_0^{1/2} + \left( \frac{q}{q_0} \right)^{1/4} \right) + x^{c/2} \left( \left( \frac{q_0^3}{q} \right)^{1/8} + 1 \right) \\
+ y^{5/7} x^{(4c-4)/7} q_0^{1/7} + y^{4/7} x^{(4c-3)/7} q_0^{3/28} q_0^{-1/28} \right\}.
\]
Theorem 1.4 Let \( q \geq 3 \), and \( 1 < c < \frac{3}{2} \) or \( \frac{3}{2} < c < 2 \). Let \( 1 \leq y \leq x \). Then
\[
S_{c,x}(x,y) \ll \mathcal{L}(q^{-1/2}x^{c-1} + q^{1/2}x^{1/2} + q^{1/6}x^{(c+1)/3} + q^{1/3}x^{(c+5)/6} + q^{5/16}y^{1/4}x^{(5c+1)/16} + q^{5/18}y^{1/9}x^{(5c+5)/18}).
\]

Theorem 1.5 Let \( q \geq 3 \) and \( 1 < c < 2 \). Let \( 1 \leq y \leq x \). Then
\[
S_{c,x}(x,y) \ll x^{c} \left( q^{-1/2}x^{c-1} + q^{1/2}x^{1/2} + (y^{9}x^{20c+2}q^{-9})^{1/40} + (y^{9}x^{23c+3}q^{-9})^{1/55} + (yx^{5c+1})^{1/11} + (y^{9}x^{24c+10})^{1/65}q^{1/5} + (y^{11}x^{26c+13}q^{-18})^{1/77} + y^{1/8}(x^{14c+11}q^{13})^{1/48} + (qx^{4c+3})^{1/10} + (yq^{2}x^{3c+2})^{1/9} \right).
\]

Theorem 1.6 Let \( q \geq 3 \). Then
(i) We have, for \( c > 1, c \neq \mathbb{Z} \),
\[
S_{c,x}(x,y) \ll \mathcal{L}(yx^{(c-3)/7}q^{3/7} + q^{1/2}y^{3/4} + q^{3/4}y^{1/4}x^{(3-c)/4}).
\]
(ii) For \( c > 2, c \neq \mathbb{Z} \),
\[
S_{c,x}(x,y) \ll q^{1/2}x^{1-b/c^2},
\]
where \( b \) is a positive constant.

Theorems 1.3–1.6 are proved by combining techniques from a well-known paper of Heath-Brown \[14\] with bounds for exponential integrals and (for Theorems 1.4–1.6) bilinear exponential sums. See in particular [14, Section 4] in which Heath-Brown develops a ‘\( qT \)-analog’ of van der Corput’s method for exponential sums. Following this approach yields Theorem 1.3. It turns out that for \( c > \frac{3}{4} \), it is more efficient not to perform a ‘Weyl shift’ before estimating a sum such as
\[
\sum_{N < n \leq N+M} \chi(n)e(kn^\gamma),
\]
where we write \( e(z) = e^{2\pi iz} \) and \( \gamma = 1/c \). This line of attack produces Theorems 1.4 and 1.5. Theorem 1.6 is proved in a very simple way by comparison.

We are unable to profit from the renowned work of Burgess [8] on character sums in Theorems 1.3–1.6. However, Burgess’s paper [7] is used indirectly in proving Theorem 1.1.

We shall write ‘\( A \asymp B \)’ to indicate ‘\( A \ll B \ll A \)’ and ‘\( m \sim M \)’ for ‘\( M < m \leq 2M \)’.

2. Preliminary lemmas for Theorems 1.3–1.6

We assemble some lemmas, some of them straightforward and others very deep. Let \( \psi(t) = t - [t] - \frac{1}{2} \).

Lemma 2.1 Let \( G \) be a real function with continuous derivative \( G' \) on \([A,B]\). Then
\[
\sum_{A < n \leq B} G(n) = \int_{A}^{B} G(t) \, dt + \int_{A}^{B} G'(t)\psi(t) \, dt - [G(t)\psi(t)]_{A}^{B}.
\]
Proof. Apply integration by parts to \( \int_A^B G' \psi \, dt \).

**Lemma 2.2** Let \( q \geq 2 \). Then

\[
\sum_{r=0}^{q-1} e \left( \frac{nr}{q} \right) \psi \left( \frac{t-r}{q} \right) = \begin{cases} 
\psi(t) & \text{if } q \mid n, \\
e(n[t]/q) & \text{if } q \nmid n.
\end{cases}
\]

**Proof.** This is an elementary (but non-trivial) calculation. \( \square \)

**Lemma 2.3** Let \( K \geq 1 \). There are functions \( a(k) \) (\( 1 \leq |k| \leq K \)), \( b(k) \) (\( 0 \leq |k| \leq K \)) such that

\[
a(k) \ll \frac{1}{k} \quad (1 \leq |k| \leq K), \quad b(k) \ll \frac{1}{K} \quad (0 \leq |k| \leq K),
\]

\[
\frac{d}{dk} (a(k)) \ll \frac{1}{k^2}, \quad \frac{d}{dk} (b(k)) \ll \frac{1}{K^2} \quad (1 \leq |k| \leq K)
\]

and

\[
\left| \psi(t) - \sum_{0 < |k| \leq K} a(k) e(kt) \right| \leq \sum_{|k| \leq K} b(k) e(kt).
\]

Explicitly,

\[
a(k) = (2\pi i k)^{-1} F \left( \frac{k}{K} \right), \quad b(k) = \frac{1}{2K+2} \left( 1 - \frac{|k|}{K+1} \right)
\]

with \( F(u) = \pi u (1 - |u|) \cot \pi u \).

**Proof.** See Vaaler [16]. \( \square \)

**Lemma 2.4** Let \( B > 0 \) and \( \min(B, q/2) \leq r < \min(B, q/2) + q \). Then

\[
B \sum_{n \equiv r \pmod{q}} \frac{1}{n^2} \ll \frac{1}{r}.
\] (2.1)

**Proof.** Suppose first that \( B \leq \frac{q}{2} \). Then \( B \leq r < B + \frac{q}{2} \). Thus, \( r \) is the least integer \( n \) counted in the sum in (2.1). Hence,

\[
B \sum_{n \equiv r \pmod{q}} \frac{1}{n^2} \lesssim \frac{B}{r^2} + B \sum_{j \geq 0} \sum_{2^j q \leq n < 2^{j+1} q \atop n \equiv r \pmod{q}} \frac{1}{n^2}
\]

\[
\lesssim \frac{B}{r^2} + B \sum_{j \geq 0} 2^j \cdot (2^j q)^{-2}
\]

\[
\ll \frac{B}{r^2} + \frac{B}{q^2} \ll \frac{1}{r}.
\]
Now suppose that $B > q/2$. Then $q/2 \leq r < 3q/2$,

$$
\begin{align*}
B \sum_{n \geq B \atop n \equiv r \pmod{q}} \frac{1}{n^2} & \leq B \sum_{j \geq 0 \atop 2q \leq B/2 \atop n \equiv r \pmod{q}} \sum_{2q \leq n < 2^{j+1}q} \frac{1}{n^2} \\
& \ll B \sum_{j \geq 0 \atop 2^{j+1}q \geq B/2} 2^{-j}q^{-2} \ll B \cdot \frac{q}{B} \cdot q^{-2} \\
& \ll q^{-1}.
\end{align*}
$$

**Lemma 2.5**  Fix $c \in (1, 2)$ and let $\gamma = 1/c$. Let $z_1, z_2, \ldots$ be complex numbers such that $z_j \ll j^{c}$. Then for $N \geq 1, M > 0$,

$$
\sum_{N \leq j < N+M \atop j = [nt]} z_j = \gamma \sum_{N \leq j < N+M} z_j j^{\gamma - 1} + \sum_{N \leq j < N+M} z_j (\psi((-j+1)^{\gamma}) - \psi(-j^{\gamma})) + O(1).
$$

**Proof.** This is a consequence of [4, Lemma 2].

**Lemma 2.6**  Let $N \geq 1$. Let $g^{(d)}$ be continuous on $(1, \infty)$ with $g'' > 0$,

$$
|g^{(d)}(x)| \asymp FN^{-j} \quad (2 \leq j \leq 4) \quad \text{for} \quad x \asymp N.
$$

Let $1 \leq M \leq N$ and

$$
I = I(N, M) = \int_N^{N+M} e(g(t)) \, dt.
$$

Then

(i) $I \ll F^{-1/2} N$.

(ii) If there is a $t_0 \asymp N$ with $g'(t_0) = 0$, then

$$
I - \frac{\sigma e(1/8 + g(t_0))}{g''(t_0)^{1/2}} \ll \min\left(\frac{N^2}{F|t_0 - N_1|}, \frac{N}{F^{1/2}}\right) + \frac{N}{F},
$$

where $N_1 \in \{N, N + M\}$, $\sigma = 1$ for $t_0 \in [N, M + N]$ and $\sigma = 0$ otherwise.

(iii) Suppose that on $[N, N + M]$, $h^{(2)}$ is continuous and $g'/h$ is monotonic with

$$
|g'(x)/h(x)| \geq \lambda > 0.
$$

Then

$$
\int_N^{N+M} h(t)e(g(t)) \, dt \ll \lambda^{-1}.
$$

**Proof.** Assertions (i) and (iii) follow from [11, Lemmas 3.1, 3.2]. In the case $t_0 \in [N, N + M]$, assertion (ii) follows from [11, Lemma 3.4]. The case $t_0 \notin [N, N + M]$ follows from (i) and (iii).
Lemma 2.7 Let $F > 0$. Let $A$ be a subset of $\mathcal{R} = [C_1H, C_2H] \times [C_3N, C_4N]$ and

$$S = \sum_{(h,n) \in A} f(h,n)e(g(h,n)),$$

where $f$ and $g$ are real functions on $\mathcal{R}$ with

$$|f^{(i,j)}(u,v)| < C_5KH^{-i}N^{-j} \quad ((u,v) \in \mathcal{R}, \ 0 \leq i,j \leq 1).$$

Then for some subrectangle $\mathcal{R}'$ of $\mathcal{R},$

$$|S| < C(C_1, \ldots, C_5)K \left| \sum_{(h,n) \in A \cap \mathcal{R}'} e(g(h,n)) \right|.$$

Proof. This is [3, Lemma 3]. \hfill \Box

We define $(\alpha)_0 = 1, (\alpha)_s = (\alpha)_{s-1}(\alpha + s - 1)$ $(s = 1, 2, \ldots)$.

Lemma 2.8 Let $\theta, \phi$ be real constants with $\begin{pmatrix} \theta \\ \phi \end{pmatrix} \neq 0, (\theta, \phi + 2) \neq 0$. (2.2)

Let $X \geq 1, LM \leq X, |a_m| \leq 1$ $(m \sim M)$. Let $L_0 = \min(M, L)$. Let $f : [L, 2L] \rightarrow \mathbb{R}$ with $f'$ continuous, $f(t) \ll 1, f'(t) \ll L^{-1}$. Let

$$T = \sum_{m \sim M} a_m \sum_{\ell \in I_m} f(\ell) e\left(\frac{E M^\theta L^\phi}{m^\theta \ell^\phi}\right),$$

where $E > 0$ and $I_m$ is a subinterval of $[L, 2L]$. Then

$$|T| < C_6(\theta, \phi, \varepsilon)X^\varepsilon (X^{11/12} + XL^{-1/2} + E^{1/8}X^{13/16}L^{-1/8} + (EX^5L^{-1}L_0^{-1})^{1/6} + XE^{-1}).$$

Proof. This is a slight generalization of Baker and Weingartner [2, Lemma 9]. The condition corresponding to (2.2) can immediately be replaced by (2.2). To accommodate the presence of $f(\ell)$ (which corresponds to 1 in [2]), we make an application of Lemma 2.7. \hfill \Box

Lemma 2.9 Let $\theta, \phi$ be real constants with $\begin{pmatrix} \theta \\ \phi \end{pmatrix} \neq 0, (\theta)_3(\phi)_3 \neq 0$.

Let $X, L, M, a_m, E$ be as in Lemma 2.8. Let $|b_\ell| \leq 1$ and let $I$ be a real interval. Let

$$T' = \sum_{m \sim M} a_m \sum_{\ell \sim L} b_\ell e\left(\frac{E M^\theta L^\phi}{m^\theta \ell^\phi}\right),$$
Then
\[
|T| < C_{8}(\theta, \phi, \varepsilon)X^{5}\{E^{1/20}L^{19/20}M^{29/40} + E^{3/46}L^{43/46}M^{16/23}
+ E^{1/10}L^{9/10}M^{3/5} + E^{3/28}L^{23/28}M^{41/56} + E^{1/11}L^{53/66}M^{17/22}
+ E^{2/21}L^{31/42}M^{17/21} + E^{1/5}L^{7/10}M^{3/5} + L^{1/2}M + E^{1/8}(LM)^{3/4}\}.
\]

**Proof.** This is a slight variant of [3, Theorem 3], where the condition \(\ell/m \in I\) is absent. This condition can be included at the cost of a log factor, as explained in Harman [13, Section 3.2]. \(\square\)

**Lemma 2.10** Let \(\chi\) be a primitive character (mod \(q\)) and write
\[
S_{\chi}(n) = \sum_{r=1}^{q} \chi(r) e \left( \frac{rn}{q} \right), \quad S_{\chi}(h, n) = \sum_{r=0}^{q-1} \chi(r + h) \bar{\chi}(r) e \left( \frac{rn}{q} \right).
\]
Then
\[
S_{\chi}(n) = S_{\chi}(1) \bar{\chi}(n) \quad \text{and} \quad |S_{\chi}(1)| = q^{1/2}.
\]

Let \(q_0\) be a divisor of \(q\). Then
\[
\sum_{1 \leq h \leq A} \sum_{1 \leq n \leq B} |S_{\chi}(4hq_0, n)| \ll q^{5/2} \left( \frac{q}{q_0} \right)^{1/2} AB + q^{3/4} q_0^{1/4} A^{1/4}
\]
and
\[
\sum_{1 \leq h \leq A} |S(4hq_0, 0)| \ll q^{6} q_0 A.
\]

**Proof.** For \(S_{\chi}(n)\), see Davenport [9, Chapter 9]. For \(S_{\chi}(4hq_0, n)\), see Heath-Brown [14, Lemma 9]. \(\square\)

**Lemma 2.11** Let \(\chi\) be a non-principal character (mod \(q\)). Then
\[
\sum_{N < n \leq N + M} \chi(n) \ll q^{1/2} \log q.
\]

**Proof.** Davenport [9, Chapter 23]. \(\square\)

**Lemma 2.12** For \(k \geq 1\), let
\[
\phi_k(m) = 1 - e(k(m^{\gamma} - (m + 1)^{\gamma})).
\]

Let \(J \geq 1, N \geq 1\). Then for any complex numbers \(u_n\) (\(n \sim N\)),
\[
J^{-1} \sum_{k=J}^{\infty} \sum_{\varepsilon N < n \leq N} u_n \phi_k(n) \ll N^{\gamma-1} \max_{\varepsilon N < N_1 \leq N} \sum_{k=J}^{\infty} \sum_{\varepsilon N < n \leq N_1} u_n.
\]

**Proof.** This requires a partial summation in \(n\). See Heath-Brown [15, pp. 246–247] for the details in a particular case. \(\square\)
**Lemma 2.13** Let

\[ L(H) = \sum_{i=1}^{\mu} A_i H^{a_i} + \sum_{j=1}^{\nu} B_j H^{-b_j}, \]

where \( A_i, B_j, a_i \) and \( b_j \) are positive. Let \( 0 \leq H_1 \leq H_2 \). Then there is some \( H \in (H_1, H_2] \) with

\[ L(H) < C g(u, v) \sum_{i=1}^{\mu} \sum_{j=1}^{\nu} (A_j B_i)^{1/(a_i + b_j)} + \sum_{i=1}^{\mu} A_i H_1^{a_i} + \sum_{j=1}^{\nu} B_j H_2^{-b_j}. \]

**Proof.** This is Graham and Kolesnik \[11, Lemma 2.4\]. \( \square \)

**Lemma 2.14** Let \( f \) be a real-valued function, and suppose that \( f^{(3)} \) is continuous on an interval \( [x, x+y] \) with endpoints in \( \mathbb{Z} \). Suppose that

\[ |f^{(3)}(z)| \asymp \lambda \quad (x \leq z < x+y). \]

Then

\[ \sum_{x \leq n < x+y} e(f(n)) \ll y^{1/6} + y^{3/4} + y^{1/4} \lambda^{-1/4}. \]

**Proof.** This is \[11, Theorem 2.6\]. \( \square \)

**Lemma 2.15** This is a constant \( a \in (0, 1) \) such that for any \( c > 2 \), \( c \notin \mathbb{Z} \), the bound

\[ \sum_{n \sim N} e(\alpha n^c) \ll N^{1-a/c^2} \]

holds for \( N^{-c/2} \leq |\alpha| \leq N^{c/2} \). The implied constant depends only on \( c \).

**Proof.** See \[4, Lemma 21\]. \( \square \)

### 3. An approximation to a hybrid sum

We prove a result that acts as a framework for Theorems 1.3–1.6.

**Theorem 3.1** Let \( p(r) \) (\( r = 1, 2, \ldots \)) be an arithmetic function of period \( q \). We write

\[ S(p, n) = \sum_{r=1}^{q} p(r) e \left( -\frac{n r}{q} \right). \]

Let \( F > 0 \). Let \( N \geq 1 \) and let \( H(t) \) be real with \( H' \) monotonic,

\[ |H'(t)| \leq Fr^{-1} \quad (N \leq t \leq 2N). \]  \( (3.1) \)
Let $1 \leq M \leq N$, $I = [N, M + N]$,

$$S(I, p, H) = \sum_{m \in I} p(m)e(H(m)).$$

Then

$$S(I, p, H) - q^{-1} \sum_{|n| < 2FqN^{-1}} S(p, n) \int_I e\left(\frac{nt}{q} + H(t)\right) \, dt \ll q^{-1}F^{-1}N |S(p, 0)|$$

$$+ \sum_{\min(2FqN^{-1}, q/2) \leq |n| \leq \max(2FqN^{-1}, q) + q} n^{-1}|S(p, n)|.$$

**Proof.** We have

$$S(I, p, H) = \sum_{r=1}^{q} p(r) \sum_{N < kq + r \leq N + M} e(H(kq + r)). \quad (3.2)$$

Let $A = A(r) = (N - r)/q$, $B = B(r) = (N + M - r)/q$, $g(t) = g(r, t) = H(tq + r)$. The inner sum in (3.2) is

$$\sum_{A < k \leq B} e(g(k)) = \int_A^B e(g(u)) \, du + \int_A^B \frac{d}{du} e(g(u))\psi(u) \, du - [e(g(u))\psi(u)]_A^B$$

$$= T_1(r) + T_2(r) - T_3(r), \quad \text{say} \quad (3.3)$$

(Lemma 2.1). After a change of variable, we obtain

$$\sum_{r=1}^{q} p(r)T_1(r) = \sum_{r=1}^{q} p(r)q^{-1} \int_I e(H(t)) \, dt$$

$$\ll q^{-1}F^{-1}N |S(p, 0)| \quad (3.4)$$

by Lemma 2.6.

In $T_2(r)$, the $L_2$ convergence of the Fourier series

$$\psi(u) = \sum_{n \neq 0} (-2\pi in)^{-1} e(nu)$$

permits a termwise integration. After a change of variable, this yields

$$T_2(r) = \sum_{n \neq 0} (-2\pi in)^{-1} e\left(-\frac{nr}{q}\right) \int_I e\left(\frac{nt}{q}\right) \frac{d}{dt}(e(H(t))) \, dt.$$
Thus,
\[
\sum_{r=1}^{q} p(r)T_2(r) = \sum_{n \neq 0} (-2\pi in)^{-1} \sum_{r=1}^{q} p(r) e \left( -\frac{nt}{q} \right) \int_I e \left( \frac{nt}{q} \right) \frac{d}{dt} (e(H(t))) \ dt
\]
\[
= \sum_{n \neq 0} (-2\pi in)^{-1} S(p,n)J_n,
\]

where
\[
J_n = \int_I 2\pi i H'(t)e \left( \frac{nt}{q} + H(t) \right) \ dt.
\]

Let us write briefly \(A = 2FqN^{-1}\). For \(|n| \geq A\),
\[
\frac{d}{dt} \left( \frac{nt}{q} + H(t) \right) \asymp \frac{|n|}{q},
\]

and Lemma 2.6(iii) yields
\[
\sum_{|n| \geq A} (-2\pi in)^{-1} S(p,n)J_n \ll \frac{qF}{N} \sum_{|n| \geq A} \frac{|S(p,n)|}{n^2}.
\]

We claim that
\[
\frac{qF}{N} \sum_{|n| \geq A} \frac{|S(p,n)|}{n^2} \ll \sum_{\min(A,q/2) \leq |n| < \max(A,q/2) + q} \frac{|S(p,n)|}{n}.
\]

By the periodicity of \(S(p,n)\), it suffices to show that
\[
\frac{qF}{N} \sum_{\min(A,q/2) \leq |n| < \max(A,q/2) + q} \frac{1}{n^2} \ll \frac{1}{r}
\]

for \(\min(A,q/2) \leq r < \max(A,q/2) + q\), and this bound is a consequence of Lemma 2.4. We deduce that
\[
\sum_{r=1}^{q} p(r)T_2(r) = \sum_{1 \leq |n| < A} (-2\pi in)^{-1} S(p,n)J_n + O \left( \sum_{\min(A,q/2) \leq n < \min(A,q/2) + q} \frac{|S(p,n)|}{n} \right). \quad (3.5)
\]

For \(1 \leq |n| < A\), integration by parts gives
\[
(-2\pi in)^{-1} J_n = (-2\pi in)^{-1} \int_I e \left( \frac{nt}{q} \right) \frac{d}{dt} (e(H(t))) \ dt
\]
\[
= \left[ (-2\pi in)^{-1} e \left( \frac{nt}{q} + H(t) \right) \right]^{N+M}_N + q^{-1} \int_N^{N+M} e \left( \frac{nt}{q} + H(t) \right) \ dt. \quad (3.6)
\]
Combining (3.2)–(3.6), we have

\[ S(I, p, H) = \sum_{r=1}^{q} p(r) T_1(r) + \sum_{r=1}^{q} p(r) T_2(r) - \sum_{r=1}^{q} p(r) T_3(r) \]

\[ = q^{-1} \sum_{1 \leq |n| < A} S(p, n) \int_{N}^{N+M} e \left( \frac{nt}{q} + H(t) \right) \, dt \]

\[ + Q + O \left( q^{-1} F^{-1} N |S(p, 0)| + \sum_{\min(A,q/2) \leq n < \min(A,q/2)+q} \frac{|S(p, n)|}{n} \right) \]  

(3.7)

with

\[ Q = - \sum_{r \equiv 1 \pmod{q}} p(r) \left[ e(H(t)) \psi \left( \frac{t-r}{q} \right) \right]_{N}^{N+M} + \sum_{0 < |n| < A} \sum_{r \equiv 0 \pmod{q}} p(r) e \left( -\frac{nr}{q} \right) \left[ e(H(t) + nt/q) \right]_{N}^{M+N} \]

\[ = \sum_{r \equiv 0 \pmod{q}} p(r) \left[ e(H(t)) \left( \sum_{0 < |n| < A} e \left( \frac{n(t-r)/q}{-2\pi in} \right) - \psi \left( \frac{t-r}{q} \right) \right) \right]_{N}^{N+M} \]  

(3.8)

Moreover,

\[ \sum_{r=0}^{q-1} p(r) \psi \left( \frac{t-r}{q} \right) = q^{-1} \sum_{n=0}^{q-1} \sum_{s=0}^{q-1} p(s) e \left( -\frac{sn}{q} \right) \sum_{r=0}^{q-1} e \left( \frac{nr}{q} \right) \psi \left( \frac{t-r}{q} \right) \]

\[ = q^{-1} \sum_{n=0}^{q-1} S(p, n) \sum_{r=0}^{q-1} e \left( \frac{nr}{q} \right) \psi \left( \frac{t-r}{q} \right) \]

\[ = q^{-1} S(p, 0) \psi(t) + q^{-1} \sum_{n=1}^{q-1} S(p, n) \frac{e(n[t]/q)}{e(-n/q) - 1}, \]

where the last equality comes from Lemma 2.2. Replacing the last range of summation by \(-q/2 < n \leq q/2, n \neq 0,\)

\[ \sum_{r=0}^{q-1} p(r) \psi \left( \frac{t-r}{q} \right) = q^{-1} S(p, 0) \psi(t) + \sum_{-q/2 < n \leq q/2} S(p, n) \left( -2\pi in \right)^{-1} e \left( \frac{nt}{q} \right) + O(q^{-1}) \]

\[ = \sum_{0 < |n| \leq q/2} \frac{S(p, n)}{-2\pi in} e \left( \frac{nt}{q} \right) + O \left( q^{-1} \sum_{|n| \leq q/2} |S(p, n)| \right). \]
We can now rewrite (3.8) in the form

\[
Q = \left[ e(H(t)) \sum_{0 < |n| < A} S(p, n)(-2\pi in)^{-1} e\left(\frac{nt}{q}\right) - e(H(t)) \sum_{0 < |n| < \frac{q}{4}} S(p, n)(-2\pi in)^{-1} e\left(\frac{nt}{q}\right) \right]^{M+N}
+ O\left( q^{-1} \sum_{0 < |n| \leq q/2} |S(p, n)| \right)
= O\left( \sum_{\min(A, q/2) \leq |n| \leq \max(A, q/2)} \frac{|S(p, n)|}{n} + \sum_{0 < |n| \leq q/2} \frac{|S(p, n)|}{q} \right).
\]

Thus, \( Q \) can be absorbed into the error term

\[
O\left( \sum_{\min(A, q/2) \leq |n| \leq \max(A, q/2) + q} n^{-1} |S(p, n)| \right),
\]

and the proof of Theorem 3.1 is complete. \( \square \)

4. Proof of Theorems 1.2–1.6

Let \( 1 \leq K \leq yx^{\epsilon - 1} \). Combining Lemma 2.5, with \( z_m = \chi(m) \), with Lemma 2.3,

\[
\sum_{x < n \leq x+y} \chi([n^\gamma]) = \sum_{x' < [n^\gamma] \leq (x+y)^\gamma} \chi([n^\gamma]) + O(1)
= y \sum_{x' < m \leq (x+y)^\gamma} \chi(m)m^{\gamma y - 1} - \sum_{x' < m \leq (x+y)^\gamma} \chi(m) \sum_{0 < |k| \leq K} a(k)\{e(-km^\gamma) - e(-k(m+1)^\gamma)\}
+ O\left( \sum_{x' < m \leq (x+y)^\gamma} \sum_{0 < |k| \leq K} b(k)e(-k(m+\lambda)^\gamma) \right) + O\left( \frac{yx^{\epsilon - 1}}{K} \right)
\]

(\( \lambda \in \{0, 1\} \))

\[
= T - U + O\left( V + \frac{yx^{\epsilon - 1}}{K} \right),
\]

say. Combining Lemma 2.11 with a partial summation,

\[
T \ll x^{1-\epsilon} q^{1/2} \mathcal{L},
\]

(We may suppose that \( \log q \ll \log x \), for otherwise all our results are trivial.)
For $\frac{1}{2} \leq J \leq K$, consider the contribution to $U$ from

\[
U_J := \sum_{|k| \sim J} a(k) \sum_{x' < m \leq (x+y)^c} \chi(m)(e(-km^\gamma) - e(-k(m+1)^\gamma))
\]

\[
= \sum_{|k| \sim J} a(k) \sum_{x' < m \leq (x+y)^c} \chi(m)e(-km^\gamma)\phi_k(m)
\]

in the notation of Lemma 2.12. We certainly have

\[
U_J \ll J^{-1} \left| \sum_{|k| \sim J} \sum_{x' < m \leq (x+y)^c} \chi(m)e(-k(m+\lambda)) \right|,
\]

where $\lambda \in \{0, 1\}$. By Lemma 2.12, then,

\[
U_J \ll \min(x^{1-c}, J^{-1}) \left| \sum_{|k| \sim J} \sum_{x' < m \leq z} \chi(m)e(-k(m+\lambda)) \right|
\]

for some $z \in (x^c, (x+y)^c]$. Let us now focus on Theorem 1.3.

We fix $k$ with $k \sim J$ and give a bound for

\[
S_k = \sum_{x' < m \leq z} \chi(m)e(-k(m+\lambda)) = \sum_m f(m),
\]

where $f(m) = \chi(m)e(-k(m+\lambda))$ if $x^c < m \leq z$ and $f(m) = 0$ otherwise. Let $H_0$ be a positive integer, to be chosen later, such that

\[
H_0 \leq yx^{c-1}/q_0.
\]

We have

\[
H_0 S_k = \sum_{h=1}^{H_0} \sum_m f(m + 4hq_0) = \sum_m \sum_{h=1}^{H_0} f(m + 4hq_0).
\]

By Cauchy’s inequality,

\[
H_0^2 |S_k|^2 \leq \left( \sum_{x' - 4H_0q_0 < m \leq z} 1 \right) \left( \sum_m H_0 \sum_{h=1}^{H_0} f(m + 4hq_0) \right)^2
\]

and

\[
|S_k|^2 \ll yx^{c-1}H_0^{-2} \sum_{1 \leq h_1, h_2 \leq H_0} \left| \sum_m f(m + 4h_1q_0)\overline{f}(m + 4h_2q_0) \right|
\]

\[
\ll y^2 x^{2c-2}H_0^{-1} + yx^{c-1}H_0^{-1} \sum_{h=1}^{H_0} \left| \sum_m f(m + 4hq_0)\overline{f}(m) \right|.
\]
For some \( H, \frac{1}{2} \leq H \leq H_0 \), a dyadic argument yields
\[
\mathcal{L}^{-1}|S_k|^2 \ll y^2 x^{2c-2} H_0^{-1} + yx^{c-1} H_0^{-1} \times \sum_{h \sim H} \sum_{m \in I_h} \chi(m + 4hq_0) \bar{\chi}(m) e(-k(m + \lambda)^\gamma + k(m + \lambda + 4hq_0)^\gamma) .
\]

Here \( I_h = \{ m : x^c < m \leq z, x^c < m + 4hq_0 \leq z \} \).

We apply Theorem 3.1 with \( N = x^c, N + M = z \),
\[
p(r) = \tilde{\chi}(r) \chi(r + 4hq_0), \quad H(t) = k((t + 4hq_0 + \lambda)^\gamma - (t + \lambda)^\gamma), \quad F = 4q_0 Hx^{1-c}.
\]

Let \( B = 2Fq^c \). We estimate \( \int_{I_h} e(-nt/q + H(t)) \, dt \) via Lemma 2.6(i). We obtain, in the notation of Lemma 2.10,
\[
\sum_{m \in I_h} \chi(m + 4hq_0) \bar{\chi}(m) e(-k(m + \lambda)^\gamma + k(m + \lambda + 4hq_0)^\gamma) \ll q^{-1}(x^cF^{-1/2}) \sum_{1 \leq n < B} |S_\chi(4hq_0, n)| + q^{-1} F^{-1} x^c |S_\chi(4hq_0, 0)| + \sum_{\min(B,q/2) \leq |n| \leq \max(B,q/2) + q} \frac{|S_\chi(4hq_0, n)|}{n} .
\]

Recalling (4.5), and applying another dyadic argument, there is a \( G \) in \([\min(B,q/2), \max(B+q/2) + q]\) such that
\[
\mathcal{L}^{-2}|S_k|^2 \ll y^2 x^{2c-2} H_0^{-1} + yx^{c-1} H_0^{-1} q^{-1} F^{-1/2} \sum_{\chi' \in \{ \chi, \tilde{\chi} \}} \sum_{h \sim H} \sum_{1 \leq n < B} |S_\chi(4hq_0, n)| + yx^{c-1} H_0^{-1} q^{-1} F^{-1} \sum_{h \sim H} |S_\chi(4hq_0, 0)| + yx^{c-1} H_0^{-1} G^{-1} \sum_{\chi' \in \{ \chi, \tilde{\chi} \}} \sum_{h \sim H} \sum_{n \sim G} |S_\chi(4hq_0, n)| .
\]

Now we apply Lemma 2.10, obtaining
\[
x^{-c} |S_k|^2 \ll y^2 x^{2c-2} H_0^{-1} + yx^{c-1} H_0^{-1} q^{-1} F^{-1/2} \left\{ \left( \frac{q}{q_0} \right)^{1/2} HB + q^{3/4} q_0^{1/4} H^{1/4} \right\} + yx^{c-1} H_0^{-1} q^{-1} F^{-1} q_0 H + yx^{c-1} H_0^{-1} G^{-1} \left\{ \left( \frac{q}{q_0} \right)^{1/2} HG + q^{3/4} q_0^{1/4} H^{1/4} \right\} .
\]
Using
\[
G^{-1} \ll F^{-1} x^c q^{-1} + q^{-1} ,
\]
and inserting $B \asymp Fq x^{-c}$, we find that

$$x^{-\varepsilon} |S_k|^2 \ll y^2 x^{2c-2} H_0^{-1} + y x^{c-1/2} H_0^{1/2} q^{1/2} k^{1/2} + y x^{(5c-3)/2} H_0^{-1} (q q_0)^{-1/4} k^{-1/2} + y x^{c-2} H_0^{-1} q^{-1} k^{-1} + y x^{-c-1} (q/q_0)^{1/2} + y x^{-c} H_0^{-3/4} (q_0/q)^{1/4} + y x^{c-2} H_0^{-1} q^{-1} q_0^{-3/4} k^{-1}$$

$$= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7,$$

(4.6)

say. It is clear that $T_6 \ll T_5$ and $T_4 \ll T_7$.

Recalling (4.3), we deduce from (4.6) that, for some $J$,

$$U x^{-\varepsilon} \ll x^{-2c/3} U J \ll y x^{c-1} H_0^{-1/2} + y^{1/2} x^{c-1/2} H_0^{1/4} K^{1/4} q^{1/4} + y^{1/2} x^{-c-1/2} H_0^{-1/2} (q q_0)^{-1/8} + y^{1/2} x^{c-1/2} q^{1/4} + y^{1/2} x^{-c-1/2} H_0^{-1/2} q^{3/8} q_0^{-1/8}$$

$$= U_1 + U_2 + U_3 + U_4 + U_5,$$

(4.7)

say. It is clear that $U_5 \ll U_3$. Since our bound for $U$ is trivial for $H_0 < 1$, Lemma 2.13 yields

$$U x^{-\varepsilon} \ll y^{1/2} x^{c-1/2} (q_0^{1/2} + (q/q_0)^{1/4}) + x^{c/2} q_0^{3/8} q^{-1/8} + y^{2/3} x^{c-1/2} q^{1/6} K^{1/6} + y^{1/2} x^{c-3/2} q^{1/8} q_0^{-1/24} K^{1/6}.$$

We can obtain the corresponding bound for $V x^{-\varepsilon}$, with $q$, $q_0$ replaced by 1, using this approach. Thus,

$$x^{-\varepsilon} (|U| + V + y x^{c-1}/K) \ll y^{1/2} x^{c-2}/2 (q_0^{1/2} + (q/q_0)^{1/4}) + x^{c/2} ((q_0^3/q)^{1/8} + 1) + y^{2/3} x^{c-1/2} q^{1/6} K^{1/6} + y^{1/2} x^{c-3/2} q^{1/8} q_0^{-1/24} K^{1/6} + y x^{-c}/K.$$

(4.8)

Since (4.8) is trivially true for $K < 1$, an application of Lemma 2.13 gives, for a suitable choice of $K$,

$$x^{-\varepsilon} (|U| + V + y x^{c-1}/K) \ll y^{1/2} x^{c-2}/2 (q_0^{1/2} + (q/q_0)^{1/4}) + x^{c/2} ((q_0^3/q)^{1/8} + 1) + y^{5/7} x^{(4c-4)/7} q^{1/7} + y^{4/7} x^{(4c-3)/7} q^{3/28} q_0^{-1/28}.$$

We can absorb the bound (4.2) for $T$ into the last expression, since if

$$x^{1-c/2} q^{1/2} > y^{5/7} x^{(4c-4)/7} q^{1/7},$$

it is readily verified that Theorem 1.3 is trivial. Thus, (4.1) yields Theorem 1.3.

Turning to the proof of Theorem 1.4, we again use (4.1) and (4.2), and it is clear that the bound for $T$ in (4.2) can be absorbed into our final estimate for $S_{c,x}(x, y)$, while our estimate for $U$ will be
seen to apply (with $q$ replaced by 1) to $V$. Thus, it remains to show that, for a suitable choice of $K$,

$$x^{-\epsilon} \max_{1/2 \leq J \leq K} |U_J| + yx^{c-1}/K \ll q^{-1/2}x^{c-1} + q^{1/2}x^{1/2} + q^{1/6}x^{(c+1)/3} + q^{1/3}x^{(c+3)/6} + q^{5/16}y^{1/4}x^{(5c+1)/16} + q^{5/18}y^{1/9}x^{(5c+5)/18}. \quad (4.9)$$

Recalling Lemma 2.3, a partial summation argument yields

$$U_J \ll J^{-1}\left|\sum_{J < k \leq J_1} \sum_{x < m \leq (x+y)^c} \chi(m)e(-k(m+\lambda)\gamma)\right| \quad (4.10)$$

for some $J_1 \leq 2J$ and $\lambda \in \{0, 1\}$. (We may need to replace $\chi$ by $\overline{\chi}$; we disregard this without loss of generality.) We apply Theorem 3.1 to the inner sum, with

$$p(r) = \chi(r), \quad H(t) = k\gamma, \quad F = 4Jx.$$ 

Define $S_\chi(n)$ as in Lemma 2.10. A dyadic argument with a simple change of variable yields

$$J^{-1} \sum_{J < k \leq J_1} \sum_{x < m \leq (x+y)^c} \chi(m)e(-k(m+\lambda)\gamma)$$

$$- (Jq)^{-1} \sum_{1 \leq |n| \leq 2D_1 Fx^{-c}} S_\chi(n) e\left(-\frac{\lambda n}{q}\right) \sum_{J < k \leq J_1} \int_{x^c + \lambda}^{(x+y)^c + \lambda} e\left(\frac{nt}{q} + k\gamma\right) dt$$

$$\ll q^{-1}F^{-1}N|S_\chi(0)| + L^{-1}G^{-1} \sum_{|n| \sim G} |S_\chi(n)| \quad (4.11)$$

for some $G$ satisfying

$$\min(Fq^{-1}x^{-c}, q) \ll G \ll \max(Fq^{-1}x^{-c}, q).$$

By Lemma 2.10, the right-hand side of (4.11) is

$$\ll J^{-1}q^{-1/2}x^{-1} + Lq^{1/2}. \quad (4.12)$$

The terms $n$ on the left-hand side of (4.11) that are of the same sign as $k$, or of the opposite sign with

$$\frac{|n|}{q} < \frac{1}{2} \gamma J(x+y)^{1-c},$$

contribute

$$\ll q^{-1/2}(Jx^{1-c})^{-1} \sum_{1 \leq |n| \leq Jx^{1-c}q} 1 \ll q^{1/2} \quad (4.13)$$

by Lemma 2.6(iii). We are left with an interval of $n$. Replacing $n$ by $-n$, we write (briefly)

$$n \asymp Jqx^{1-c} \quad (4.14)$$
to denote this interval. We apply Lemma 2.6(ii) to the complex conjugate of the integral in (4.11). Thus,

$$g(t) = -kt^\gamma + \frac{nt}{q}, \quad t_0 = t_0(k, n) = \left(\frac{n}{\gamma qk}\right)^{1/\gamma-1},$$

$$g''(t_0) = \gamma(1-\gamma)k \left(\frac{n}{\gamma qk}\right)^{(\gamma-2)/(\gamma-1)},$$

$$\left(\int_{x'}^{x+y+\lambda} e\left(-\frac{nt}{q} + kt^\gamma\right)dt\right)^-$$

$$= \sigma(k, n)e\left(\frac{1}{8} + \frac{n}{q} \left(\frac{n}{\gamma qk}\right)^{1/\gamma-1} - k \left(\frac{n}{\gamma qk}\right)^{1/(\gamma-1)}\right) g''(t_0)^{-1/2} + E(k, n), \quad (4.15)$$

where \(\sigma(k, n) = 1\) if \(t_0(k, n) \in [x', (x+y)^\epsilon + \lambda]\), \(\sigma(k, n) = 0\) otherwise;

$$E(k, n) \ll \min\left(\frac{x^{2\epsilon-1}}{J|t_0 - ((x+y)^\epsilon + \lambda)|}, \frac{x^{\epsilon-1/2}}{J^{1/2}}\right) + \frac{x^{\epsilon-1}}{J};$$

and \(y_1 \in \{0, y\}\). Note that

$$-\frac{n}{q} \left(\frac{n}{\gamma qk}\right)^{1/\gamma-1} + k \left(\frac{n}{\gamma qk}\right)^{\gamma/(\gamma-1)} = \alpha n^{\gamma/(\gamma-1)} q^{-\gamma/(\gamma-1)} k^{1/(\gamma-1)},$$

where \(\alpha = \alpha(c) \neq 0\).

Let \(f(t) = f(k, t) = kt^\gamma, r(k) = qf''((x+y_1)^\epsilon + \lambda)\). We observe that

$$Jx^{1-2\epsilon}|t_0 - ((x+y_1)^\epsilon + \lambda)| \approx |f'(t_0) - f'((x+y_1)^\epsilon + \lambda)|$$

(by the Mean Value Theorem)

$$= \frac{|n - r(k)|}{q}.$$

We may now rewrite (4.15) in the form

$$\int_{x'}^{x+y+\lambda} e\left(-\frac{nt}{q} + kn^\gamma\right)dt - \frac{\sigma(k, n)e(-n\lambda/q)e(-1/8 + \alpha n^{\gamma/(\gamma-1)} q^{-\gamma/(\gamma-1)} k^{1/(\gamma-1)})}{g''((n/\gamma qk)^{1/(\gamma-1)})^{1/2}}$$

$$\ll \min\left(\frac{q}{|n - r(k)|}, \frac{x^{\epsilon-1/2}}{J^{1/2}}\right) + \frac{x^{\epsilon-1}}{J}.$$
We deduce from (4.10)–(4.14) that

\[ U_{j} \ll (Jq)^{-1} \sum_{n=Jq^{1-\epsilon}}^{Jq} \sum_{J<k \leq 1} \left| \frac{S_{\chi}(n)e(-n\lambda/q - 1/8 + \alpha n^{\nu/(1-\gamma)} q^{-\nu/(1-\gamma)} k^{-1/(1-\gamma)})}{g''((n/\gamma qk)^{1/(1-\gamma)})^{1/2}} \right| ^{J-1/2} \]

\[ + J^{-1} q^{-1/2} \sum_{J<k \leq 1} \left( \min \left( \frac{x^{c-1/2}}{J^{1/2}}, \frac{q}{|n - r(k)|} \right) + x^{c-1} \right) J^{-1} q^{-1/2} x^{c-1} + \mathcal{L} q^{1/2} \]

\[ = V_{1} + V_{2} + J^{-1} q^{-1/2} x^{c-1} + \mathcal{L} q^{1/2}, \quad (4.16) \]

say. It is clear that

\[ V_{2} \ll q^{-1/2} (x^{c-1/2} J^{-1/2} + q \mathcal{L}). \]

In fact, the sums in \( V_{1} \) and \( V_{2} \) are empty unless

\[ Jq \gg x^{c-1}, \]

so we see that

\[ V_{2} + J^{-1} q^{-1/2} x^{c-1} + \mathcal{L} q^{1/2} \ll q^{-1/2} x^{c-1} + \mathcal{L} q^{1/2}. \quad (4.17) \]

Since \( |S_{\chi}(n)e(-n\lambda/q)| = q^{1/2} \), it follows from Lemma 2.8 that

\[ V_{1} x^{-c} \ll q^{-1/2} J^{-3/2} x^{c-1/2} \left( (J^{2} q x^{1-c})^{11/12} + J^{3/2} q x^{1-c} \right. \]

\[ + (Jx)^{1/8} (J^{2} q x^{1-c})^{13/16} J^{-1/8} + (Jx(J^{2} q x^{1-c})^{4})^{1/6} + (Jx(J^{2} q x^{1-c})^{5} J^{-2})^{1/6} \]

\[ = q^{5/12} J^{1/3} x(c+5)/12 + q^{1/2} x^{1/2} + J^{1/8} q^{5/16} x(c+7)/16 + q^{1/6} x^{(c+1)/3} + q^{1/3} x^{(c+3)/6}. \]

Combining this with (4.16) and (4.17),

\[ x^{-c} \max_{1/2 \leq j \leq k} |U_{j}| + \frac{yx^{c-1}}{K} \ll q^{1/2} x^{1/2} + q^{-1/2} x^{c-1} + q^{1/6} x^{(c+1)/3} + q^{1/3} x^{(c+3)/6} \]

\[ + q^{5/12} K^{1/3} x^{(c+5)/12} + q^{5/16} K^{1/8} x^{(c+7)/16} + \frac{yx^{c-1}}{K}. \]

This bound is trivial for \( 0 < K < 1 \). An application of Lemma 2.13 leads to (4.9) and completes the proof of Theorem 1.4.

To prove Theorem 1.5, we need only show in place of (4.9) that, for a suitable choice of \( K \),

\[ \max_{1/2 \leq j \leq k} U_{j} + \frac{yx^{c-1}}{K} \ll x^c (q^{-1/2} x^{c-1} + q^{1/2} x^{1/2} + (y^9 x^{20c+2} q^{9})^{1/49} + (y^9 x^{23c+23} q^{9})^{1/55} \]

\[ + (yx^{5c+1} q^{1/11}) + (y^9 x^{24c+10} q^{5})^{1/65} q^{1/5} + (y^{11} x^{26c+13} q^{18})^{1/77} \]

\[ + y^{1/8} (x^{14c+11} q^{13})^{1/48} + (x^{4c+3} q^{1})^{1/10} + (y q x^{3c+2})^{1/9}. \]
To obtain this, we estimate $V_1$ using Lemma 2.9 instead of Lemma 2.8:

$$x^{-\varepsilon} V_1 \ll q^{-1/2} J^{-3/2} x^{\varepsilon - 1/2} \{(Jx)^{1/20} (Jqx^{-c})^{29/40} + (Jx)^{3/46} J^{43/46} (Jqx^{-c})^{16/23}$$

$$+ (Jx)^{1/10} J^{9/10} (Jqx^{-c})^{3/5} + (Jx)^{3/28} J^{23/28} (Jqx^{-c})^{41/56} + (Jx)^{1/11} J^{53/66} (Jqx^{-c})^{17/22}$$

$$+ (Jx)^{2/21} J^{31/42} (Jqx^{-c})^{17/21} + (Jx)^{1/5} J^{7/10} (Jqx^{-c})^{3/5} + J^{3/2} q x^{-c} + (Jx)^{1/8} (J^2 q x^{-c})^{3/4}\}.$$  

Combining this with (4.16) and (4.17),

$$x^{-\varepsilon} \max_{1/2 \leq J \leq k} |U_J| + \frac{y x^{\varepsilon - 1}}{K} \ll \left( q^9 x^{11c+11} K^9 \right)^{1/40} + \left( q^9 x^{14c+12} K^9 \right)^{1/46} + \left( q x^{4c+2} K \right)^{1/10}$$

$$+ \left( q^{13} x^{15c+19} K^9 \right)^{1/56} + \left( q^6 x^{5c+8} K \right)^{1/22} + \left( q x^{13} K \right)^{1/142} K^{1/7}$$

$$+ \left( q x^{4c+3} \right)^{1/10} + q^{1/2} x^{1/2} + \left( q^2 x^{3c+3} K \right)^{1/8} + q^{-1/2} x^{\varepsilon - 1} + \frac{y x^{\varepsilon - 1}}{K}.$$  

The proof is now completed in the same way as the preceding proof.

For Theorem 1.6, we change to a simpler strategy. We have

$$S_{c,\chi}(x, y) = \sum_{a=1}^{q} \chi(a) \sum_{x \leq \alpha < x+y \mod q} 1.$$  

Now $[n^c] \equiv a \mod q$ can be rewritten as

$$mq + a \leq n^c < mq + a + 1 \quad \text{(some } m \in \mathbb{Z})$$

or equivalently

$$\left\lfloor \frac{n^c - a}{q} \right\rfloor - \left\lfloor \frac{n^c - a - 1}{q} \right\rfloor = 1.$$  

Hence,

$$\sum_{x \leq n^c < x+y \mod q} 1 = \sum_{x \leq \alpha < x+y} \left( \frac{1}{q} - \frac{1}{q} \left( \psi \left( \frac{n^c - a}{q} \right) + \psi \left( \frac{n^c - a - 1}{q} \right) \right) \right).$$

Since $\chi$ is non-principal,

$$S_{c,\chi}(x, y) = \sum_{a=1}^{q} \chi(a) \sum_{x \leq \alpha < x+y} \left( \psi \left( \frac{n^c - a - 1}{q} \right) - \psi \left( \frac{n^c - a}{q} \right) \right)$$

$$\ll \sum_{a=1}^{q} \chi(a) \sum_{x \leq \alpha < x+y} \psi \left( \frac{n^c - a - \alpha}{q} \right).$$
where $\sigma \in [0, 1]$. Now we apply Lemma 2.3 (with $K \geq q$ at our disposal). Writing $S(k) = \sum_{x \leq n < x + y} e(kn^c/q)$,

$$|S_{c,x}(x, y)| \leq \sum_{r=1}^{q} \chi(r) \sum_{0 < |k| \leq K} a(k) \sum_{x \leq n < x + y} e\left(k \left(\frac{n^c - r - \sigma}{q}\right)\right)$$

$$+ \frac{qy}{K} + \sum_{0 < |k| \leq K} b(k) \sum_{r=1}^{q} \sum_{x \leq n < x + y} e\left(k \left(\frac{n^c - r - \sigma}{q}\right)\right)$$

$$\ll \sum_{0 < |k| \leq K} k^{-1} \sum_{r=1}^{q} \chi(r)e\left(-\frac{kr}{q}\right) \left|S(k)\right| + \frac{qy}{K} + K^{-1} \sum_{0 < |k| \leq K} \sum_{r=1}^{q} e\left(-\frac{kr}{q}\right) \left|S(k)\right|$$

$$\ll q^{1/2} \sum_{0 < k \leq K} k^{-1} \left|S(k)\right| + \frac{qy}{K} + \frac{q}{K} \sum_{0 < k \leq K, k \equiv 0 \mod q} \left|S(k)\right|. \quad (4.18)$$

From Lemma 2.14, for $k > 0$,

$$S(k) \ll y \left(\frac{kx^{c-3}}{q}\right)^{1/6} + y^{3/4} + y^{1/4} \left(\frac{kx^{c-3}}{q}\right)^{-1/4}. \label{eq:4.18}$$

Hence,

$$q^{1/2} \sum_{0 < |k| \leq K} \left|S(k)\right| \ll q^{1/3} y K^{1/6} x^{(c-3)/6} + \mathcal{L} q^{1/2} y^{3/4} + y^{1/4} q^{3/4} x^{(3-c)/4},$$

while

$$\frac{q}{K} \sum_{0 < k \leq K, k \equiv 0 \mod q} \left|S(k)\right| \ll qK^{-1} \sum_{0 < j \leq K/q} (y(jx^{c-3})^{1/6} + y^{3/4} + y^{1/4} (jx^{c-3})^{-1/4})$$

$$\ll q^{-1/6} y K^{1/6} x^{(c-3)/6} + y^{3/4} + y^{1/4} q^{1/4} K^{-1/4} x^{(3-c)/4}.$$
with a positive constant $a$. Now (4.18) yields

$$S_{c,x}(x,y) \ll q^{1/2} L \cdot x^{1-a/c^2} + qx^{1/2} + x^{1-a/c^2} \ll x^{1-a/(3c^2)}.$$  

For Theorem 1.2, it clearly suffices to show that for $q < x^{\theta(c)-\epsilon}$,

$$S_{c,x}(x,x) \ll x^{1-\epsilon/12}.$$  

We obtain this for $1 < c \leq \frac{5}{4}$ from Theorem 1.3 with $q_0 = 1$; for $\frac{5}{4} < c \leq \frac{87}{55}$ from Theorem 1.5; for $\frac{87}{55} < c < \frac{9}{5}$ from Theorem 1.4; for $\frac{9}{5} < c < 3$, $c \neq 2$, $q \leq x^{1-c/3-\epsilon}$ from Theorem 1.6(i) and for $c > 2$, $c \notin \mathbb{Z}$, $q \leq x^{b/c^2}$ from Theorem 1.6(ii).

5. Proof of Theorem 1.1

We require further lemmas.

**Lemma 5.1**

(i) Let $L \geq 1$ and let $q$ be prime. The interval $[1, L]$ contains at least $cL$ quadratic residues (mod $q$). Here $c$ is a positive constant.

(ii) Let $q$ be prime and let $L \geq q^{1/4}\sqrt{e} + \epsilon$. Then for $q > C(\epsilon)$, the interval $[1, L]$ contains at least $c(\epsilon)L$ quadratic non-residues. Here $c(\epsilon) > 0$.

**Proof.** Part (i) is due to Hall [12] and part (ii) to Banks et al. [6].

**Lemma 5.2** Let $f$ be a real function on $[a, b]$, $b - a \geq 1$, with $f''$ continuous,

$$|f''(t)| \geq \lambda \quad (a \leq t \leq b),$$

where $\lambda > 0$. Then

$$\sum_{a < n \leq b} e(f(n)) \ll (b - a)\lambda^{1/2} + \lambda^{-1/2}.$$  

**Proof.** See [11, Theorem 2.2].

**Lemma 5.3** Let $(\kappa, \lambda)$ be an exponent pair. Let $H \geq \frac{1}{2}$, $M \geq \frac{1}{2}$, $L \geq \frac{1}{2}$, $|b_{h,m}| \leq 1$, $|c_{\ell}| \leq 1$, $E \gg MH$, $T > 0$. Let $\alpha\beta\theta \neq 0$, $\theta < 1$,

$$S = \sum_{h \sim H} \sum_{m \sim M} \sum_{\ell \sim L} a_{h,m} c_{\ell} e \left( \frac{EH^\alpha M^\beta L^\theta}{h^\alpha m^\beta \ell^\theta} \right).$$

Then

$$|S| < C_{10}(\alpha, \beta, \theta) HML \log^2(HML + 2) \left( (MH)^{-1/2} + \left( \frac{E}{MH} \right)^{\kappa/(2+2\kappa)} L^{-(1+\kappa-\lambda)/(2+2\kappa)} \right).$$
Proof. See Baker [1, Theorem 2.1]. The condition $\ell m \leq T$ is absent there, but as noted earlier, can be inserted at the cost of a log factor. \qed

**Proof of Theorem 1.1.** Let $L = q^{1/4\sqrt{\varepsilon} + \varepsilon/2}$, $M = L^{-1} (\log q)^9$. We may suppose that $q$ is large. By Lemma 5.1, we may choose a set $\mathcal{L} \subset (\varepsilon L, L)$, $\# \mathcal{L} \gg L$, consisting of quadratic non-residues (mod $q$) and a set $\mathcal{M} \subset (\varepsilon M, M)$, $\# \mathcal{M} \gg M$, consisting of quadratic residues (mod $q$).

Note that $$Y := LM \ll q^{(1/4\sqrt{\varepsilon} + \varepsilon)}$$ so that $n \ll q^{1/4\sqrt{\varepsilon} + \varepsilon}$ whenever $[n^\varepsilon] \ll Y$.

In Lemma 2.5, with $N, N + M$ replaced by $e^2 ML, ML$, and

$$z_j = \sum_{\ell m = j \atop \ell \in \mathcal{L}, m \in \mathcal{M}} 1,$$

it suffices to show the positivity of

$$\sum_{e^2 ML < j \leq ML} z_j = S_1 + S_2 + O(1),$$

where

$$S_1 = \gamma \sum_{e^2 ML < j \leq ML} z_j^{\varepsilon^{-1}},$$

$$S_2 = \sum_{\ell \in \mathcal{L}} \sum_{m \in \mathcal{M}} \{\psi(- (\ell m + 1)^\varepsilon) - \psi(- (\ell m)^\varepsilon)\}.$$

Since $S_1 \gg Y^\varepsilon$, it remains to show that

$$S_2 \ll \frac{Y^\varepsilon}{\log q}.$$

We apply Lemma 2.3 with $K = Y^{1-\gamma} \log q$.

This yields

$$S_2 - \sum_{0 < |k| \leq K} a(k) \sum_{\ell \in \mathcal{L}} \sum_{m \in \mathcal{M}} \left\{e(- k(\ell m + 1)^\varepsilon) - e(- k(\ell m)^\varepsilon)\right\} \ll \frac{Y^\varepsilon}{\log q} + D \left| \sum_{0 < |k| \leq K} b(k) \sum_{\ell^2 Y + \lambda < n \leq Y + \lambda} e(k n^\varepsilon) \right|,$$

where $\lambda = 0$ or $1$ and

$$D = \max_j \sum_{\ell m = j \atop \ell \in \mathcal{L}, m \in \mathcal{M}} 1 \ll Y^\varepsilon.$$
By Lemma 5.2,
\[
\sum_{\varepsilon^2 Y^\lambda < n \leq Y + \lambda} e(kn^\gamma) \ll Y(|k|Y^\gamma)^{-1/2} + (|k|Y^\gamma - 1)^{-1/2} = Y^{\gamma/2}|k|^{-1/2} + Y^{1-\gamma/2}|k|^{-1/2}.
\]

Since \( b(k) \ll K^{-1} \),
\[
D \sum_{0 < |k| \leq K} b(k) \sum_{\varepsilon^2 Y^\lambda < n \leq Y + \lambda} e(kn^\gamma) \ll Y^{\gamma/2} K^{1/2} + Y^{1-\gamma/2} K^{-1/2} \ll \frac{Y^\gamma}{\log q}.
\]

Since \( a(-k) = a(k) \ll k^{-1} \), it now suffices to show that for \( \frac{1}{2} \leq H \leq K \),
\[
S(H) := \frac{H^{-1}}{Y} \sum_{k \sim H} \sum_{\ell \in \mathcal{L}} \sum_{m \in \mathcal{M}} \left| e(-k(\ell m + 1)^\gamma) - e(-k(\ell m)^\gamma) \right| \ll \frac{Y^\gamma}{(\log q)^2}.
\]

By Lemma 2.12,
\[
S(H) \ll Y^{\gamma-1} \max_{Z \geq Y} \sum_{k \sim H} \sum_{\ell \in \mathcal{L}} \sum_{m \in \mathcal{M}} \sum_{\ell m \leq Z} e(-k(\ell m)^\gamma)
\]
\[
= Y^{\gamma-1} \sum_{k \sim H} e_k \sum_{\ell \in \mathcal{L}} \sum_{m \in \mathcal{M}} \sum_{\ell m \leq Z} e(-k(\ell m)^\gamma)
\]

for a suitable choice of \( Z \approx Y \) and \( e_k, |e_k| \leq 1 \).

Let \( (\kappa, \gamma) = BA^B(0, 1) = (\frac{19}{32}, \frac{32}{63}) \). By Lemma 5.3,
\[
S \ll HML(\log q)^2 \left( (MH)^{-1/2} + \left( \frac{E}{MH} \right)^{\kappa/(2+2\kappa)} L^{-((1+\kappa-\lambda)/(2+2\kappa))} \right).
\]

Here \( E := H(ML)^\gamma \) is easily seen to satisfy the condition \( E \gg HM \).

We have
\[
HML(\log q)^2 (MH)^{-1/2} \ll \frac{ML}{(\log q)^2}
\]
since
\[
H^{1/2}M^{-1/2}(\log q)^4 \ll (KM^{-1})^{1/2}(\log q)^4 \ll 1.
\]

It now suffices to verify that
\[
HML(\log q)^2 (YM^{-1})^{\kappa/(2+2\kappa)} L^{-((1+\kappa-\lambda)/(2+2\kappa))} \ll \frac{ML}{(\log q)^2}.
\]
Since \( L = Y^{\gamma} (\log q)^{-9\gamma} \), \( M = Y^{1-\gamma} (\log q)^{9\gamma} \), this reduces to

\[
y^{(2 + 2\kappa)(1 - \gamma) + \kappa (2\gamma - 1) - \gamma(1 + \kappa - \lambda)} \ll (\log q)^{-C}
\]

for a suitable constant \( C \). The exponent of \( Y \) is

\[
2 + \kappa - \gamma(3 + \kappa - \lambda).
\]

Since \( c < \frac{371}{369} = (3 + \kappa - \lambda)/(2 + \kappa) \), this exponent is negative and the proof of Theorem 1.1 is complete. \( \square \)

References