On repeated values of the Riemann zeta function on the critical line

WILLIAM D. BANKS
Dept. of Mathematics, University of Missouri
Columbia, MO 65211, USA
bankswd@missouri.edu

SARAH KANG
Dept. of Mathematics, University of Missouri
Columbia, MO 65211, USA
sk244@mail.missouri.edu

September 27, 2011

Abstract

Let $\zeta(s)$ be the Riemann zeta function. In this paper, we study repeated values of $\zeta(s)$ on the critical line, and we give evidence to support our conjecture that for every nonzero complex number $z$, the equation $\zeta(1/2 + it) = z$ has at most two solutions $t \in \mathbb{R}$. We prove a number of related results, some of which are unconditional, and some of which depend on the truth of the Riemann hypothesis. We also propose some related conjectures which are implied by Montgomery’s pair correlation conjecture.
1 Introduction

The Riemann zeta function $\zeta(s)$ is well known and lies at the heart of analytic number theory. In the half-plane $\{s = \sigma + it \in \mathbb{C} : \sigma > 1\}$ it can be defined either as a Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

or (equivalently) as an Euler product

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$ 

In the extraordinary memoir of Riemann [14] it is shown that $\zeta(s)$ extends to a meromorphic function on the whole complex plane with its only singularity being a simple pole at $s = 1$, and it satisfies the functional equation relating its values at $s$ and $1 - s$. There are many excellent accounts of the theory of the Riemann zeta function; we refer the reader to [1, 3, 7, 8, 9, 13, 16] and the references contained therein.

In the half-plane $\mathcal{H} = \{\sigma > \frac{1}{2}\}$ the zeta function $\zeta(s)$ takes every nonzero complex value infinitely often (cf. [16, Theorem 11.10]), whereas the Riemann hypothesis (RH) asserts that $\zeta(s) \neq 0$ for any $s \in \mathcal{H}$; in particular,

$$\text{RH} \implies \{z \in \mathbb{C} : \zeta(s) = z \text{ for infinitely many } s \in \mathcal{H}\} = \mathbb{C} \setminus \{0\}.$$ 

In March 2008, at the Analytic Number Theory workshop in Oberwolfach, the first author gave empirical evidence that the complementary result

$$\{z \in \mathbb{C} : \zeta(s) = z \text{ for infinitely many } s \in \mathcal{L}\} = \{0\} \quad (1)$$

is likely to hold on the boundary $\mathcal{L}$ of the half-plane $\mathcal{H}$, that is, on the critical line $\mathcal{L} = \{\sigma = \frac{1}{2}\}$; this result had been conjectured earlier by Selberg in a footnote to his 1989 paper on Dirichlet series [15]. The purpose of the present note is to describe some (albeit limited) numerical evidence that we have obtained in support of the Selberg’s conjecture (1). Moreover, our findings suggest that the following stronger statement may be true.

**Conjecture 1.** For every complex number $z \neq 0$ the equation $\zeta(1/2 + it) = z$ has at most two solutions $t \in \mathbb{R}$. 


Note that this conjecture implies (1) in view of the famous result of Hardy [6] that \( \zeta(s) \) has infinitely many zeros on the critical line.

The analysis of data from our numerical computations has also led us to some unconditional results which show that there are many complex numbers \( z \neq 0 \) such that \( \zeta(1/2 + it) = z \) has at least two solutions \( t \in \mathbb{R} \) (and thus we expect there are precisely two solutions for every such \( z \)). We shall say that a closed interval \([a, b]\) is **good** if there exist two infinite sequences of real numbers, \((t_k)_{k=1}^{\infty}\) and \((t_k^*)_{k=1}^{\infty}\), such that

1. \((t_k)_{k=1}^{\infty}\) is contained in \([a, b]\);
2. \((t_k^*)_{k=1}^{\infty}\) is unbounded;
3. \(\zeta(1/2 + it_k) = \zeta(1/2 + it_k^*) \neq 0\) for every \(k\).

**Theorem 1.** Let \( \lambda = 3.4362182260 \cdots \) be the least positive real number for which \( \zeta(1/2 + i\lambda) \in \mathbb{R} \). Then, the interval \([-\lambda, \lambda]\) is good.

This is proved in §5 using a criterion for “goodness” (Lemma 1) that is given in §4. In a similar spirit, we prove the following statement in §6:

**Theorem 2.** Let \( \gamma_{126} = 279.2292509277 \cdots \) and \( \gamma_{127} = 282.4651147650 \cdots \) be the ordinates of the two zeros \( \rho = 1/2 + i\gamma \) of \( \zeta(s) \) with \( 279 < \gamma < 283 \). Then, the interval \([\gamma_{126}, \gamma_{127}]\) is good.

In §5 we prove a conditional result concerning loops in the graph of \( \zeta(1/2 + it) \), \( t \in \mathbb{R} \). To formulate the theorem, suppose that RH is true, and let \( 0 < \tau_1 < \tau_2 < \cdots \) be the sequence of distinct ordinates of the zeros \( \rho = 1/2 + i\gamma \) of \( \zeta(s) \) with \( \gamma > 0 \). For each \( n \geq 1 \), the **loop** \( \mathcal{L}_n \) is the collection of complex numbers given by

\[
\mathcal{L}_n = \{ \zeta(1/2 + it) : \tau_n < t < \tau_{n+1} \}
\]

(see §3 for a more general definition of \( \mathcal{L}_n \) which does not require the assumption of RH). Note that \( 0 \notin \mathcal{L}_n \), but \( \mathcal{L}_n \cup \{0\} \) is a closed curve.

**Theorem 3.** Assume RH. Then, there are infinitely many \( n \) such that \( \mathcal{L}_n \) does not intersect itself, and there are infinitely many \( n \) for which \( \mathcal{L}_n \) has a self-intersection.

**Corollary 1.** Assume RH. Then, for every \( \varepsilon > 0 \) there are real numbers \( t_1 \neq t_2 \) with \( |t_1 - t_2| < \varepsilon \) and \( \zeta(1/2 + it_1) = \zeta(1/2 + it_2) \neq 0 \).
Corollary 2. Assume RH. Then, for every $\varepsilon > 0$ there exists a good interval $[a, b]$ of length $b - a < \varepsilon$.

We also propose the following conjecture, which follows from the truth of RH and Montgomery’s pair correlation conjecture via a variant of Theorem 3.

Conjecture 2. For any $k \geq 1$ there is a loop $\mathcal{L}_n$ with $k$ self-intersections.

A related conjecture for pairs of loops is given in §6.

\section{Ordinates of zeros on the critical line}

Let 
\[ \cdots < \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 < \cdots \]
be the sequence of distinct real solutions to the equation $\zeta(1/2 + it) = 0$, arranged in increasing order, with $\tau_1 = 14.1347 \cdots$ being the least positive solution. According to this definition we have
\[ \zeta(1/2 + it) \neq 0 \quad (\tau_n < t < \tau_{n+1}). \quad (2) \]

Note that
\[ \tau_n = -\tau_{-n} \quad (n \in \mathbb{Z}), \quad (3) \]
which follows from the fact that $\overline{\zeta(1/2 + it)} = \zeta(1/2 - it)$ for all $t \in \mathbb{R}$.

As usual, we also arrange the zeros $\beta + i\gamma$ of $\zeta(s)$ with $\gamma > 0$ in a sequence $\rho_n = \beta_n + i\gamma_n$ so that $\gamma_{n+1} \geq \gamma_n$. From the computations of Gourdon and Demichel [4] it is known that $\beta_n = 1/2$ and $\gamma_{n+1} > \gamma_n$ for all natural numbers $n \leq 10^{13}$, hence $\tau_n = \gamma_n$ for every such $n$. By (3) we also have $\tau_n = -\gamma_{1-n}$ in the range $-10^{13} < n \leq 0$. Thus, for small values of $|n|$ the number $\tau_n$ can be evaluated with arbitrary numerical precision using, e.g., the function \texttt{ZetaZero} in Mathematica. The following table gives values of $\tau_n$ with $-5 \leq n \leq 5$:
For every $n \in \mathbb{Z}$ we define the loop $\mathcal{L}_n$ to be the collection of complex numbers given by

$$\mathcal{L}_n = \{ \zeta(1/2 + it) : \tau_n < t < \tau_{n+1} \}.$$ 

In view of (3) it follows that $\mathcal{L}_n = \mathcal{L}_{-n}$ for all $n \in \mathbb{Z}$. Also, by (2) we see that zero is not contained in any set $\mathcal{L}_n$, hence $\bigcup_{n \in \mathbb{Z}} \mathcal{L}_n$ is the complete set of nonzero values taken by $\zeta(s)$ on the critical line.

### 4 Two criteria for goodness

**Lemma 1.** Let $a, b \in \mathbb{R}$ with $a < b$ and $\zeta(1/2 + ia) = \zeta(1/2 + ib)$. Suppose that

$$\mathcal{C} = \{ \zeta(1/2 + it) : a \leq t \leq b \}$$

is a Jordan curve in $\mathbb{C}$ which encloses an open neighborhood of zero. Then, the interval $[a, b]$ is good.

**Proof.** Let $M = \max_{t \in [a, b]} |\zeta(1/2 + it)|$. Since $\zeta(s)$ is unbounded on the critical line, there is a sequence $(t^o_k)_{k=1}^\infty$ such that $|\zeta(1/2 + it^o_k)| > M + k$ for all $k$. Let $n_k$ be the integer for which $\tau_{n_k} < t^o_k < \tau_{n_k+1}$; clearly, the sequence $(n_k)_{k=1}^\infty$ is unbounded. For each $k$, since $\zeta(1/2 + i\tau_{n_k}) = 0$ lies inside the curve $\mathcal{C}$ and $\zeta(1/2 + it^o_k)$ lies outside, there is a real number $t^*_k$ in the range $\tau_{n_k} < t^*_k < t^o_k$ such that $\zeta(1/2 + it^*_k)$ lies on the curve $\mathcal{C}$; that is,
ζ(1/2 + it_k^*) = ζ(1/2 + it_k) ≠ 0 for some t_k ∈ [a, b]. As the sequence (n_k)_{k=1}^∞ is unbounded, the same is true of (t_k^*)_{k=1}^∞, and hence the sequences (t_k)_{k=1}^∞ and (t_k^*)_{k=1}^∞ satisfy the conditions (i), (ii) and (iii) in §1.

With a slight modification to the above proof, the interval [a, b] in Lemma 1 can be replaced by any finite union of closed intervals.

**Lemma 2.** Let \( \mathcal{U} \) be a finite union of closed intervals in \( \mathbb{R} \), and suppose that the set

\[ \mathcal{C} = \{ \zeta(1/2 + it) : t \in \mathcal{U} \} \]

is a Jordan curve in \( \mathbb{C} \) which encloses an open neighborhood of zero. Then, \( \mathcal{C} \cap \mathcal{L}_n \neq \emptyset \) for infinitely many \( n \in \mathbb{Z} \).

### 5 Self-intersecting loops

![Figure 1: The loop \( \mathcal{L}_0 \)](image)

**Proof of Theorem 1.** The loop \( \mathcal{L}_0 \) (see Figure 1) has four self-intersections, which are given in the following table:

<table>
<thead>
<tr>
<th>( t_1 )</th>
<th>( t_2 )</th>
<th>( \zeta(1/2 + it_1) ) and ( \zeta(1/2 + it_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-13.26322741 ( \cdots )</td>
<td>-1.33231317 ( \cdots )</td>
<td>0.30051216 ( \cdots ) + ( i \cdot 0.55357158 ) ( \cdots )</td>
</tr>
<tr>
<td>-9.66690805 ( \cdots )</td>
<td>9.66690805 ( \cdots )</td>
<td>1.53182067 ( \cdots )</td>
</tr>
<tr>
<td>-3.43621822 ( \cdots )</td>
<td>3.43621822 ( \cdots )</td>
<td>0.56415097 ( \cdots )</td>
</tr>
<tr>
<td>13.26322741 ( \cdots )</td>
<td>1.33231317 ( \cdots )</td>
<td>0.30051216 ( \cdots ) - ( i \cdot 0.55357158 ) ( \cdots )</td>
</tr>
</tbody>
</table>
If $\lambda = 3.4362 \cdots$ is the least positive real number such that $\zeta(1/2 + i\lambda) \in \mathbb{R}$, then $\mathcal{C} = \{\zeta(1/2 + it) : -\lambda \leq t \leq \lambda\}$ is a Jordan curve in $\mathbb{C}$ which encloses an open neighborhood of zero, as seen in Figure 2. Applying Lemma 1 we immediately obtain the statement of Theorem 1.

![Figure 2: The Jordan curve $\mathcal{C} = \{\zeta(1/2 + it) : -\lambda \leq t \leq \lambda\}$](image)

In our numerical investigation, we were originally concerned only with intersections between distinct loops $\mathcal{L}_m \neq \mathcal{L}_n$ (see §6 below). However, to have a complete understanding of the repeated values of $\zeta(s)$ on the critical line, one must also consider loops with self-intersections. Initially, we did not expect to find self-intersecting loops other than the loop $\mathcal{L}_0$; however, in studying the question we were led to Theorem 3, which suggests the existence of infinitely many such loops. Using a specialized search we subsequently found the self-intersecting loop $\mathcal{L}_{379}$, which is shown in Figure 3.

![Figure 3: The self-intersecting loop $\mathcal{L}_{379}$](image)
Proof of Theorem 3. As usual, if \( t \) is not the ordinate of a zero of the zeta function, we define \( \arg \zeta(s) = \Im \log \zeta(s) \) by continuous variation from \( \infty + it \) to \( \sigma + it \), and if \( t > 0 \) we denote by \( N(t) \) the number of zeros \( \rho = \beta + i\gamma \) of \( \zeta(s) \) in the rectangle \( 0 < \beta < 1, 0 < \gamma < t \). To simplify the notation somewhat, we put \( \vartheta(t) = \arg \zeta(1/2 + it) \) for any such \( t \).

Let \( b > a \geq 1 \) and suppose that the closed interval \([a, b]\) does not contain the ordinate of a zero of \( \zeta(s) \). We begin with the well known identity (cf. [12, Theorem 14.1])

\[
\vartheta(t) = \pi(N(t) - 1) - \arg \Gamma(1/4 + it/2) + \frac{t}{2} \log \pi.
\]

Since \( N(t) \) is constant on \([a, b]\) it follows that

\[
\vartheta'(t) = -\frac{1}{2} \Re \frac{\Gamma'}{\Gamma}(1/4 + it/2) + \frac{1}{2} \log \pi \quad (t \in [a, b]).
\]

Using the estimate (cf. [12, Theorem C.1])

\[
\frac{\Gamma'}{\Gamma}(s) = \log s + O(1/|s|)
\]

in the special case that \( s = 1/4 + it/2 \), we derive that

\[
\vartheta'(t) = -\frac{1}{2} \log t + \frac{1}{2} \log 2\pi + O(t^{-1}) \quad (t \in [a, b]),
\]

which in turn yields the estimate

\[
\vartheta(b) - \vartheta(a) = -\frac{1}{2}(b - a) \log a + O(b - a). \tag{5}
\]

Note that the implied constants in (4) and (5) are absolute.

Assuming RH we can take \( a \to \gamma_n^+ \) and \( b \to \gamma_{n+1}^- \) in the discussion above. From (5) we see that

\[
\theta_{1,n} - \theta_{2,n} = \frac{1}{2}(\gamma_{n+1}^- - \gamma_n^+) \log \gamma_n + o(1) \quad (n \to \infty), \tag{6}
\]

where

\[
\theta_{1,n} = \lim_{a \to \gamma_n^+} \vartheta(a) \quad \text{and} \quad \theta_{2,n} = \lim_{b \to \gamma_{n+1}^-} \vartheta(b).
\]
Montgomery [11] has shown (under RH) that the number of simple zeros
\[ \rho = 1/2 + i \gamma \] of \( \zeta(s) \) with \( 0 < \gamma \leq T \) is not less than \( (2/3 + o(1))N(T) \); from
this it follows there are at least \( (3+o(1))N(T) \) simple zeros with \( T < \gamma \leq 6T \). Since there are at least as many gaps between zeros as there are simple zeros,
it follows that there are infinitely many \( n \) such that
\[
0 < \gamma_{n+1} - \gamma_n < \frac{5}{3} \cdot \frac{2\pi}{\log \gamma_n}. \tag{7}
\]
For any sufficiently large \( n \) with this property, combining (6) and (7) we deduce that
\[
0 < \theta_{1,n} - \theta_{2,n} < 2\pi,
\]
and (4) implies that \( \vartheta(t) \) is strictly decreasing on the interval \( (\gamma_n, \gamma_{n+1}) \);
consequently, the map
\[
t \mapsto e^{i\vartheta(t)} = \frac{\zeta(1/2 + it)}{|\zeta(1/2 + it)|}
\]
is injective on \( (\gamma_n, \gamma_{n+1}) \), i.e., the loop \( \mathcal{L}_n \) does not intersect itself.

In the other direction, Conrey, Ghosh and Gonek [2] have shown that the
truth of RH implies that
\[
\limsup(\gamma' - \gamma) \frac{\log \gamma}{2\pi} > 2.337
\]
where \( \gamma \leq \gamma' \) are consecutive ordinates of zeros of \( \zeta(s) \). Hence, under RH there are infinitely many \( n \) such that
\[
\gamma_{n+1} - \gamma_n > \frac{7}{3} \cdot \frac{2\pi}{\log \gamma_n}. \tag{8}
\]
If \( n \) has this property and is large enough, combining (6) and (8) we deduce that
\[
\theta_{1,n} - \theta_{2,n} > 2\pi.
\]
As before, using (4) we see that \( \vartheta(t) \) is strictly decreasing on the interval
\( (\gamma_n, \gamma_{n+1}) \) for large \( n \), thus the map \( \vartheta : (\gamma_n, \gamma_{n+1}) \to (\theta_{2,n}, \theta_{1,n}) \) is invertible;
let \( \vartheta^{-1} : (\theta_{2,n}, \theta_{1,n}) \to (\gamma_n, \gamma_{n+1}) \) be the inverse map. Let
\[
f(\theta) = |\zeta(1/2 + i\vartheta^{-1}(\theta))| - |\zeta(1/2 + i\vartheta^{-1}(\theta + 2\pi))| \quad (\theta_{2,n} < \theta < \theta_{1,n} - 2\pi).
\]
Since
\[ f(\theta_{2,n}^+) = -\left| \zeta(1/2 + i\vartheta^{-1}(\theta_{2,n} + 2\pi)) \right| < 0 \]
and
\[ f(\theta_{1,n}^- - 2\pi) = \left| \zeta(1/2 + i\vartheta^{-1}(\theta_{1,n} - 2\pi)) \right| > 0, \]
there is a number \( \theta \in (\theta_{2,n}, \theta_{1,n} - 2\pi) \) such that \( f(\theta) = 0 \); that is,
\[ \left| \zeta(1/2 + it_1) \right| = \left| \zeta(1/2 + it_2) \right|, \]
where
\[ t_1 = \vartheta^{-1}(\theta) \quad \text{and} \quad t_2 = \vartheta^{-1}(\theta + 2\pi). \] (9)
Since
\[ \arg \zeta(1/2 + it_1) = \theta \quad \text{and} \quad \arg \zeta(1/2 + it_2) = \theta + 2\pi, \]
it follows that
\[ \zeta(1/2 + it_1) = \zeta(1/2 + it_2). \]
6 Intersections between distinct loops

In this section, we discuss our results about intersections between distinct loops \( \mathcal{L}_m \neq \mathcal{L}_n \). Figure 4a illustrates the fairly typical situation in which a loop pair \( (\mathcal{L}_m, \mathcal{L}_n) \) has a single intersection. We also found many loop pairs with no intersections; this is illustrated in Figure 4b, which graphs the loops \( \mathcal{L}_n \) with \( n \in \{-64, -43, 1, 2, 3, 4, 8, 16, 33, 53, 55\} \), no two of which intersect.

The next table discloses, for various values of \( N \), the total number of loop pairs \( (\mathcal{L}_m, \mathcal{L}_n) \) with \( -N \leq m < n \leq N \), the overall number of intersections that are found amongst such pairs, and the number of such pairs having precisely 0, 1, 2, 3, 4 or 5 intersections:
We did not encounter any loop pairs with more than five intersections in our limited investigation. Nevertheless, we propose the following conjecture, which is related to Conjecture 2 (see §1).

**Conjecture 3.** Let $\iota_k(N)$ be the number of pairs $(m, n)$, $-N \leq m < n \leq N$, such that the loops $L_m$ and $L_n$ have precisely $k$ intersections. Then, there are constants $N_0(k)$ and $c_k > 0$ such that $\iota_k(N) \geq c_k N$ for all $N \geq N_0(k)$.

The primary aim of our numerical experiment was to gather evidence in support of Conjecture 1 (stated in §1). It is easy to see that if $z \neq 0$ and the equation $\zeta(1/2 + it) = z$ has more than two solutions $t \in \mathbb{R}$, then at least one of the following possibilities must occur:

(i) there is a loop which intersects itself three times at the same point;

![Figure 4: Intersecting and non-intersecting loops](image-url)
(ii) there is a loop which has a self-intersection at a point that also lies on another loop;

(iii) there is a point that lies on three distinct loops.

Figure 5: Intersections of the loops $L_{-19}$, $L_{39}$ and $L_{100}$

Our focus was on the loops $L_n$ with $|n| \leq 100$, and we did not encounter any loops satisfying (i). To eliminate the possibilities (ii) and (iii) within our loop set, we located and precisely evaluated 8933 points of intersection: the four self-intersections on loop $L_0$ together with additional 8929 intersections between distinct loops $L_m \neq L_n$ (see §7 for a description of our methods). We found no instance of a point satisfying either (ii) or (iii). Of the 8933 intersection points we considered, the closest pair is separated by a distance exceeding $5.28687 \times 10^{-7}$, and this pair occurs where $L_{-19}$ intersects $L_{39}$ and $L_{100}$ (see Figure 5). The following table gives information about the relevant points of intersection:

<table>
<thead>
<tr>
<th>Loop pair</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$\zeta(1/2 + it_1)$ and $\zeta(1/2 + it_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>($L_{-19}, L_{39}$)</td>
<td>$-76.38206310\cdots$</td>
<td>$121.71273069\cdots$</td>
<td>$0.61023434\cdots + i \cdot 0.16504225\cdots$</td>
</tr>
<tr>
<td>($L_{-19}, L_{100}$)</td>
<td>$-76.38206243\cdots$</td>
<td>$236.70765230\cdots$</td>
<td>$0.61023446\cdots + i \cdot 0.16504173\cdots$</td>
</tr>
<tr>
<td>($L_{39}, L_{100}$)</td>
<td>$121.71273203\cdots$</td>
<td>$236.70765293\cdots$</td>
<td>$0.61023638\cdots + i \cdot 0.16504150\cdots$</td>
</tr>
</tbody>
</table>

Since $\mathscr{L}_n = L_{-n}$ for every $n \in \mathbb{Z}$, the loops $L_n$ and $L_{-n}$ have a real point of intersection whenever $L_n$ crosses the real axis; these were first studied
by Gram [5] and are called Gram points. Of the 8933 intersection points considered, only the Gram points were found to lie on the real axis, although some points of intersection lie quite close to the real axis (see Figure 6). We propose the following conjecture, which is consequence of Conjecture 1.

**Conjecture 4.** If $\mathcal{L}_m$ and $\mathcal{L}_n$ are distinct loops with a point of intersection on the real axis, then $m = -n$.

To finish this section, let us now turn to the proof of Theorem 2.

**Proof of Theorem 2.** We first observe that the loops $\mathcal{L}_{126}$ and $\mathcal{L}_{-126}$ have two intersections, which are given in the following table:

<table>
<thead>
<tr>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$\zeta(1/2 + it_1)$ and $\zeta(1/2 + it_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>280.80242937 $\cdots$</td>
<td>-280.80242937 $\cdots$</td>
<td>7.00315163 $\cdots$</td>
</tr>
<tr>
<td>282.45472082 $\cdots$</td>
<td>-282.45472082 $\cdots$</td>
<td>280.80242937 $\cdots$</td>
</tr>
</tbody>
</table>

Let $\lambda_1 = 280.8024 \cdots$ and $\lambda_2 = 282.4547 \cdots$ be the two real numbers $t_1$ in the open interval $(\gamma_{126}, \gamma_{127})$ such that $\zeta(1/2 + it_1) \in \mathbb{R}$. If $\mathcal{U} = [\lambda_1, \lambda_2] \cup [-\lambda_2, -\lambda_1]$, then one verifies that $\mathcal{C} = \{\zeta(1/2 + it) : t \in \mathcal{U}\}$ is a Jordan curve in $\mathbb{C}$ which encloses an open neighborhood of zero. Applying Lemma 2 we see that $\mathcal{C} \cap \mathcal{L}_n \neq \emptyset$ for infinitely many $n \in \mathbb{Z}$. As $\mathcal{C}$ is a subset of $\mathcal{L}_{126} \cup \mathcal{L}_{-126}$, it follows that one of the two cases

(i) $\mathcal{L}_{126} \cap \mathcal{L}_n \neq \emptyset$

(ii) $\mathcal{L}_{-126} \cap \mathcal{L}_n \neq \emptyset$
occurs for infinitely many \( n \). However, since \( \mathcal{L}_{126} \cap \mathcal{L}_n \neq \emptyset \) if and only if \( \mathcal{L}_{-126} \cap \mathcal{L}_{-n} \neq \emptyset \), both cases (i) and (ii) must occur for infinitely many \( n \), which finishes the proof of Theorem 2.

\[ \square \]

7 Description of numerical methods

In this section we briefly describe our method for computing intersections between loops. Our computations were performed using Mathematica, which we selected for its ease of use, its built-in library, and its display capabilities.

The built-in function \texttt{FindRoot} in Mathematica is exceedingly convenient for numerically evaluating intersections between loops (i.e., repeated values of the zeta function) to any desired level of precision. However, to insure that no intersections were missed among the 20000+ loop pairs under consideration, and in order to automate our use of \texttt{FindRoot}, we first needed to find crude approximations for the locations of the intersections. To do so, our basic data object was an ordered quadruple of real numbers called a \texttt{quad}. Each quad was given in the form of a list \( Q = \{t_{1,b}, t_{1,e}, t_{2,b}, t_{2,e}\} \) and represented two arcs on the graph of \( \zeta(s) \) defined by

\begin{align*}
\mathcal{A}_1 &= \{ \zeta(1/2 + it) : t_{1,b} < t < t_{1,e} \}, \\
\mathcal{A}_2 &= \{ \zeta(1/2 + it) : t_{2,b} < t < t_{2,e} \}.
\end{align*}

For a given loop pair \( (\mathcal{L}_m, \mathcal{L}_n) \) with \( m \neq n \) we defined an initial quad \( Q \) by taking

\begin{align*}
t_{1,b} &= \tau_m + \delta = \gamma_m + \delta, & t_{1,e} &= \tau_{m+1} - \delta = \gamma_{m+1} - \delta, \\
t_{2,b} &= \tau_n + \delta = \gamma_n + \delta, & t_{2,e} &= \tau_{n+1} - \delta = \gamma_{n+1} - \delta,
\end{align*}

where \( \delta > 0 \) was a predetermined parameter. Note that \( \mathcal{A}_1 \approx \mathcal{L}_m \) and \( \mathcal{A}_2 \approx \mathcal{L}_n \) when \( \delta \) is small. In our computation, the value of \( \delta \) was chosen to be small enough so that all intersections between the loops \( \mathcal{L}_m \) and \( \mathcal{L}_n \) would still be present as intersections between the arcs \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \). On the other hand, \( \delta \) was large enough so that the \texttt{FindRoot} command, once it was invoked, would not converge to the value zero, i.e., to the limiting value of \( \zeta(s) \) near the boundaries of the intervals \( (\tau_m, \tau_{m+1}) \) and \( (\tau_n, \tau_{n+1}) \).

Once defined, the initial quad was placed into a list (containing only the one quad), the following cycle was performed twenty times:
(i) Each quad $Q = \{t_{1,b}, t_{1,e}, t_{2,b}, t_{2,e}\}$ in the current list was split into four distinct subquads by splitting both arcs $A_1$ and $A_2$ into two pieces:

$$Q_1 = \{t_{1,b}, t_{1,m}, t_{2,b}, t_{2,m}\}, \quad Q_2 = \{t_{1,b}, t_{1,m}, t_{2,m}, t_{2,e}\},$$
$$Q_3 = \{t_{1,m}, t_{1,e}, t_{2,b}, t_{2,m}\}, \quad Q_4 = \{t_{1,m}, t_{1,e}, t_{2,m}, t_{2,e}\},$$

where

$$t_{1,m} = \frac{1}{2} (t_{1,b} + t_{1,e}) \quad \text{and} \quad t_{2,m} = \frac{1}{2} (t_{2,b} + t_{2,e}).$$

(ii) For each subquad $Q_j$ a crude (and fast) test was used to determine whether the arcs $A_{1,j}$ and $A_{2,j}$ represented by $Q_j$ were close enough so that an intersection might be possible; if not, the subquad was eliminated from further consideration. Although this particular test allowed for false positives, it proved to be fairly effective in practice.

(iii) All of the subquads which survived step (ii) were collected into a new list for processing during the next cycle.

In principle, at the end of this twenty-cycle procedure, every remaining quad would give rise to a point of intersection for the loops $L_m$ and $L_n$. Our verification procedure can be summarized as follows:

(i) For each quad $Q$ we considered not only the arcs $A_1$ and $A_2$ defined by (10) but also the line segments connecting the endpoints of the arcs, namely,

$$L_1 = \{L_1(t) : t_{1,b} < t < t_{1,e}\} \quad \text{and} \quad L_2 = \{L_2(t) : t_{2,b} < t < t_{2,e}\},$$

where

$$L_j(t) = \zeta(1/2 + it_{j,b}) \frac{t - t_{j,e}}{t_{j,b} - t_{j,e}} + \zeta(1/2 + it_{j,e}) \frac{t_{j,b} - t}{t_{j,b} - t_{j,e}} \quad (j = 1, 2).$$

(ii) To find an intersection of $A_1$ and $A_2$, we used first found numbers $t_1^*$ and $t_2^*$ such that $L_1(t_1^*) = L_2(t_2^*)$; these numbers became our initial “guess” when using the command `FindRoot` to locate an intersection between $A_1$ and $A_2$. If `FindRoot` returned a values $t_1, t_2$ such that $t_1 \not\in (t_{1,b}, t_{1,e})$ or $t_2 \not\in (t_{1,b}, t_{1,e})$, then this step of the algorithm returned with FAIL (in this case, the quad was not eliminated, but instead it was subjected to more iterations of the cycling process described earlier); otherwise, the algorithm proceeded to step (iii).
In this step, a special routine was used to eliminate the possibility of multiple intersections between the arcs $A_1$ and $A_2$. When a multiple intersection was deemed possible, this step of the algorithm returned with \texttt{FAIL} (and as before, this quad would then be subjected to further iterations of the cycling process).

In this manner, every quad which did not fail the verification procedure gave rise to a pair of numbers $t_1, t_2$ for which $\zeta(1/2 + t_1) = \zeta(1/2 + it_2) \neq 0$.

The same techniques were used to find self-intersections, but the initial quad was defined in a slightly different way before the cycling process began.

8 Concluding remarks

Using analytic properties of $\vartheta(t) = \arg \zeta(1/2 + it)$ it should be possible to prove that Conjecture 1 holds with only countably many exceptions.

Let $\mathcal{R} = \{\sigma + it : |\sigma - 1/2| \leq \delta, \ 1 \leq t \leq T\}$. Levinson [10] has shown that the number of solutions in $\mathcal{R}$ to the equation $\zeta(\sigma + it) = z$ is equal to $(T/2\pi) \log T + O_{\delta}(T)$, whereas the number of solutions with $|\sigma - 1/2| > \delta$ is $O_{\delta}(T)$. The latter result can be improved if and only if $z = 0$, which shows that the clustering of the zeros of $\zeta(s)$ near the critical line is more pronounced than the clustering of $z$-values for any $z \neq 0$. This can be viewed as weak evidence for the conjecture (1), which asserts that the equation $\zeta(1/2 + it) = z$ has infinitely many solutions if and only if $z = 0$.

It would be interesting to see whether our numerical investigation could be performed on a much larger scale to obtain more compelling evidence in support of our conjectures. We leave this project to the interested reader!

References


