Congruences and exponential sums with the sum of aliquot divisors function

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Abstract

We give bounds on the number of integers $1 \leq n \leq N$ such that $p \mid s(n)$, where $p$ is a prime and $s(n)$ is the sum of aliquot divisors function given by $s(n) = \sigma(n) - n$, where $\sigma(n)$ is the sum of divisors function. Using this result we obtain nontrivial bounds in certain ranges for rational exponential sums of the form

$$S_p(a, N) = \sum_{n \leq N} \exp(2\pi i a s(n)/p), \quad \gcd(a, p) = 1.$$ 

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1 Introduction

For every positive integer $n$, let $s(n)$ be the sum of the aliquot divisors of $n$:

$$s(n) = \sum_{d \mid n; \; d \neq n} d = \sigma(n) - n,$$

where $\sigma(n)$ is the sum of divisors function. In this paper we consider arithmetic properties of the aliquot sequence $(s(n))_{n \geq 1}$. In particular, for a fixed prime $p$ we obtain nontrivial upper bounds in certain ranges for exponential sums of the form:

$$S_p(a, N) = \sum_{n=1}^{N} e_p(2\pi i a s(n)/p), \quad (a \in \mathbb{Z}, \; N \geq 1),$$

where

$$e_p(x) = \exp(2\pi i x/p) \quad (x \in \mathbb{R}).$$

Our results for the sums $S_p(a, N)$ rely on upper bounds for the cardinalities $\#T_p(N)$ of the sets

$$T_p(N) = \{1 \leq n \leq N \mid s(n) \equiv 0 \pmod{p} \} \quad (N \geq 1).$$

We remark that analogous results for the Euler function $\varphi(n)$ have been obtained in [1, 2, 3], and we apply similar methods in the present paper. Various modifications are needed, however, since $s(n)$ is not a multiplicative function.
Theorem 1. For \( v = (\log N)/(\log p) \to \infty \), the following bound holds:

\[
\#T_p(N) \ll Nv^{-u/2+o(v)} + \frac{Nv}{p}.
\]

Using this result we show:

Theorem 2. The following bound holds:

\[
\max_{\gcd(a,p)=1} |S_p(a,N)| \ll N \left( \frac{\log^4 N}{p^{1/2}} + \frac{\log p \log \log N}{\log \log \log N} \right).
\]

In the statements above and throughout the paper, any implied constants in the symbols \( \ll, \gg \) and \( O \) are absolute unless indicated otherwise. We recall that for positive functions \( F \) and \( G \) the notations \( F = O(G) \), \( F \ll G \) and \( G \gg F \) are all equivalent to the assertion that the inequality \( F \leq cG \) holds for some constant \( c > 0 \).

Throughout the paper, the letters \( p, q \) are used to denote prime numbers, and \( m, n \) are positive integers.

2 Preliminaries

Let \( P(n) \) be the largest prime factor of an integer \( n \geq 2 \), and put \( P(1) = 1 \). An integer \( n \geq 1 \) is said to be \( y \)-smooth if \( P(n) \leq y \). As usual, we define

\[ \psi(x, y) = \# \{ n \leq x : n \text{ is } y\text{-smooth} \} \quad (x \geq y > 1). \]

The following bound is a relaxed and simplified version of [7, Corollary 1.3] (see also [4]):

Lemma 3. For \( u = (\log x)/(\log y) \to \infty \) with \( u \leq y^{1/2} \), we have

\[ \psi(x, y) \ll xu^{-u+o(u)}. \]

The next statement is a simplified form of the Brun-Titchmarsh theorem; see, for example, [5, Section 2.3.1, Theorem 1] or [6, Chapter 3, Theorem 3.7].

Lemma 4. Let \( \pi(x; k, a) \) be the number of primes \( p \leq x \) such that \( p \equiv a \pmod k \). Then, for any \( x > k \) we have

\[ \pi(x; k, a) \ll \frac{x}{\varphi(k) \log(2x/k)}. \]
Finally, our principal tool is the following bound for exponential sums with prime numbers, which follows immediately from Theorem 2 of [8].

**Lemma 5.** For any prime $p$ and real number $x \geq 2$, the following bound holds:

$$\max_{\gcd(a,p)=1} \left| \sum_{q \leq x} e_p(aq) \right| \ll (p^{-1/2} + x^{-1/4} p^{1/8} + x^{-1/2} p^{1/2}) x \log^3 x.$$

## 3 Proof of Theorem 1

We can assume that $v \leq p$ since the result is trivial otherwise. Thus, taking

$$u = \frac{v}{2} = \frac{\log N}{2 \log p},$$

we see that

$$2u \log u \leq v \log p = \log N.$$

Defining the smoothness bound $K = N^{1/u} = p^2$, it follows that $u \leq K^{1/2}$. In particular, if $\mathcal{E}_1$ is the set of integers $n \leq N$ such that $n$ is $K$-smooth, then we can apply Lemma 3 to derive the bound

$$\# \mathcal{E}_1 = \psi(N, K) \ll Nu^{-u+o(u)} = N v^{-v/2+o(v)}.$$

Next, let $\mathcal{E}_2$ be the set of integers $n \leq N$ such that $q^2 \mid n$ for some prime $q > K$. Then,

$$\# \mathcal{E}_2 \leq \sum_{q > K} \sum_{n \leq N \atop q^2 \mid n} 1 \leq \sum_{q > K} N/q^2 \ll N/K \leq N/p^2.$$

Finally, let $\mathcal{E}_3$ be the set of integers $n \leq N$ which are multiples of $p$. Then,

$$\# \mathcal{E}_3 = \lfloor N/p \rfloor \leq N/p.$$

Now let $\mathcal{N} = \{1, \ldots, N\} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3)$. Using the bounds established above, it follows that

$$\# \mathcal{T}_p(N) \ll N v^{-v/2+o(v)} + N/p + \# (\mathcal{T}_p(N) \cap \mathcal{N}). \quad \text{(1)}$$
For any \( n \in \mathcal{T}_p(N) \cap \mathcal{N} \), we write \( n = mq \), where \( q = P(n) > P(m) \). Since \( s(n) = \sigma(n) - n \), and \( \sigma(n) \) is multiplicative, the condition \( s(n) \equiv 0 \pmod{p} \) implies
\[
mq \equiv \sigma(mq) \equiv \sigma(m)(q + 1) \pmod{p}.
\]
Then \( \sigma(m) \not\equiv 0 \pmod{p} \) since \( p \nmid n \), hence the same relation also implies that \( \sigma(m) \not\equiv m \pmod{p} \); consequently, \( q \equiv a_m \pmod{p} \) for any integer \( a_m \equiv \sigma(m)(m - \sigma(m))^{-1} \pmod{p} \). Since \( q > K \) we see that
\[
\#(\mathcal{T}_p(N) \cap \mathcal{N}) \leq \sum_{\sigma(m) \not\equiv m \pmod{p}} \sum_{q \equiv a_m \pmod{p}} 1.
\]
For the inner sum, we have by Lemma 4:
\[
\sum_{K < q \leq N/m \atop q \equiv a_m \pmod{p}} 1 \leq \frac{N}{mp \log(2N/mp)} \leq \frac{N}{mp \log(2K/p)} \leq \frac{N}{mp \log K},
\]
where we have used the inequality \( p \leq K^{1/2} \) in the last step. Therefore,
\[
\#(\mathcal{T}_p(N) \cap \mathcal{N}) \ll \frac{N}{p \log K} \sum_{m \leq N/K} \frac{1}{m} \ll \frac{N \log(N/K)}{p \log K} \ll \frac{Nu}{p} \ll \frac{Nv}{p}.
\]
Inserting this bound into (1), we obtain the desired result.

4 Proof of Theorem 2

We can assume that \( p \geq \log^8 N \) and that \( v = (\log N)/(\log p) \to \infty \) as \( N \to \infty \) since the result is trivial otherwise.

Let \( u, K \) and the sets \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \) be defined as in the proof of Theorem 1. Then,
\[
\#(\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3) \ll Nv^{-v/2+o(v)} + N/p.
\]
Also, put \( M = N^{1/w} \), where \( w \geq 2 \) is a parameter to be specified later, and let \( \mathcal{E}_4 \) be the set of integers \( n \leq N \) for which \( P(n) \geq n/M \). Every integer \( n \in \mathcal{E}_4 \) can factored as \( n = mq \), where
\[
P(m) \leq P(n) = q \leq N/m \quad \text{and} \quad m \leq M.
\]
Therefore,
\[
\#E_4 \leq \sum_{m \leq M} \sum_{q \leq N/m} 1 \ll \sum_{m \leq M} \frac{N/m}{\log(N/m)} \ll \frac{N}{\log N} \sum_{m \leq M} \frac{1}{m} \ll \frac{N \log M}{\log N} = \frac{N}{w}.
\]

Now let \( N = \{1, \ldots, N\} \setminus (E_1 \cup E_2 \cup E_3 \cup E_4) \). From the bounds above it follows that
\[
S_p(a, N) = \sum_{n \in N} e_p(as(n)) + O(N(v^{-v/2+o(v)} + p^{-1} + w^{-1})). \tag{2}
\]

Every integer \( n \in N \) can be uniquely represented in the form \( n = mq \), where \( M < m < N/K \) and \( \max\{K, P(m)\} < q \leq N/m \).

Conversely, if the numbers \( m, q \) satisfy these inequalities, then \( n = mq \) lies in \( N \). Observing that \( s(mq) = s(m)q + \sigma(m) \), we have
\[
\sum_{n \in N} e_p(as(n)) = \sum_{M < m < N/K} \sum_{L_m < q \leq N/m} e_p(as(mq)) = \Sigma_1 + \Sigma_2, \tag{3}
\]
where \( L_m = \max\{K, P(m)\} \), and
\[
\Sigma_1 = \sum_{M < m < N/K} e_p(a\sigma(m)) \sum_{p \nmid s(m)} e_p(as(m)q),
\]
\[
\Sigma_2 = \sum_{M < m < N/K} e_p(a\sigma(m)) \sum_{L_m < q \leq N/m} 1.
\]

Write
\[
\sum_{L_m < q \leq N/m} e_p(as(m)q) = \sum_{q \leq N/m} e_p(as(m)q) - \sum_{q < L_m} e_p(as(m)q),
\]
and observe that the right side of the bound in Lemma 5 is a monotonically increasing function of \( x \); thus, if \( p \nmid s(m) \) we have
\[
\sum_{L_m < q \leq N/m} e_p(as(m)q) \ll (p^{-1/2} + (N/m)^{-1/4}p^{1/8} + (N/m)^{-1/2}p^{1/2}) \frac{N \log^3 N}{m}.
\]
For \( m < N/K = N/p^2 \) the first term inside the parentheses dominates the other two; therefore,

\[
\Sigma_1 \ll \frac{N \log^3 N}{p^{1/2}} \sum_{M < m < N/K \atop p | s(m)} \frac{1}{m} \ll \frac{N \log^4 N}{p^{1/2}}.
\]  

(4)

Next, we turn our attention to the problem of bounding \( \Sigma_2 \). Writing

\[ I = \lfloor \log M \rfloor + 1 \quad \text{and} \quad J = \lfloor \log(N/K) \rfloor + 1, \]

we have trivially:

\[
\Sigma_2 \ll N \sum_{M < m < N/K \atop p | s(m)} \frac{1}{m} \ll N \sum_{j=1}^{J} \sum_{m \leq e^j \atop p | s(m)} \frac{1}{m} \ll N \sum_{j=1}^{J} e^{-j} \sum_{m \leq e^j \atop p | s(m)} 1 = N \sum_{j=1}^{J} e^{-j} \# \pi_p(e^j).
\]

Define

\[
v_j = \frac{j}{\log p} \quad (I \leq j \leq J),
\]

and note that

\[
\frac{v}{w} = \frac{\log M}{\log p} < v_j = \frac{j}{\log p} \leq \frac{\log N + 1}{\log p} \ll w \quad (I \leq j \leq J).
\]

Thus if

\[
v/w \to \infty
\]

then Theorem 1 implies that

\[
e^{-j} \# \pi_p(e^j) \ll v_j^{-v_j/2+o(v_j)} + \frac{v_j}{p}.
\]

Hence,

\[
\Sigma_2 \ll N \sum_{j=1}^{J} \left( v_j^{-v_j/2+o(v_j)} + \frac{v_j}{p} \right) \ll N \left( (v/w)^{-v/(2w)+o(v/w)} + \frac{w}{p} \right) \log N.
\]
Now, combining the previous bound with (2), (3) and (4), and dropping terms which are clearly dominated by other terms, it follows that

$$\frac{S_p(a,N)}{N} \ll \frac{\log^4 N}{p^{1/2}} + w^{-1} + (v/w)^{-v/(2w)+o(v/w)} \log N + \frac{w \log N}{p}. \quad (6)$$

Note that the last term in this bound can also be dropped. Indeed, we can assume that $w \leq v$, for otherwise the bound is trivial, and thus

$$\frac{w \log N}{p} \leq \frac{v \log N}{p} = \frac{\log^2 N}{p \log p} \leq \frac{\log^4 N}{p^{1/2}}.$$

We now choose

$$w = \frac{v \log \log \log N}{6 \log \log N}$$

to (essentially) balance the middle two terms in (6). We also note that the condition (5) is satisfied. With this choice of $w$, it is easily seen that

$$(v/w)^{-v/(2w)+o(v/w)} \log N = (\log N)^{-2+o(1)} \ll (\log N)^{-3/2},$$

whereas for $p \geq \log^8 N$ we have

$$w^{-1} = \frac{6 \log p}{\log N \log \log \log N} \gg (\log N)^{-3/2}.$$

Therefore, the third term in (6) can be dropped, and the result follows.

References


