HEREDITY OF WHITTAKER MODELS
ON THE METAPLECTIC GROUP
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In this paper, Rodier’s theorem on the heredity of Whittaker models is generalized to
non-algebraic setting of the n-fold metaplectic cover of the general linear group \( GL_r(F) \),
where \( F \) is nonarchimedean local field containing the \( n \)-th roots of unity.

§1. Introduction.

Let \( F \) be a nonarchimedean local field, let \( G \) be the general linear group \( GL_r(F) \)
for some positive integer \( r \), and let \( P \) be a standard parabolic subgroup of \( G \) with Levi
component \( M \). Given an admissible representation \( \pi_M \) of \( M \), extend \( \pi_M \) to a representation
\( \pi_P \) of \( P \) by letting the unipotent radical of \( P \) act trivially, and let \( \pi_G \) be the normalized
full-induced representation \( \text{Ind}(P, G; \pi_P) \). Then by a well-known result of F. Rodier, there
exists a correspondence between the Whittaker models of the induced representation \( \pi_G \)
and the Whittaker models of the inducing representation \( \pi_M \) (cf. Theorem 2 of [4]).

In this paper, Rodier’s theorem on the “heredity” of Whittaker models is extended
to the non-algebraic setting of the \( n \)-fold metaplectic cover \( \tilde{G} \) of \( G \), where \( n \) is a positive
integer such that \( F \) contains all of the \( n \)-th roots of unity. The main result is stated as a
theorem in §2. In order to illustrate the situation, consider the example of a representation
of \( \tilde{G} \) induced from the metaplectic preimage \( \tilde{B} \) of the standard Borel subgroup \( B \) of \( G \).
Since the Levi component \( T \) of \( B \) is a (maximal) torus in \( G \), its metaplectic preimage \( \tilde{T} \) is a
Heisenberg group. Consequently, the dimension of any irreducible representation \( \pi_{\tilde{T}} \)
of \( \tilde{T} \) is equal to the index \([\tilde{T}:\tilde{T}_*]\), where \( \tilde{T}_* \) is an arbitrary maximal abelian subgroup of \( \tilde{T} \). In this
example, every linear functional on the space of \( \pi_{\tilde{T}} \) is a Whittaker functional, hence the
inducing representation \( \pi_{\tilde{T}} \) has precisely \([\tilde{T}:\tilde{T}_*]\) distinct Whittaker models. Now extend
\( \pi_{\tilde{T}} \) to a representation \( \pi_{\tilde{B}} \) of \( \tilde{B} \) (see §2 below), and let \( \pi_G \) be the normalized, full-induced
representation \( \text{Ind}(\tilde{B}, \tilde{G}; \pi_{\tilde{B}}) \) of \( \tilde{G} \). By Lemma I.3.2 of [3], it follows that \( \pi_G \) also has \([\tilde{T}:\tilde{T}_*]\)
distinct Whittaker models, thus Rodier’s theorem evidently extends to this example.

While the main techniques of proof employed in this paper are contained in [4], it is unclear a priori that those techniques carry over to the metaplectic group. Here the situation is clarified by a close examination of various aspects of Rodier’s proof.

The results of this paper will be relevant to the generalization of F. Shahidi’s theory of local coefficients (cf. [5]) to the metaplectic setting, to the construction of certain non-principal theta functions (cf. [3]), and to the eventual classification of metaplectic representations.

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§2. The Metaplectic Group, Whittaker Models, and Heredity.

Let $n$ and $r$ be fixed positive integers. Let $F$ be a nonarchimedean local field, and let $\mu_n$ denote the group of $n$-th roots of unity in $F$. *We will assume that $\mu_n$ has cardinality $n$.*

Let $\tilde{G}$ denote the $n$-fold metaplectic cover of $G := GL_r(F)$ (cf. §0.I of [3]). As a set, $\tilde{G} = G \times \mu_n$, with multiplication defined by:

$$(g, \zeta) \cdot (g', \zeta') = (gg', \zeta \zeta' \sigma(g, g')),$$

$\forall g, g' \in G, \zeta, \zeta' \in \mu_n$.

Here $\sigma : G \times G \to \mu_n$ is the Matsumoto 2-cocycle in $H^2(G; \mu_n)$. Let $s : G \to \tilde{G}$ be the preferred section $g \mapsto (g, 1)$, and let $p : \tilde{G} \to G$ be the canonical projection $(g, \zeta) \mapsto g$.

Let $N$ be the unipotent radical of the standard Borel subgroup $B$ of $G$, and let $N^s := s(N)$. Since $\sigma|_{N \times N} = 1$, $s : N \to N^s$ is a group isomorphism. Once and for all, let $\psi$ be a fixed principal character of $N$ (cf. §2 of [4]). Then for any positive simple root $\alpha$ of $G$, the restriction of $\psi$ to the unipotent root group $N_\alpha$ is nontrivial. Let $\psi^*$ denote the corresponding character $\psi \circ p = \psi \circ s^{-1}$ of $N^s$, and let $\bar{\psi}^*$ be the character obtained from $\psi^*$ by complex conjugation.

Let $W$ be the Weyl group of permutation matrices in $G$, and let $W^s := s(W)$. If the $n$-th order Hilbert symbol $(\cdot, \cdot)_F : \mathbb{F}_n^x \times \mathbb{F}_n^x \to \mu_n$ satisfies the relation $(-1, -1)_F = 1$, then $s : W \to W^s$ is a group isomorphism, but we will proceed without this assumption.
Let $M \subseteq G$ be the Levi component of an arbitrary standard parabolic subgroup $P$ of $G$. Then $P = MU$, and $M \cap U = \{e\}$, where $U \subseteq N$ is the unipotent radical of $P$, and $e$ is the identity of $G$. Let $\tilde{P}$, $\tilde{M}$, and $U^s$ denote the subgroups $p^{-1}(P)$, $p^{-1}(M)$, and $s(U)$ of $\tilde{G}$, respectively. Then $\tilde{P} = \tilde{M}U^s$, and $\tilde{M} \cap U^s = \{\tilde{e}\}$, where $\tilde{e}$ is the identity of $\tilde{G}$.

Let $W_M := W \cap M$. As in §1 of [2], let:

$$[W/W_M] := \{w \in W \mid w \text{ is of minimal length in } wW_M \in W/W_M\},$$

and let $w_M$ denote the longest element of $[W/W_M]$. Let $N_M := N \cap M$, and let $N_M^s := s(N_M)$. Since $w_M N_M w_M^{-1} \subseteq N$, we can define a character $\psi_M^s : N_M^s \to \mathbb{C}^\times$ by:

$$\psi_M^s(n) := \psi\left(w_M p(n) w_M^{-1}\right), \quad \forall n \in N_M^s.$$ 

Let $\overline{\psi}_M^s$ be the character obtained from $\psi_M^s$ by complex conjugation. In particular, we have that $N_G^s = N^s$, and $w_G = e$, hence $\psi_G^s = \psi^s$, and $\overline{\psi}_G^s = \overline{\psi}^s$.

Let $\Pi_M^\psi$ denote the full-induced representation $\text{Ind}(N_M^s, \tilde{M}; \psi_M^s)$ of $\tilde{M}$. The space $\mathcal{W}_M^\psi$ of $\Pi_M^\psi$ consists of the locally-constant functions $f : \tilde{M} \to \mathbb{C}$ that satisfy $f(ng) = \psi_M^s(n) f(g)$ for all $n \in N_M^s$ and $g \in \tilde{M}$, and $\tilde{M}$ acts on $\mathcal{W}_M^\psi$ by right translation. Similarly, let $\circ \Pi_M^\psi$ denote the compactly-induced representation $\text{Ind}^c(N_M^s, \tilde{M}; \overline{\psi}_M^s)$ of $\tilde{M}$. The space $\circ \mathcal{W}_M^\psi$ of $\circ \Pi_M^\psi$ consists of the locally-constant functions $f : \tilde{M} \to \mathbb{C}$ that are compactly supported modulo $N_M^s$ and satisfy $f(ng) = \overline{\psi}_M^s(n) f(g)$ for all $n \in N_M^s$ and $g \in \tilde{M}$. Then $\tilde{M}$ also acts on $\circ \mathcal{W}_M^\psi$ by right translation. By Proposition 2.25(c) of [1], $\circ \Pi_M^\psi$ is the contragredient of the representation $\Pi_M^\psi$.

Let $\pi_M$ be a smooth representation of $\tilde{M}$. A subspace $\mathcal{W}$ of $\mathcal{W}_M^\psi$ is said to be a $\psi_M^s$-Whittaker model for $\pi_M$ if $\mathcal{W}$ is $\tilde{M}$-invariant, and the restriction of $\Pi_M^\psi$ to $\mathcal{W}$ is a representation that is equivalent to $\pi_M$. In other words, $\mathcal{W}$ is the image of an injective element of $\text{Hom}_M(\pi_M, \Pi_M^\psi)$.

The following theorem is a generalization to the metaplectic group of Rodier’s theorem on the heredity of Whittaker models (cf. Theorem 2 of [4]).
Theorem. Let $\pi_m$ be an admissible representation of $\tilde{M}$. Extend $\pi_m$ to a representation $\pi_p$ of $\tilde{P}$ by letting $U^s$ act trivially, and let $\pi_G$ be the normalized induced representation $\text{Ind}(\tilde{P}, \tilde{G}; \pi_p)$ of $\tilde{G}$. Then $\text{Hom}_{\tilde{G}}(\pi_G, \Pi_\psi) \cong \text{Hom}_{\tilde{M}}(\pi_m, \Pi_\psi)$.

Proof: A topological space $X$ is an $l$-space if it is Hausdorff, locally-compact, and zero-dimensional (cf. §1.1 of [1]). For any $l$-space $X$ and any complex vector space $V$, let $\mathcal{S}(X; V)$ denote the space of locally-constant, compactly-supported functions from $X$ to $V$, and let $\mathcal{D}(X; V)$ be the linear dual of $\mathcal{S}(X; V)$. When $V = \mathbb{C}$, we will simply write $\mathcal{S}(X)$ and $\mathcal{D}(X)$, respectively. Any element of $\mathcal{D}(X; V)$ [resp. $\mathcal{D}(X)$] is called a $V$-distribution [resp. distribution].

Let $\mathcal{V}$ denote the space of $\pi_m$. For any $l$-subspace $X$ of $\tilde{G}$ such that $N^sX \tilde{P} = X$, let $\mathcal{D}^1(X)$ denote the space of $\mathcal{V}$-distributions $D \in \mathcal{D}(X; \mathcal{V})$ that satisfy:

$$D(\lambda_1^1(n)p^1_1(p)\varphi) = \tilde{\psi}^*(n)\delta(p)^{-1/2}D(\pi_p(p^{-1})\circ\varphi), \quad \forall n \in N^s, p \in \tilde{P}, \varphi \in \mathcal{S}(X; V).$$

Here $\lambda_1^1 : N^s \rightarrow \text{Aut}(\mathcal{S}(X; \mathcal{V}))$ and $p^1_1 : \tilde{P} \rightarrow \text{Aut}(\mathcal{S}(X; \mathcal{V}))$ are the representations defined in the usual way by left and right translation, respectively, and $\delta : \tilde{P} \rightarrow \mathbb{C}^\times$ is the modular character of $\tilde{P}$.

By a theorem of F. Bruhat (cf. Theorem 4 of [4]), $\mathcal{D}^1(\tilde{G})$ is isomorphic to the space $\text{Bil}_{\tilde{G}}(\gamma\Pi_\psi, \pi_G)$ of $\tilde{G}$-invariant bilinear forms on $^0\mathcal{W}_G^{\psi} \times \mathcal{V}$ (i.e., intertwining forms in the sense of §1 of [4]). Here we have used the fact that $\tilde{G} = \tilde{P}K$ for some compact, open subset $K$ of $\tilde{G}$, and that $\tilde{\psi}^* = \tilde{\psi}^*$. Since $^0\Pi_\psi$ is the contragredient of the representation $\Pi_\psi$, it can also be shown that $\text{Hom}_{\tilde{G}}(\pi_G, \Pi_\psi) \cong \text{Bil}_{\tilde{G}}(\gamma\Pi_\psi, \pi_G)$. Hence, $\mathcal{D}^1(\tilde{G}) \cong \text{Hom}_{\tilde{G}}(\pi_G, \Pi_\psi)$, and it remains to show that $\mathcal{D}^1(\tilde{G}) \cong \text{Hom}_{\tilde{M}}(\pi_m, \Pi_\psi)$.

For every $w \in [W/W_M]$, let $\tilde{w} := s(w)$. Starting from the Bruhat decomposition for $G$, one can show that $\tilde{G} = \bigsqcup_{\tilde{w}} N^s\tilde{w}\tilde{P}$, where the disjoint union is taken over all $w \in [W/W_M]$ (cf. §1 of [2]). In order to describe $\mathcal{D}^1(\tilde{G})$, it will suffice to study each space $\mathcal{D}^1(N^s\tilde{w}\tilde{P})$ separately. Thus, let $w$ be a fixed element of $[W/W_M]$. For every $\varphi \in \mathcal{S}(N^s\tilde{w}\tilde{P}; \mathcal{V})$, let $\tilde{\varphi} \in \mathcal{S}(N^s\tilde{w}\tilde{P}; \mathcal{V})$ be defined by:

$$\tilde{\varphi}(n\tilde{w}p) := \int_{N^s \cap \tilde{w}\tilde{P}\tilde{w}^{-1}} \varphi(nn_\circ, \tilde{w}^{-1}n_\circ^{-1}\tilde{w}p) \, dn_\circ, \quad \forall n \in N^s, p \in \tilde{P},$$
where $dn_\circ$ is a Haar measure for $N^s \cap \tilde{w}\tilde{P}\tilde{w}^{-1}$. The map $\varphi \mapsto \varphi$ is surjective, hence by duality it follows that $D^1(N^s\tilde{w}\tilde{P})$ is isomorphic to the space $D^2$ of $\mathcal{V}$-distributions $D \in D(N^s \times \tilde{P}; \mathcal{V})$ that satisfy:

$$D(\lambda_n^3(\varphi) \rho^3_n(p) \varphi) = \psi^*(n) \delta(p)^{-1/2} D(\pi_p(p^{-1}) \circ \varphi), \quad \forall \ n \in N^s, \ p \in \tilde{P},$$

$$D(\rho^3_n(n) \varphi) = D(\lambda_n^3(\tilde{w}^{-1}n_\circ \tilde{w}) \varphi), \quad \forall \ n_\circ \in N^s \cap \tilde{w}\tilde{P}\tilde{w}^{-1},$$

for all $\varphi \in S(N^s \times \tilde{P}; \mathcal{V})$. Here $\lambda_n^3$ and $\lambda_n^2$ are representations defined by left translation:

$$\lambda_n^3 : N^s \to \text{Aut}(S(N^s \times \tilde{P}; \mathcal{V})), \quad \lambda_n^2 : \tilde{P} \to \text{Aut}(S(N^s \times \tilde{P}; \mathcal{V})), $$

and $\rho_n^3$ and $\rho_n^2$ are representations defined by right translation:

$$\rho_n^3 : N^s \to \text{Aut}(S(N^s \times \tilde{P}; \mathcal{V})), \quad \rho_n^2 : \tilde{P} \to \text{Aut}(S(N^s \times \tilde{P}; \mathcal{V})).$$

Next, we identify $S(N^s \times \tilde{P}; \mathcal{V})$ with $S(N^s) \otimes S(\tilde{P}; \mathcal{V})$ in the usual way, that is, for every $\varphi_n \in S(N^s)$ and $\varphi_p \in S(\tilde{P}; \mathcal{V})$, let $\varphi_n \otimes \varphi_p \in S(N^s \times \tilde{P}; \mathcal{V})$ be defined by:

$$(\varphi_n \otimes \varphi_p)(n, p) := \varphi_n(n) \varphi_p(p), \quad \forall \ n \in N^s, \ p \in \tilde{P}.$$ 

Let $\lambda_n^3$ and $\lambda_n^3$ be the representations defined by left translation:

$$\lambda_n^3 : N^s \to \text{Aut}(S(N^s)), \quad \lambda_n^3 : \tilde{P} \to \text{Aut}(S(\tilde{P}; \mathcal{V})), $$

and let $\rho_n^3$ and $\rho_n^3$ be the representations defined by right translation:

$$\rho_n^3 : N^s \to \text{Aut}(S(N^s)), \quad \rho_n^3 : \tilde{P} \to \text{Aut}(S(\tilde{P}; \mathcal{V})).$$

For every $D \in D^2$ and $\varphi_p \in S(\tilde{P}; \mathcal{V})$, let $D_{\varphi_p} \in D(N^s)$ be the distribution defined by $D_{\varphi_p}(\varphi_n) := D(\varphi_n \otimes \varphi_p)$ for all $\varphi_n \in S(N^s)$. Then $D_{\varphi_p}(\lambda^3_n(n) \varphi_n) = \psi^*(n) D_{\varphi_p}(\varphi_n)$ for all $\varphi_n \in S(N^s)$ and $n \in N^s$, since $\lambda^3_n(n)(\varphi_n \otimes \varphi_p) = \lambda^3_n(n) \varphi_n \otimes \varphi_p$. By the uniqueness of left quasi-invariant distributions on $N^s$ (cf. §1.18 of [1] – the proof for quasi-invariant distributions is similar), it follows that $D_{\varphi_p}$ is a constant multiple of the distribution $\tilde{\psi}^*\ dn \in D(N^s)$ defined by:

$$\varphi_n \mapsto \int_{N^s} \varphi_n(n) \tilde{\psi}^*(n) \ dn, \quad \forall \ \varphi_n \in S(N^s).$$
Hence, $D_{\varphi_p}(\rho^3_n(n_0)\varphi_N) = \psi^*(n_0)D_{\varphi_p}(\varphi_N)$ for all $\varphi_N \in S(N^s)$ and $n_0 \in N^s \cap \tilde{w}\tilde{P}\tilde{w}^{-1}$, and it follows that $\mathcal{D}^2$ is isomorphic to the space $\mathcal{D}^3$ of $\mathcal{V}$-distributions $D \in \mathcal{D}(\tilde{P}; \mathcal{V})$ that satisfy:

$$D(\lambda^3_n(p_0)\rho^3_p(p)\varphi) = \psi^*(\tilde{w)p_0\tilde{w}^{-1}) \delta(p)^{-1/2}D(\pi_p(p^{-1}) \circ \varphi)$$

for all $p_0 \in \tilde{w}^{-1}N^s\tilde{w} \cap \tilde{P}$, $p \in \tilde{P}$, and $\varphi \in S(\tilde{P}; \mathcal{V})$.

Proceeding as above, we next identify $S(\tilde{P}; \mathcal{V})$ with $S(\tilde{M}; \mathcal{V}) \otimes S(U^s)$. Thus, for every $\varphi_M \in S(\tilde{M}; \mathcal{V})$ and $\varphi_U \in S(U^s)$, $\varphi_M \otimes \varphi_U \in S(\tilde{P}; \mathcal{V})$ is defined by:

$$(\varphi_M \otimes \varphi_U)(mu) := \varphi_M(m) \varphi_U(u), \quad \forall m \in \tilde{M}, u \in U^s.$$  

Let $\lambda^4_M$ and $\lambda^4_U$ be the representations defined by left translation:

$$\lambda^4_M : \tilde{M} \to \text{Aut}(S(\tilde{M}; \mathcal{V})), \quad \lambda^4_U : U^s \to \text{Aut}(S(U^s)),$$

and let $\rho^4_M$ and $\rho^4_U$ be the representations defined by right translation:

$$\rho^4_M : \tilde{M} \to \text{Aut}(S(\tilde{M}; \mathcal{V})), \quad \rho^4_U : U^s \to \text{Aut}(S(U^s)).$$

If $D \in \mathcal{D}^3$ and $\varphi_M \in S(\tilde{M}; \mathcal{V})$, let $D_{\varphi_M} \in \mathcal{D}(U^s)$ be defined by $D_{\varphi_M} \varphi_U := D(\varphi_M \otimes \varphi_U)$ for all $\varphi_U \in S(U^s)$. Since $\delta |_{U^s} = 1$ and $\pi_p |_{U^s} = 1$:

$$D_{\varphi_M}(\rho^4_M(u)\varphi_U) = D(\varphi_M \otimes \rho^4_M(u)\varphi_U) = D(\rho^3_p(u)(\varphi_M \otimes \varphi_U)) = D(\varphi_M \otimes \varphi_U) = D_{\varphi_M}(\varphi_U)$$

for all $\varphi_U \in S(U^s)$ and $u \in U^s$. Then by the uniqueness of right-invariant distributions on $U^s$ (cf. §1.18 of [1]), it follows that $D_{\varphi_M}$ is a constant multiple $D'(\varphi_M)$ of the Haar measure $du \in \mathcal{D}(U^s)$:

$$D(\varphi_M \otimes \varphi_U) = D'(\varphi_M) \int_{U^s} \varphi_U(u) du, \quad \forall \varphi_M \in S(\tilde{M}; \mathcal{V}), \varphi_U \in S(U^s),$$

and $D'$ is a $\mathcal{V}$-distribution in $\mathcal{D}(\tilde{M}; \mathcal{V})$.

We will now show that if $w \neq w_M$, then $\mathcal{D}^1(N^s\tilde{w}\tilde{P}) = 0$. Let $D \in \mathcal{D}^3$ be fixed, and let $D' \in \mathcal{D}(\tilde{M}; \mathcal{V})$ be as above. For every $\varphi \in S(\tilde{P}; \mathcal{V})$, let $\varphi' \in S(\tilde{M}; \mathcal{V})$ be defined by:

$$\varphi'(m) := \int_{U^s} \varphi(mu) du, \quad \forall m \in \tilde{M}.$$
Then $D'(\varphi') = D(\varphi)$ for all $\varphi \in S(\tilde{P}; V)$. Indeed, this is easy to check when $\varphi$ is of the form $\varphi_M \otimes \varphi_U$, and the general case follows by linearity. Since $\tilde{M}$ normalizes $U^s$, it follows that $(\lambda_p^3(u)\varphi)' = \varphi'$ for all $\varphi \in S(\tilde{P}; V)$ and $u \in U^s$. In particular:

$$D(\varphi) = D'(\varphi') = D'(\lambda_p^3(u_0)\varphi) = \psi^*(\tilde{w}u_0\tilde{w}^{-1})D(\varphi)$$

for all $u_0 \in \tilde{w}^{-1}N^s\tilde{w} \cap U^s$. Since $w$ is not the longest element in $[W/W_M]$, there exists a positive simple root $\alpha$ such that the root group $N_\alpha$ is contained in $wu_0w^{-1}$. Since $\psi$ is principal, we have that $\psi^*|_{N_\alpha} \neq 1$. Moreover, from the definition of the Matsumoto 2-cocycle $\sigma$, it follows that $\tilde{w}^{-1}N^s\tilde{w} = s(w^{-1}N_\alpha w)$, and $\tilde{w}^{-1}N^s\tilde{w} \cap U^s = s(w^{-1}Nw \cap U)$. Hence there exists a $u_0 \in \tilde{w}^{-1}N^s\tilde{w} \subseteq \tilde{w}^{-1}N^s\tilde{w} \cap U^s$ such that $\psi^*(\tilde{w}u_0\tilde{w}^{-1}) \neq 1$. This shows that $D = 0$, and therefore $D^1(N^s\tilde{w}P) \cong D^2 \cong D^3 = 0$.

Now suppose that $w = w_M$. Then $\tilde{w}^{-1}N^s\tilde{w} \cap \tilde{P} = \tilde{w}^{-1}N^s\tilde{w} \cap \tilde{M}$, and $D^3$ is isomorphic to the space $D^4$ of $V$-distributions $D \in D(\tilde{M}; V)$ that satisfy:

$$D(\lambda^4_M(m_\circ)\rho^4_M(m)\varphi_M) = \psi^*(\tilde{w}m_\circ\tilde{w}^{-1})\delta(m)^{1/2}D(\pi^*_M(m^{-1}) \circ \varphi_M)$$

for all $m_\circ \in \tilde{w}^{-1}N^s\tilde{w} \cap \tilde{M}$, $m \in \tilde{M}$, and $\varphi_M \in S(\tilde{M}; V)$. Here we have used the fact that:

$$\rho^3_M(m)(\varphi_M \otimes \varphi_U) = \rho^4_M(m)\varphi_M \otimes \lambda^4_M(m^{-1})\rho^4_M(m)\varphi_U$$

for all $m \in \tilde{M}$, $\varphi_M \in S(\tilde{M}; V)$, and $\varphi_U \in S(U^s)$, which implies that:

$$(\rho^3_M(m)(\varphi_M \otimes \varphi_U))' = \delta(m)(\rho^4_M(m)\varphi_M \otimes \varphi_U)'$$

By another straightforward calculation with the 2-cocycle $\sigma$, $\tilde{w}^{-1}N^s\tilde{w} \cap \tilde{M} = N^s_M$. Thus, from the definition of $\psi^*_M$, it follows that complex conjugation provides an isomorphism between $D^4$ and the space $D^5$ of $V$-distributions $D \in D(\tilde{M}; V)$ that satisfy:

$$D(\lambda^4_M(m_\circ)\rho^4_M(m)\varphi) = \psi^*_M(n_\circ)\delta(m)^{-1/2}D((\delta^{-1} \otimes \pi_M)(m^{-1}) \circ \varphi), \quad \forall \ n_\circ \in N^s_M, \ m \in \tilde{M},$$

for all $\varphi \in S(\tilde{M}; V)$. By Bruhat’s theorem, $D^5 \cong \text{Bil}_M(\psi_M^\psi, \delta^{-1} \otimes \pi_M)$, the space of intertwining forms of $\psi_M^\psi$ and $\delta^{-1} \otimes \pi_M$, which is isomorphic to $\text{Hom}_M(\delta^{-1} \otimes \pi_M, \psi_M^\psi)$. Finally, since $\delta|_{N^s_M} = 1$, the spaces $\text{Hom}_M(\delta^{-1} \otimes \pi_M, \psi_M^\psi)$ and $\text{Hom}_M(\pi_M, \psi_M^\psi)$ are isomorphic.
(although the representations $\delta^{-1} \otimes \pi_M$ and $\pi_M$ need not be). Thus, in the case $w = w_M$, we have that $D_1(N^s \tilde{w} \tilde{P}) \cong \text{Hom}_{\tilde{M}}(\pi_M, \Pi^\psi_M)$.

As $\tilde{G}$ is a finite covering of $G$, we have that $N^s \tilde{w}_M \tilde{P}$ is open in $\tilde{G}$, since $Nw_M P$ is open in $G$. Similarly, $X^1 := \bigsqcup_{w \neq w_M} N^s \tilde{w} \tilde{P}$ is closed in $\tilde{G}$, and the sequence:

$$0 \rightarrow D_1(X^1) \rightarrow D_1(\tilde{G}) \rightarrow D_1(N^s \tilde{w}_M \tilde{P}) \rightarrow 0$$

is exact (cf. §1.9 of [1]). We will now show that $D_1(X^1) = 0$. Indeed, for $i \geq 1$, let $X^{i+1} := X^i - Y^i$, where:

$$Y^i := \bigsqcup_{w \in W^i} N^s \tilde{w} \tilde{P}, \quad W^i := \{ w \in W \mid N^s \tilde{w} \tilde{P} \text{ is an open subset of } X^i \}.$$  

Then $W - \{ w_M \} = \bigsqcup_{i \geq 1} W^i$, and $X^1 = \bigsqcup_{i \geq 1} Y^i$. As each $Y^i$ is open in $X^i$, and each $X^{i+1}$ is closed in $X^i$, the sequence:

$$0 \rightarrow D_1(X^{i+1}) \rightarrow D_1(X^i) \rightarrow D_1(Y^i) \rightarrow 0$$

is also exact. But $N^s \tilde{w} \tilde{P}$ is also an open subset of $Y^i$ for every $w \in W^i$, hence it follows that $D_1(Y^i) \cong \bigoplus_{w \in W^i} D_1(N^s \tilde{w} \tilde{P}) = 0$. Consequently, $D_1(X^1) \cong D_1(X^i)$ for all $i \geq 1$, and since $X^i = \emptyset$ for $i \gg 0$, this shows that $D_1(X^1) = 0$. From the exact sequence (**), it now follows that $D_1(\tilde{G}) \cong D_1(N^s \tilde{w}_M \tilde{P}) \cong \text{Hom}_{\tilde{M}}(\pi_M, \Pi^\psi_M)$, and this completes the proof. \(\square\)


