Solutions to Exercises 8.1

1. \( u_{xx} + u_{xy} = 2u \) is a second order, linear, and homogeneous partial differential equation. \( u_x(0, y) = 0 \) is linear and homogeneous.

5. \( u_t u_x + u_{xt} = 2u \) is second order and nonlinear because of the term \( u_t u_x \). \( u(0, t) + u_x(0, t) = 0 \) is linear and homogeneous.

9. (a) Let \( u(x, y) = e^{ax} e^{by} \). Then

\[
\begin{align*}
  u_x &= a e^{ax} e^{by} \\
  u_y &= b e^{ax} e^{by} \\
  u_{xx} &= a^2 e^{ax} e^{by} \\
  u_{yy} &= b^2 e^{ax} e^{by} \\
  u_{xy} &= ab e^{ax} e^{by}.
\end{align*}
\]

So

\[
Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0
\]

\[
\Leftrightarrow \quad Aa^2 e^{ax} e^{by} + 2Bab e^{ax} e^{by} + Cb^2 e^{ax} e^{by} + Dae^{by} + Ebe^{ax} e^{by} + F = 0
\]

\[
\Leftrightarrow \quad e^{ax} e^{by} (Aa^2 + 2Bab + Cb^2 + Da + Eb + F) = 0
\]

\[
\Leftrightarrow \quad Aa^2 + 2Bab + Cb^2 + Da + Eb + F = 0
\]

because \( e^{ax} e^{by} \neq 0 \) for all \( x \) and \( y \).

(b) By (a), in order to solve

\[
u_{xx} + 2u_{xy} + u_{yy} + 2u_x + 2u_y + u = 0,
\]

we can try \( u(x, y) = e^{ax} e^{by} \), where \( a \) and \( b \) are solutions of

\[
a^2 + 2ab + b^2 + 2a + 2b + 1 = 0.
\]

But

\[
a^2 + 2ab + b^2 + 2a + 2b + 1 = (a + b + 1)^2.
\]

So \( a + b + 1 = 0 \). Clearly, this equation admits infinitely many pairs of solutions \( (a, b) \). Here are four possible solutions of the partial differential equation:

\[
\begin{align*}
  a &= 1, \quad b = -2 \Rightarrow u(x, y) = e^x e^{-2y} \\
  a &= 0, \quad b = -1 \Rightarrow u(x, y) = e^{-y} \\
  a &= -1/2, \quad b = -1/2 \Rightarrow u(x, y) = e^{-x/2} e^{-y/2} \\
  a &= -3/2, \quad b = 1/2 \Rightarrow u(x, y) = e^{-3x/2} e^{y/2}
\end{align*}
\]

13. We follow the outlined solution in Exercise 12. We have

\[
A(u) = \ln(u), \quad \phi(x) = e^x, \quad \Rightarrow \quad A(u(x(t), t)) = A(\phi(x(0))) = \ln(e^{x(0)}) = x(0).
\]

So the characteristic lines are

\[
x = tx(0) + x(0) \quad \Rightarrow \quad x(0) = L(x_t) = \frac{x}{t+1}.
\]
So $u(x, t) = f(L(x, t)) = f\left(\frac{x}{t+1}\right)$. The condition $u(x, 0) = e^x$ implies that $f(x) = e^x$ and so

$$u(x, t) = e^{\frac{x}{t+1}}.$$

Check: $u_t = -e^{\frac{x}{t+1}} \frac{x}{(t+1)^2}$, $u_x = e^{\frac{x}{t+1}} \frac{1}{t+1}$.

$$u_t + \ln(u)u_x = -e^{\frac{x}{t+1}} \frac{x}{(t+1)^2} + \frac{x}{t+1}e^{\frac{x}{t+1}} \frac{1}{t+1} = 0.$$

17. We have

$$A(u) = u^2, \quad \phi(x) = \sqrt{x}, \quad \Rightarrow \quad A(u(x(t)), t)) = A(\phi(x(0))) = x(0).$$

So the characteristic lines are

$$x = tx(0) + x(0) \quad \Rightarrow \quad x(0)(t+1) - x = 0.$$

Solving for $x(0)$, we find

$$x(0) = \frac{x}{t+1},$$

and so

$$u(x, t) = f\left(\frac{x}{t+1}\right).$$

Now

$$u(x, 0) = f(x) = \sqrt{x}.$$

So

$$u(x, t) = \sqrt{\frac{x}{t+1}}.$$
Solutions to Exercises 8.2

1. The solution is

\[ u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( b_n \cos \frac{n\pi t}{L} + b_n^* \sin \frac{n\pi t}{L} \right), \]

where \( b_n \) are the Fourier sine coefficients of \( f \) and \( b_n^* \) are \( \frac{L}{cn} \) times the Fourier coefficients of \( g \). In this exercise, \( b_n^* = 0 \), since \( g = 0 \), \( b_1 = 0.05 \); and \( b_n = 0 \) for all \( n > 1 \), because \( f \) is already given by its Fourier sine series (period 2). So \( u(x, t) = 0.05 \sin \pi x \cos t \).

5. (a) The solution is

\[ u(x, t) = \sum_{n=1}^{\infty} \sin(n\pi x) \left( b_n \cos(4n\pi t) + b_n^* \sin(4n\pi t) \right), \]

where \( b_n \) is the \( n \)th sine Fourier coefficient of \( f \) and \( b_n^* \) is \( L/(cn) \) times the Fourier coefficient of \( g \), where \( L = 1 \) and \( c = 4 \). Since \( g = 0 \), we have \( b_n^* = 0 \) for all \( n \). As for the Fourier coefficients of \( f \), we can get them by using Exercise 17, Section 2.4, with \( p = 1 \), \( h = 1 \), and \( a = 1/2 \). We get

\[ b_n = \frac{8}{\pi^2} \sin \frac{n\pi}{2n^2}. \]

Thus

\[ u(x, t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{2n^2} \sin(n\pi x) \cos(4n\pi t)}{\sin(2k + 1)\pi x \cos(4(2k + 1)\pi t)}. \]

(b) Here is the initial shape of the string. Note the new Mathematica command that we used to define piecewise a function. (Previously, we used the If command.)

```
Clear[f]
f[x_] := 2 x /; 0 < x < 1/2
f[x_] := 2 (1 - x) /; 1/2 < x < 1
Plot[f[x], {x, 0, 1}]
```

Because the period of \( \cos(4(2k + 1)\pi t) \) is \( 1/2 \), the motion is periodic in \( t \) with period \( 1/2 \). This is illustrated by the following graphs. We use two different ways to plot the graphs: The first uses simple Mathematica commands; the second one is more involved and is intended to display the graphs in a convenient array.
Clear[partsum]
partsum[x_, t_] := 
8/Pi^2 Sum[Sin[(-1)^k (2 k + 1) Pi x]Cos[4 (2 k + 1) Pi t]/(2 k + 1)^2, {k, 0, 10}]
Plot[Evaluate[{partsum[x, 0], f[x]}], {x, 0, 1}]

Approximation of the initial shape of the string by the Fourier series solution at t = 0

Here is the motion in an array.
9. The solution is

\[ u(x, t) = \sum_{n=1}^{\infty} \sin(n\pi x) \left( b_n \cos(n\pi t) + b^*_n \sin(n\pi t) \right), \]

where \( b^*_1 = \frac{1}{\pi} \) and all other \( b^*_n = 0 \). The Fourier coefficients of \( f \) are

\[ b_n = 2 \int_0^1 x(1-x) \sin(n\pi x) \, dx. \]

To evaluate this integral, we will use integration by parts to derive first the formula: for \( a \neq 0 \),

\[ \int x \sin(ax) \, dx = -\frac{x \cos(ax)}{a} + \frac{\sin(ax)}{a^2} + C, \]

and

\[ \int x^2 \sin(ax) \, dx = \frac{2 \cos(ax)}{a^3} - \frac{x^2 \cos(ax)}{a} + \frac{2x \sin(ax)}{a^2} + C; \]
thus
\[
\int x(1-x) \sin(ax) \, dx = -\frac{2 \cos(ax)}{a^3} - \frac{x \cos(ax)}{a} + \frac{x^2 \cos(ax)}{a^2} - \frac{2x \sin(ax)}{a^2} + C.
\]

Applying the formula with \( a = n\pi \), we get
\[
\int_0^1 x(1-x) \sin(n\pi x) \, dx = -\frac{2 \cos(n\pi x)}{(n\pi)^3} \bigg|_0^1 = \left\{ \begin{array}{l}
\frac{4}{(n\pi)^3} \quad \text{if } n \text{ is odd}, \\
0 \qquad \text{if } n \text{ is even}.
\end{array} \right.
\]

Thus
\[
b_n = \left\{ \begin{array}{l}
\frac{8}{(n\pi)^3} \quad \text{if } n \text{ is odd}, \\
0 \qquad \text{if } n \text{ is even},
\end{array} \right.
\]

and so
\[
u(x, t) = \frac{8}{\pi^3} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x) \cos((2k+1)\pi t)}{(2k+1)^3} + \frac{1}{\pi} \sin(\pi x) \sin(\pi t).
\]

13. To solve
\[
\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},
\]
\[
u(0, t) = u(\pi, t) = 0,
\]
\[
u(x, 0) = \sin x, \quad \frac{\partial u}{\partial t}(x, 0) = 0,
\]

we follow the method of the previous exercise. We have \( c = 1, \, k = .5, \, L = \pi, \, f(x) = \sin x, \)
and \( g(x) = 0 \). Thus the real number \( \frac{kL}{c} = .5 \) is not an integer and we have \( n > \frac{kL}{c} \pi \) for all \( n \). So only Case III from the solution of Exercise 12 needs to be considered. Thus
\[
u(x, t) = \sum_{n=1}^{\infty} e^{-5t} \sin nx (a_n \cos \lambda_n t + b_n \sin \lambda_n t),
\]

where
\[
\lambda_n = \sqrt{(5n)^2 - 1}.
\]

Setting \( t = 0 \), we obtain
\[
\sin x = \sum_{n=1}^{\infty} a_n \sin nx.
\]
Hence $a_1 = 1$ and $a_n = 0$ for all $n > 1$. Now since

$$b_n = \frac{ka_n}{\lambda_n} + \frac{2}{\lambda_n L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx, \quad n = 1, 2, \ldots,$$

it follows that $b_n = 0$ for all $n > 1$ and the solution takes the form

$$u(x, t) = e^{-5t} \sin x \left( \cos \lambda_1 t + b_1 \sin \lambda_1 t \right),$$

where $\lambda_1 = \sqrt{(\frac{5}{2})^2 - 1} = \sqrt{\frac{9}{2}} = \frac{3\sqrt{3}}{2}$ and

$$b_1 = \frac{ka_1}{\lambda_1} = \frac{1}{\sqrt{3}}.$$

So

$$u(x, t) = e^{-5t} \sin x \left( \cos \left(\frac{3\sqrt{3}}{2} t\right) + \frac{1}{\sqrt{3}} \sin \left(\frac{3\sqrt{3}}{2} t\right) \right).$$

17. (a) That $G$ is even follows from its Fourier series representation that we derive in a moment. That $G$ is $2L$-periodic follows from the fact that $g$ is $2L$-periodic and its integral over one period is 0, because it is odd (see Section 7.1, Exercise 15).

Since $G$ is an antiderivative of $g^*$, to obtain its Fourier series, we apply Exercise 33, Section 7.3, and get

$$G(x) = A_0 - \frac{L}{\pi} \sum_{n=1}^{\infty} b_n(g) \cos \frac{n\pi}{L} x,$$

where $b_n(g)$ is the $n$th Fourier sine coefficient of $g^*$,

$$b_n(g) = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

and

$$A_0 = \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{b_n(g)}{n}.$$

In terms of $b_n^*$, we have

$$\frac{L}{\pi} \frac{b_n(g)}{n} = \frac{2}{n\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx = cb_n^*,$$

and so

$$G(x) = \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{b_n(g)}{n} - \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{b_n(g)}{n} \cos \frac{n\pi}{L} x$$

$$= \sum_{n=1}^{\infty} cb_n^* \left( 1 - \cos \frac{n\pi}{L} x \right).$$
Since the Fourier series of $G$ contains only cosine terms, it follows that $G$ is even.

(b) From (a), it follows that

$$G(x + ct) - G(x - ct) = \sum_{n=1}^{\infty} cb_n^* \left[ (1 - \cos\left(\frac{n\pi}{L}(x + ct)\right)) - (1 - \cos\left(\frac{n\pi}{L}(x - ct)\right)) \right]$$

$$= \sum_{n=1}^{\infty} -cb_n^* \left[ \cos\left(\frac{n\pi}{L}(x + ct)\right) - \cos\left(\frac{n\pi}{L}(x - ct)\right) \right]$$

(c) Continuing from (b) and using the notation in the text, we obtain

$$\frac{1}{2c} \int_{x-ct}^{x+ct} g^*(s) \, ds = \frac{1}{2c} [G(x + ct) - G(x - ct)]$$

$$= \sum_{n=1}^{\infty} -b_n^* \frac{1}{2} \left[ \cos\left(\frac{n\pi}{L}(x + ct)\right) - \cos\left(\frac{n\pi}{L}(x - ct)\right) \right]$$

$$= \sum_{n=1}^{\infty} b_n^* \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L}ct\right)$$

$$= \sum_{n=1}^{\infty} b_n^* \sin\left(\frac{n\pi}{L}x\right) \sin(\lambda_n t).$$

(d) To derive d’Alembert’s solution, proceed as follows:

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \cos(\lambda_n t) + \sum_{n=1}^{\infty} b_n^* \sin\left(\frac{n\pi}{L}x\right) \sin(\lambda_n t)$$

$$= \frac{1}{2} \left[ f^*(x - ct) + f^*(x + ct) \right] + \frac{1}{2c} [G(x + ct) - G(x - ct)],$$

where in the last equality we used Exercise 16 and part (c).
Solutions to Exercises 8.3

1. Multiply the solution in Example 1 by $\frac{78}{100}$ to obtain

$$u(x, t) = \frac{312}{\pi} \sum_{k=0}^{\infty} \frac{e^{- (2k+1)^2 t}}{2k+1} \sin(2k + 1)x.$$ 

5. We have

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{- (n\pi)^2 t} \sin(n\pi x),$$

where

$$b_n = 2 \int_0^1 x \sin(n\pi x) \, dx = 2 \left[ \frac{x \cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2 \pi^2} \right]_0^1$$

$$= -2 \cos n\pi \pi = 2 \frac{(-1)^{n+1}}{n\pi}.$$

So

$$u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{- (n\pi)^2 t} \sin(n\pi x)}{n}.$$

9. (a) The steady-state solution is a linear function that passes through the points (0, 0) and (1, 100). Thus, $u(x) = 100x$.

(b) The steady-state solution is a linear function that passes through the points (0, 100) and (1, 100). Thus, $u(x) = 100$. This is also obvious: If you keep both ends of the insulated bar at 100 degrees, the steady-state temperature will be 100 degrees.

13. We have $u_1(x) = -\frac{50}{\pi} x + 100$. We use (13) and the formula from Exercise 10, and get (recall the Fourier coefficients of $f$ from Exercise 3)

$$u(x, t) = -\frac{50}{\pi} x + 100$$

$$+ \sum_{n=1}^{\infty} \left[ \frac{132}{\pi} \frac{\sin(n\pi x)}{n^2} - 100 \left\{ \frac{2 - (-1)^n}{n\pi} \right\} \right] e^{- n^2 t} \sin nx.$$ 

17. Fix $t_0 > 0$ and consider the solution at time $t = t_0$:

$$u(x, t_0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{- \lambda_n^2 t_0}.$$ 

We will show that this series converges uniformly for all $x$ (not just $0 \leq x \leq L$) by appealing to the Weierstrass $M$-test. For this purpose, it suffices to establish the following two inequalities:

(a) $\left| b_n \sin \frac{n\pi x}{L} e^{- \lambda_n^2 t_0} \right| \leq M_n$ for all $x$; and
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(b) \( \sum_{n=1}^{\infty} M_n < \infty \).

To establish (a), note that

\[
|b_n| = \left| \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx \right| \leq \frac{2}{L} \int_0^L |f(x) \sin \frac{n\pi}{L} x| \, dx
\]

(The absolute value of the integral is \( \leq \) the integral of the absolute value.)

\[
\leq \frac{2}{L} \int_0^L |f(x)| \, dx = A \quad (\text{because } |\sin u| \leq 1 \text{ for all } u).
\]

Note that \( A \) is a finite number because \( f \) is bounded, so its absolute value is bounded and hence its integral is finite on \([0, L]\). We have

\[
|b_n \sin \frac{n\pi}{L} x e^{-\lambda_n^2 t_0}| \leq A e^{-\lambda_n^2 t_0} = A e^{-\frac{e^{\frac{\lambda_n^2}{L^2} t_0}}{L^2} n^2}
\]

\[
\leq A \left(e^{-\frac{e^{\frac{\lambda_n^2}{L^2} t_0}}{L^2}}\right)^n = Ar^n,
\]

where \( r = e^{-\frac{e^{\frac{\lambda_n^2}{L^2} t_0}}{L^2}} < 1 \). Take \( M_n = Ar^n \). Then (a) holds and \( \sum M_n \) is convergent because it is a geometric series with ratio \( r < 1 \).
Solutions to Exercises 8.4

1. Since the bar is insulated and the temperature inside is constant, there is no exchange
of heat, and so the temperature remains constant for all \( t > 0 \). Thus \( u(x, t) = 100 \) for all
\( t > 0 \). This is also a consequence of (2), since in this case all the \( a_n \)'s are 0 except \( a_0 = 100 \).

5. The solution is given by (4) with \( c = L = 1 \), and where the coefficients \( c_n \) are the
Fourier cosine coefficients of the function \( f(x) = \cos \pi x \). This function is clearly its own
cosine series with \( a_n = 0 \) for all \( n \neq 1 \) and \( a_1 = 1 \). Thus
\[
u(x, t) = e^{-\lambda t^2} \cos \pi x = e^{-\pi^2 t} \cos \pi x.
\]

9. This is a straightforward application of Exercise 7. For Exercise 1 the average is 100.
For Exercise 2 the average is \( a_0 = 0 \).

13. The solution is given by (8), where \( c_n \) is given by (11). We have
\[
\int_0^1 \sin^2 \mu_n x \, dx = \frac{1}{2} \int_0^1 (1 - \cos(2\mu_n x)) \, dx
\]
\[
= \frac{1}{2} \left( x - \frac{1}{2\mu_n} \sin(2\mu_n x) \right) \bigg|_0^1 = \frac{1}{2} \left( 1 - \frac{1}{2\mu_n} \sin(2\mu_n) \right).
\]
Since \( \mu_n \) is a solution of \( \tan \mu = -\mu \), we have \( \sin \mu_n = -\mu_n \cos \mu_n \), so
\[
\sin 2\mu_n = 2 \sin \mu_n \cos \mu_n = -2\mu_n \cos^2 \mu_n,
\]
and hence
\[
\int_0^1 \sin^2 \mu_n x \, dx = \frac{1}{2} (1 + \cos^2 \mu_n).
\]
Also,
\[
\int_0^{\frac{1}{2}} \sin \mu_n x \, dx = \frac{1}{\mu_n} (1 - \cos \frac{\mu_n}{2}).
\]
Applying (11), we find
\[
c_n = \frac{100}{\mu_n} \left( 1 - \cos \frac{\mu_n}{2} \right) \div \frac{1}{2} \left( 1 + \cos^2 \mu_n \right)
\]
\[
= \frac{200 \left( 1 - \cos \frac{\mu_n}{2} \right)}{\mu_n \left( 1 + \cos^2 \mu_n \right)}.
\]
Thus the solution is
\[
u(x, t) = \sum_{n=1}^{\infty} \frac{200 \left( 1 - \cos \frac{\mu_n}{2} \right)}{\mu_n \left( 1 + \cos^2 \mu_n \right)} e^{-\mu_n^2 t} \sin \mu_n x.
\]
17. Part (a) is straightforward as in Example 2. We omit the details that lead to the separated equations:

\[ T' - kT = 0, \]
\[ X'' - kX = 0, \quad X'(0) = -X(0), \quad X'(1) = -X(1), \]

where \( k \) is a separation constant.

(b) If \( k = 0 \) then

\[ X'' = 0 \quad \Rightarrow \quad X = ax + b, \]
\[ X'(0) = -X(0) \quad \Rightarrow \quad a = -b \]
\[ X'(1) = -X(1) \quad \Rightarrow \quad a = -(a + b) \quad \Rightarrow \quad 2a = -b; \]
\[ \Rightarrow \quad a = b = 0. \]

So \( k = 0 \) leads to trivial solutions.

(c) If \( k = \alpha^2 > 0 \), then

\[ X'' - \mu^2 X = 0 \quad \Rightarrow \quad X = c_1 \cosh \mu x + c_2 \sinh \mu x; \]
\[ X'(0) = -X(0) \quad \Rightarrow \quad \mu c_2 = -c_1 \]
\[ X'(1) = -X(1) \quad \Rightarrow \quad \mu c_1 \sinh \mu + \mu c_2 \cosh \mu = -c_1 \cosh \mu - c_2 \sinh \mu \]
\[ \Rightarrow \quad \mu c_1 \sinh \mu - c_1 \cosh \mu = -c_1 \cosh \mu - c_2 \sinh \mu \]
\[ \Rightarrow \quad \mu c_1 \sinh \mu = -c_2 \sinh \mu \]
\[ \Rightarrow \quad \mu c_1 \sinh \mu = \frac{c_1}{\mu} \sinh \mu. \]

Since \( \mu \neq 0 \), \( \sinh \mu \neq 0 \). Take \( c_1 \neq 0 \) and divide by \( \sinh \mu \) and get

\[ \mu c_1 = \frac{c_1}{\mu} \quad \Rightarrow \quad \mu^2 = 1 \quad \Rightarrow \quad k = 1. \]

So \( X = c_1 \cosh x + c_2 \sinh x \). But \( c_1 = -c_2 \), so

\[ X = c_1 \cosh x + c_2 \sinh x = c_1 \cosh x - c_1 \sinh x = c_1 e^{-x}. \]

Solving the equation for \( T \), we find \( T(t) = e^t \); thus we have the product solution

\[ c_0 e^{-x} e^t, \]

where, for convenience, we have used \( c_0 \) as an arbitrary constant.

(d) If \( k = -\alpha^2 < 0 \), then

\[ X'' + \mu^2 X = 0 \quad \Rightarrow \quad X = c_1 \cos \mu x + c_2 \sin \mu x; \]
\[ X'(0) = -X(0) \quad \Rightarrow \quad \mu c_2 = -c_1 \]
\[ X'(1) = -X(1) \quad \Rightarrow \quad -\mu c_1 \sin \mu + \mu c_2 \cos \mu = -c_1 \cos \mu - c_2 \sin \mu \]
\[ \Rightarrow \quad -\mu c_1 \sin \mu - c_1 \cos \mu = -c_1 \cos \mu - c_2 \sin \mu \]
\[ \Rightarrow \quad -\mu c_1 \sin \mu = -c_2 \sin \mu \]
\[ \Rightarrow \quad -\mu c_1 \sin \mu = \frac{c_1}{\mu} \sin \mu. \]
Since $\mu \neq 0$, take $c_1 \neq 0$ (otherwise you will get a trivial solution) and divide by $c_1$ and get

$$\mu^2 \sin \mu = -\sin \mu \Rightarrow \sin \mu = 0 \Rightarrow \mu = n\pi,$$

where $n$ is an integer. So $X = c_1 \cos n\pi x + c_2 \sin n\pi x$. But $c_1 = -c_2\mu$, so

$$X = -c_1(n\pi \cos n\pi x - \sin n\pi x).$$

Call $X_n = n\pi \cos n\pi x - \sin n\pi x$.

(c) To establish the orthogonality of the $X_n$'s, treat the case $k = 1$ separately. For $k = -\mu^2$, we refer to the boundary value problem

$$X'' + \mu^2 X = 0, \quad X(0) = -X'(0), \quad X(1) = -X'(1),$$

that is satisfied by the $X_n$'s, where $\mu_n = n\pi$. We establish orthogonality using a trick from Sturm-Liouville theory (Chapter 6, Section 6.2). Since

$$X''_m = \mu^2_m X_m \quad \text{and} \quad X''_n = \mu^2_n X_n,$$

multiplying the first equation by $X_n$ and the second by $X_m$ and then subtracting the resulting equations, we obtain

$$X_nX''_m = \mu^2_m X_m X_n \quad \text{and} \quad X_mX''_n = \mu^2_n X_n X_m$$

$$X_nX''_m - X_mX''_n = (\mu^2_n - \mu^2_m)X_n X_m$$

$$(X_nX'_m - X_mX'_n)' = (\mu^2_n - \mu^2_m)X_n X_n$$

where the last equation follows by simply checking the validity of the identity $X_nX''_m - X_mX''_n = (X_nX'_m - X_mX'_n)'$. So

$$(\mu^2_n - \mu^2_m) \int_0^1 X_m(x)X_n(x) \, dx = \int_0^1 (X_n(x)X'_m(x) - X_m(x)X'_n(x))' \, dx$$

$$= X_n(x)X'_m(x) - X_m(x)X'_n(x) \bigg|_0^1,$$

because the integral of the derivative of a function is the function itself. Now we use the boundary conditions to conclude that

$$X_n(x)X'_m(x) - X_m(x)X'_n(x) \bigg|_0^1$$

$$= X_n(1)X'_m(1) - X_m(1)X'_n(1) - X_n(0)X'_m(0) + X_m(0)X'_n(0)$$

$$= -X_n(1)X_m(1) + X_m(1)X_n(1) + X_n(0)X_m(0) - X_m(0)X_n(0)$$

$$= 0.$$

Thus the functions are orthogonal. We still have to verify the orthogonality when one of the $X_n$'s is equal to $e^{-x}$. This can be done by modifying the argument that we just gave.

(f) Superposing the product solutions, we find that

$$u(x, t) = c_0 e^{-x} e^t + \sum_{n=1}^{\infty} c_n T_n(t) X_n(x).$$
Using the initial condition, it follows that

\[ u(x, 0) = f(x) = c_0 e^{-x} + \sum_{n=1}^{\infty} c_n X_n(x). \]

The coefficients in this series expansion are determined by using the orthogonality of the \( X_n \)'s in the usual way. Let us determine \( c_0 \). Multiplying both sides by \( e^{-x} \) and integrating term by term, it follows from the orthogonality of the \( X_n \) that

\[ \int_0^1 f(x) e^{-x} \, dx = c_0 \int_0^1 e^{-2x} \, dx + \sum_{n=1}^{\infty} c_n \int_0^1 X_n(x) e^{-x} \, dx. \]

Hence

\[ \int_0^1 f(x) e^{-x} \, dx = c_0 \int_0^1 e^{-2x} \, dx = c_0 \frac{1 - e^{-2}}{2}. \]

Thus

\[ c_0 = \frac{2e^2}{e^2 - 1} \int_0^1 f(x) e^{-x} \, dx. \]

In a similar way, we prove that

\[ c_n = \frac{1}{\kappa_n} \int_0^1 f(x) X_n(x) \, dx \]

where

\[ \kappa_n = \int_0^1 X_n^2(x) \, dx. \]

This integral can be evaluated as we did in Exercise 15 or by straightforward computations, using the explicit formula for the \( X_n \)'s, as follows:

\[
\int_0^1 X_n^2(x) \, dx = \int_0^1 (n\pi \cos n\pi x - \sin n\pi x)^2 \, dx \\
= \int_0^1 (n^2 \pi^2 \cos^2 n\pi x + \sin^2 n\pi x - 2n\pi \cos(n\pi x) \sin(n\pi x)) \, dx \\
= (n^2 \pi^2)/2 + 1/2 \\
= n^2 \pi^2 \cos^2 n\pi x \, dx + \int_0^1 \sin^2 n\pi x \, dx \\
= 0 - 2n\pi \int_0^1 \cos(n\pi x) \sin(n\pi x) \, dx \\
= n^2 \pi^2 + 1./2.
\]
Solutions to Exercises 8.5

5. We proceed as in Exercise 3. We have

\[ u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (B_{mn} \cos \lambda_{mn} t + B_{mn}^* \sin \lambda_{mn} t) \sin m\pi x \sin n\pi y, \]

where \( \lambda_{mn} = \sqrt{m^2 + n^2} \), \( B_{mn} = 0 \), and

\[
B_{mn}^* = \frac{4}{\sqrt{m^2 + n^2}} \int_0^1 \int_0^1 \sin m\pi x \sin n\pi y \, dx \, dy
\]

\[
= \frac{4}{\sqrt{m^2 + n^2}} \int_0^1 \sin m\pi x \, dx \int_0^1 \sin n\pi y \, dy
\]

\[
= \begin{cases} 
\frac{16}{\sqrt{m^2 + n^2}(mn)^{\pi^2}} & \text{if } m \text{ and } n \text{ are both odd}, \\
0 & \text{otherwise}.
\end{cases}
\]

Thus

\[ u(x, y, t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{16 \sin((2k + 1)\pi x) \sin((2l + 1)\pi y)}{\sqrt{(2k + 1)^2 + (2l + 1)^2}(2k + 1)(2l + 1)^{\pi^2}}} \sin \sqrt{(2k + 1)^2 + (2l + 1)^2} t \]
Solutions to Exercises 8.6

1. The solution is given by

\[ u(x, y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) \sinh(n\pi y), \]

where

\[
B_n = \frac{2}{\sinh(2n\pi)} \int_0^1 x \sin(n\pi x) \, dx

= \frac{2}{\sinh(2n\pi)} \left[ -\frac{x \cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2 \pi^2} \right]_0^1

= \frac{2}{\sinh(2n\pi)} \frac{(-1)^n}{n\pi} = \frac{2}{\sinh(2n\pi)} \frac{(-1)^{n+1}}{n\pi}.
\]

Thus,

\[ u(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sinh(2n\pi)} \sin(n\pi x) \sinh(n\pi y). \]

5. Start by decomposing the problem into four subproblems as described by Figure 3. Let \( u_j(x, y) \) denote the solution to problem \( j \) (\( j = 1, 2, 3, 4 \)). Each \( u_j \) consists of only one term of the series solutions, because of the orthogonality of the sine functions. For example, to compute \( u_1 \), we have

\[ u_1(x, y) = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh[n\pi(1 - y)], \]

where

\[
A_n = \frac{2}{\sinh n\pi} \int_0^1 \sin 7\pi x \sin n\pi x \, dx.
\]

Since the integral is 0 unless \( n = 7 \) and, when \( n = 7 \),

\[ A_7 = \frac{2}{\sinh 7\pi} \int_0^1 \sin^2 7\pi x \, dx = \frac{1}{\sinh 7\pi}. \]

Thus

\[ u_1(x, y) = \frac{1}{\sinh 7\pi} \sin 7\pi x \sinh[7\pi(1 - y)]. \]
In a similar way, appealing to the formulas in the text, we find

\[ u_2(x, y) = \frac{1}{\sinh \pi} \sin \pi x \sinh(\pi y) \]
\[ u_3(x, y) = \frac{1}{\sinh 3\pi} \sinh[3\pi(1 - x)] \sin(3\pi y) \]
\[ u_4(x, y) = \frac{1}{\sinh 6\pi} \sinh 6\pi x \sin(6\pi y) \]
\[ u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y) \]
\[ = \frac{1}{\sinh 7\pi} \sin 7\pi x \sinh[7\pi(1 - y)] + \frac{1}{\sinh 3\pi} \sin(3\pi x) \sinh(\pi y) \]
\[ + \frac{1}{\sinh 3\pi} \sinh[3\pi(1 - x)] \sin(3\pi y) + \frac{1}{\sinh 6\pi} \sinh(6\pi x) \sin(6\pi y) \]

13. Consider the function

\[ f(z) = iA \cos[\alpha(z - z_0)], \]

where \( A \) and \( \alpha \) and the given real constants and \( z_0 = \beta + i\gamma \), where \( \beta \) and \( \gamma \) are the given real constants. Clearly, \( f \) is entire (analytic for all \( z \)). Appealing to (15), Section 1.6, we find that

\[ f(z) = iA \{ \cos[\text{Re}(\alpha(z - z_0))] \cosh[\text{Im}(\alpha(z - z_0))] \]
\[ - \sin[\text{Re}(\alpha(z - z_0))] \sinh[\text{Im}(\alpha(z - z_0))]) \}
\[ = iA \{ \cos[\alpha(x - \beta)] \cosh[\alpha(y - \gamma)] - \sin[\alpha(x - \beta)] \sinh[\alpha(y - \gamma)] \}
\[ = A \sin[\alpha(x - \beta)] \sinh[\alpha(y - \gamma)] + iA \cos[\alpha(x - \beta)] \cosh[\alpha(y - \gamma)]. \]

Thus the real part of \( f \) is

\[ \phi(x, y) = A \sin[\alpha(x - \beta)] \sinh[\alpha(y - \gamma)] \]

and its imaginary part is

\[ \psi(x, y) = A \cos[\alpha(x - \beta)] \cosh[\alpha(y - \gamma)]. \]

This completes the solution, since the real and imaginary parts are harmonic functions and the imaginary part is a harmonic conjugate of the real part.
Solutions to Exercises 8.7

1. We apply (2), with \(a = b = 1\):

\[
u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y,
\]

where

\[
E_{mn} = \frac{-4}{\pi^2(m^2 + n^2)} \int_0^1 \int_0^1 x \sin m\pi x \sin n\pi y \, dx \, dy
\]

\[
= \frac{-4}{\pi^2(m^2 + n^2)} \int_0^1 x \sin m\pi x \, dx \int_0^1 \sin n\pi y \, dy
\]

\[
= \frac{-4}{\pi^2(m^2 + n^2)} \frac{1 - (-1)^n}{n} \left( -\frac{x \cos(m\pi x)}{m} + \frac{\sin(m\pi x)}{m^2\pi} \right) \bigg|_0^1
\]

\[
= \frac{4}{\pi^2(m^2 + n^2)} \frac{1 - (-1)^n (-1)^m}{n}.
\]

Thus

\[
u(x, y) = \frac{8}{\pi^4} \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2 + (2k + 1)^2} \frac{1}{m(2k + 1)} \sin m\pi x \sin((2k + 1)\pi y).
\]

5. We will use an eigenfunction expansion based on the eigenfunctions \(\phi(x, y) = \sin m\pi x \sin n\pi y\), where \(\Delta\pi(x, y) = -\pi^2(m^2 + n^2) \sin m\pi x \sin n\pi y\). So plug

\[
u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y
\]

into the equation \(\Delta u = 3u - 1\), proceed formally, and get

\[
\Delta \left( \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y \right) = 3 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y - 1
\]

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \Delta \left( \sin m\pi x \sin n\pi y \right) = 3 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y - 1
\]

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} -E_{mn} \pi^2(m^2 + n^2) \sin m\pi x \sin n\pi y
\]

\[
= 3 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y - 1
\]

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( 3 + \pi^2(m^2 + n^2) \right) E_{mn} \sin m\pi x \sin n\pi y = 1.
\]

Thinking of this as the double sine series expansion of the function identically 1, it follows that \(3 + \pi^2(m^2 + n^2)\) \(E_{mn}\) are double Fourier sine coefficients, given by (see (8), Section 3.7)

\[
(3 + \pi^2(m^2 + n^2)) E_{mn} = 4 \int_0^1 \int_0^1 \sin m\pi x \sin n\pi y \, dx \, dy
\]

\[
= 4 \frac{1 - (-1)^n}{m\pi} \frac{1 - (-1)^m}{n\pi}
\]

\[
= \begin{cases} 
0 & \text{if either } m \text{ or } n \text{ is even}, \\
\frac{16}{\pi^2 mn} & \text{if both } m \text{ and } n \text{ are even}.
\end{cases}
\]
Thus

\[ E_{mn} = \begin{cases} 
0 & \text{if either } m \text{ or } n \text{ is even}, \\
\frac{16}{\pi^2 mn(3 + \pi^2(m^2 + n^2))} & \text{if both } m \text{ and } n \text{ are even}, 
\end{cases} \]

and so

\[ u(x, y) = \frac{16}{\pi^2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\sin((2k + 1)\pi x) \sin((2l + 1)\pi y)}{(2k + 1)(2l + 1)(3 + \pi^2((2k + 1)^2 + (2l + 1)^2))}. \]
Solutions to Exercises 8.8

1. We use a combination of solutions from (2) and (3) and try a solution of the form

\[ u(x, y) = \sum_{n=1}^{\infty} \sin mx [A_m \cosh m(1 - y) + B_m \sinh my]. \]

(If you have tried a different form of the solution, you can still do the problem, but your answer may look different from the one derived here. The reason for our choice is to simplify the computations that follow.) The boundary conditions on the vertical sides are clearly satisfied. We now determine \( A_m \) and \( B_m \) so as to satisfy the conditions on the other sides.

Starting with \( u(1, 0) = 100 \), we find that

\[ 100 = \sum_{m=1}^{\infty} A_m \cosh m \sin mx. \]

Thus \( A_m \cosh m \) is the sine Fourier coefficient of the function \( f(x) = 100 \). Hence

\[ A_m \cosh m = \frac{2}{\pi} \int_0^\pi 100 \sin mx \, dx \quad \Rightarrow \quad A_m = \frac{200}{\pi m \cosh m} [1 - (-1)^m]. \]

Using the boundary condition \( u_y(x, 1) = 0 \), we find

\[ 0 = \sum_{m=1}^{\infty} \sin mx \left[ A_m (-m) \sinh(m(1 - y)) \right] \bigg|_{y=1}. \]

Thus

\[ 0 = \sum_{m=1}^{\infty} mB_m \sin m x \cosh m. \]

By the uniqueness of Fourier series, we conclude that \( mB_m \cosh m = 0 \) for all \( m \). Since \( m \cosh m \neq 0 \), we conclude that \( B_m = 0 \) and hence

\[ u(x, y) = \frac{200}{\pi} \sum_{m=1}^{\infty} \frac{1 - (-1)^m}{m \cosh m} \sin mx \cosh(m(1 - y)) \]

\[ = \frac{400}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k + 1)x)}{(2k + 1) \cosh(2k + 1)} \cosh((2k + 1)(1 - y)). \]

5. We combine solutions of different types from Exercise 4 and try a solution of the form

\[ u(x, y) = A_0 + B_0 y + \sum_{m=1}^{\infty} \cos \frac{m\pi}{a} x [A_m \cosh \frac{m\pi}{a} (b - y)] + B_m \sinh \frac{m\pi}{a} y]. \]

Using the boundary conditions on the horizontal sides, starting with \( u_y(x, b) = 0 \), we find that

\[ 0 = B_0 + \sum_{m=1}^{\infty} \frac{m\pi}{a} B_m \cos \frac{m\pi}{a} x \cosh \frac{m\pi}{a} b. \]
Thus $B_0 = 0$ and $B_m = 0$ for all $m \geq 1$ and so
\[ A_0 + \sum_{m=1}^{\infty} A_m \cos \frac{m\pi}{a} x \cosh \left[ \frac{m\pi}{a} (b - y) \right]. \]

Now, using $u(x, 0) = g(x)$, we find
\[ g(x) = A_0 + \sum_{m=1}^{\infty} A_m \cosh \left[ \frac{m\pi}{a} b \right] \cos \frac{m\pi}{a} x. \]

Recognizing this as a cosine series, we conclude that
\[ A_0 = \frac{1}{a} \int_0^a g(x) \, dx \]
and
\[ A_m \cosh \left[ \frac{m\pi}{a} b \right] = \frac{2}{a} \int_0^a g(x) \cos \frac{m\pi}{a} x \, dx; \]
equivalently, for $m \geq 1$,
\[ A_m = \frac{2}{a \cosh \left[ \frac{m\pi}{a} b \right]} \int_0^a g(x) \cos \frac{m\pi}{a} x \, dx. \]

9. We follow the solution in Example 3. We have
\[ u(x, y) = u_1(x, y) + u_2(x, y), \]
where
\[ u_1(x, y) = \sum_{m=1}^{\infty} B_m \sin mx \sinh my, \]
with
\[ B_m = \frac{2}{\pi m \cosh(m\pi)} \int_0^\pi \sin mx \, dx = \frac{2}{\pi m^2 \cosh(m\pi)} (1 - (-1)^m); \]
and
\[ u_2(x, y) = \sum_{m=1}^{\infty} A_m \sin mx \cosh[m(\pi - y)], \]
with
\[ A_m = \frac{2}{\pi \cosh(m\pi)} \int_0^\pi \sin mx \, dx = \frac{2}{\pi m \cosh(m\pi)} (1 - (-1)^m). \]
Hence
\[ u(x, y) = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(1 - (-1)^m)}{m \cosh(m\pi)} \sin mx \left[ \frac{\sinh my}{m} + \cosh[m(\pi - y)] \right] \]
\[ = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k + 1)x)}{(2k + 1) \cosh((2k + 1)\pi)} \left[ \frac{\sinh[(2k + 1)y]}{(2k + 1)} + \cosh[(2k + 1)(\pi - y)] \right]. \]