Solutions to Exercises 7.1

1. (a) $\cos x$ has period $2\pi$. (b) $\cos \pi x$ has period $T = \frac{2\pi}{\pi} = 2$. (c) $\cos \frac{2}{3}x$ has period $T = \frac{2\pi}{\frac{2}{3}} = 3\pi$. (d) $\cos x$ has period $2\pi$, $\cos 2x$ has period $\pi$, $2\pi$, $3\pi$. A common period of $\cos x$ and $\cos 2x$ is $2\pi$. So $\cos x + \cos 2x$ has period $2\pi$.

5. The function is periodic with period 1. To describe it, we use any interval of length one period. Let us use the interval $[0, 1)$. On that interval, the function is given by $f(x) = x$. Thus, the complete description of the function is

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ f(x+1) & \text{otherwise.} \end{cases}$$

9. (a) Suppose that $f$ and $g$ are $T$-periodic. Then $f(x+T) \cdot g(x+T) = f(x) \cdot g(x)$, and so $f \cdot g$ is $T$ periodic. Similarly,

$$\frac{f(x+T)}{g(x+T)} = \frac{f(x)}{g(x)},$$

and so $f/g$ is $T$ periodic.

(b) Suppose that $f$ is $T$-periodic and let $h(x) = f(x/a)$. Then

$$h(x+aT) = f\left(\frac{x+aT}{a}\right) = f\left(\frac{x}{a} + T\right) = f\left(\frac{x}{a}\right) \quad \text{(because } f \text{ is } T\text{-periodic)} = h(x).$$

Thus $h$ has period $aT$. Replacing $a$ by $1/a$, we find that the function $f(ax)$ has period $T/a$.

(c) Suppose that $f$ is $T$-periodic. Then $g(f(x+T)) = g(f(x))$, and so $g(f(x))$ is also $T$-periodic.

13. We have

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = \int_{0}^{\pi/2} 1\,dx = \pi/2.$$

17. By Exercise 16, $F$ is 2 periodic, because $\int_{0}^{2} f(t)\,dt = 0$ (this is clear from the graph of $f$). So it is enough to describe $F$ on any interval of length 2. For $0 < x < 2$, we have

$$F(x) = \int_{0}^{x} (1-t)\,dt = x - \frac{x^2}{2}\bigg|_{0}^{x} = x - \frac{x^2}{2}.$$

For all other $x$, $F(x+2) = F(x)$. (b) The graph of $F$ over the interval $[0, 2]$ consists of the arch of a parabola looking down, with zeros at 0 and 2. Since $F$ is 2-periodic, the graph is repeated over and over.
Fourier Series

Solutions to Exercises 7.2

1. The graph of the Fourier series is identical to the graph of the function, except at the points of discontinuity where the Fourier series is equal to the average of the function at these points, which is $\frac{1}{2}$.

- Function
- Fourier series

5. We compute the Fourier coefficients using the Euler formulas. Let us first note that since $f(x) = |x|$ is an even function on the interval $-\pi < x < \pi$, the product $f(x) \sin nx$ is an odd function. So

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin nx \, dx = 0,$$

because the integral of an odd function over a symmetric interval is 0. For the other coefficients, we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{0} (-x) \, dx + \frac{1}{2\pi} \int_{0}^{\pi} x \, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} x \, dx = \frac{1}{2\pi} x^2 \bigg|_{0}^{\pi} = \frac{\pi}{2}.$$

In computing $a_n (n \geq 1)$, we will need the formula

$$\int x \cos ax \, dx = \frac{\cos(ax)}{a^2} + \frac{x \sin(ax)}{a} + C \quad (a \neq 0),$$

which can be derived using integration by parts. We have, for $n \geq 1$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi} \left[ \frac{x \sin nx + \frac{1}{n^2} \cos nx}{n^2} \right]_{0}^{\pi}$$

$$= \frac{2}{\pi} \left[ \frac{(-1)^n}{n^2} - \frac{1}{n^2} \right] = \frac{2}{\pi n^2} \left[ (-1)^n - 1 \right]$$

$$= \begin{cases} 
0 & \text{if } n \text{ is even} \\
-\frac{4}{n^2} & \text{if } n \text{ is odd}.
\end{cases}$$

Thus, the Fourier series is

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos(2k+1)x.$$
The function $g(x) = |x|$ and its periodic extension

Partial sums of the Fourier series. Since we are summing over the odd integers, when $n = 7$, we are actually summing the 15th partial sum.

9. Just some hints:
(1) $f$ is even, so all the $b_n$’s are zero.
(2) $a_0 = \frac{1}{\pi} \int_0^\pi x^2 \, dx = \frac{\pi^2}{3}$.
(3) Establish the identity

$$\int x^2 \cos(ax) \, dx = \frac{2x \cos(ax)}{a^2} + \frac{(-2 + a^2 x^2) \sin(ax)}{a^3} + C \quad (a \neq 0),$$

using integration by parts.

13. You can compute directly as we did in Example 1, or you can use the result of Example 1 as follows. Rename the function in Example 1 $g(x)$. By comparing graphs, note that $f(x) = -2g(x+\pi)$. Now using the Fourier series of $g(x)$ from Example, we get

$$f(x) = -2 \sum_{n=1}^{\infty} \frac{\sin(n\pi + x)}{n} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$
where we have used \( \cos n\pi = (-1)^n \). Simplifying, we find

\[
\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

(b) Setting \( x = \frac{\pi}{2} \) in the Fourier series expansion in Exercise 13 and using the fact that the Fourier series converges to \( f(x) = x \) for \(-\pi < x < \pi\), we obtain

\[
\frac{\pi}{2} = f(\pi/2) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2}
\]

\[
= 2 \left( \frac{(-1)^{1+1}}{1} \sin \frac{\pi}{2} + \frac{(-1)^{2+1}}{2} \sin \frac{2\pi}{2} + \frac{(-1)^{3+1}}{3} \sin \frac{3\pi}{2} + \frac{(-1)^{4+1}}{4} \sin \frac{4\pi}{4} + \cdots \right)
\]

\[
= 2 \left( 1 + 0 + \frac{1}{3}(-1) + 0 + \cdots \right) = 2 \left( 1 - \frac{1}{3} + \cdots \right).
\]

Dividing by 2 both sides, we get the desired identity.

21. We simply repeat the solution of Example 6, making some obvious simplifications. The idea is to realize that the function

\[
f(\theta) = \frac{1}{3 + \cos \theta}
\]

is the restriction to the unit circle of a function that is analytic on the unit circle. This function has a Laurent series expansion, that, when restricted to the unit circle, gives the Fourier series of \( f \).

Write \( \cos \theta = \frac{z + \frac{1}{z}}{2} \), where \( z = e^{i\theta} \). Then

\[
\frac{1}{3 + \cos \theta} = \frac{2z}{3 + z^2 + \frac{1}{z^2}} = \frac{2z}{z^2 + 6z + 1} = g(z).
\]

We have

\[
z^2 + 6z + 1 = (z - z_1)(z - z_2) \quad \text{where} \quad z_1 = -3 - \sqrt{8} \quad \text{and} \quad z_2 = -3 + \sqrt{8}.
\]

Since \(|z_2| < 1 < |z_1|\), the function \( g(z) \) is analytic on the annulus \(|z_2| < |z| < |z_1|\), that contains the unit circle. We next find the Laurent series expansion of \( g \) in that annulus. The computation is not straightforward, but it is done in complete detail in Example 6. I will simply recall the result of this example (but you should go over the details leading to the cited result). We have, in the annulus \(|z_2| < |z| < |z_1|\), (take \( a = 3 \) and \( z_1 \) and \( z_2 \) as previously)

\[
\frac{2z}{z^2 + 6z + 1} = \frac{1}{\sqrt{8} - 1} \left[ 1 + \sum_{n=1}^{\infty} z^n \left( \frac{z^n}{z^n} + \frac{1}{z^n} \right) \right]
\]

\[
= \frac{1}{\sqrt{8}} \left[ 1 + \sum_{n=1}^{\infty} \left( -3 + \sqrt{8} \right)^n \left( z^n + \frac{1}{z^n} \right) \right]
\]

Now set \( z = e^{i\theta} \) (which is a complex number in the domain of the Laurent series), then

\[
\frac{1}{3 + \cos \theta} = \frac{2z}{z^2 + 6z + 1}
\]

\[
= \frac{1}{\sqrt{8}} \left[ 1 + \sum_{n=1}^{\infty} \left( -3 + \sqrt{8} \right)^n \left( e^{in\theta} + e^{-in\theta} \right) \right]
\]

\[
= \frac{1}{\sqrt{8}} \left[ 1 + 2 \sum_{n=1}^{\infty} \left( -3 + \sqrt{8} \right)^n \cos n\theta \right],
\]
which is the desired Fourier series.

25. The Fourier series in Example 6 is valid for all $\theta$. Replacing $\theta$ by $\theta - \pi$ in the Fourier series of Example 6, we obtain the desired Fourier series, since $\cos(\theta - \pi) = -\cos \theta$ and $\cos[n(\theta - \pi)] = (-1)^n \cos n\theta$. 
Solutions to Exercises 7.3

1. (a) and (b) Since \( f \) is odd, all the \( a_n \)'s are zero and

\[
b_n = \frac{2}{p} \int_{0}^{p} \sin \left( \frac{n\pi x}{p} \right) dx = 0.
\]

Thus the Fourier series is

\[
\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \sin \left( \frac{(2k+1)\pi}{p} \right)x.
\]

At the points of discontinuity, the Fourier series converges to the average value of the function. In this case, the average value is 0 (as can be seen from the graph).

5. (a) and (b) The function is even. It is also continuous for all \( x \). All the \( b_n \)'s are 0. Also, by computing the area between the graph of \( f \) and the \( x \)-axis, from \( x = 0 \) to \( x = p \), we see that \( a_0 = 0 \).

Now, using integration by parts, we obtain

\[
a_n = \frac{2}{p} \int_{0}^{p} \left( \frac{2c}{p} \right) (x-p/2) \cos \left( \frac{n\pi x}{p} \right) dx = -\frac{4c}{p^2} \int_{0}^{p} (x-p/2) \sin \left( \frac{n\pi x}{p} \right) dx
\]

\[
= -\frac{4c}{p^2} \left[ \frac{p}{n\pi} (x-p/2) \sin \left( \frac{n\pi x}{p} \right) \bigg|_{x=0}^{p} \right] - \frac{p}{n\pi} \int_{0}^{p} \sin \left( \frac{n\pi x}{p} \right) dx
\]

\[
= -\frac{4c}{p^2 n^2 \pi^2} \cos \left( \frac{n\pi x}{p} \right) \bigg|_{x=0}^{p} = \frac{4c}{n^2 \pi^2} (1 - \cos \pi n)
\]

Thus the Fourier series is

\[
f(x) = \frac{8c}{\pi^2} \sum_{k=0}^{\infty} \cos \left( \frac{(2k+1)\pi x}{p} \right) \left( \frac{2k+1}{p} \right)^2.
\]

9. The function is even; so all the \( b_n \)'s are 0,

\[
a_0 = \frac{1}{p} \int_{0}^{p} e^{-cx} dx = \frac{1 - e^{-cp}}{cp} = \frac{1 - e^{-cp}}{cp};
\]

and with the help of the integral formula from Exercise 15, Section 2.2, for \( n \geq 1 \),

\[
a_n = \frac{2}{p} \int_{0}^{p} e^{-cx} \cos \left( \frac{n\pi x}{p} \right) dx
\]

\[
= \frac{2}{p} \int_{0}^{p} \cos \left( \frac{n\pi x}{p} \right) \frac{1}{p^2 n^2 \pi^2 + p^2 c^2} \left[ npe^{-cx} \sin \left( \frac{n\pi x}{p} \right) - p^2 ce^{-cx} \cos \left( \frac{n\pi x}{p} \right) \right] dx
\]

\[
= \frac{2pc}{n^2 \pi^2 + p^2 c^2} \left[ 1 - (-1)^n e^{-cp} \right].
\]

Thus the Fourier series is

\[
\frac{1}{pc} (1 - e^{-cp}) + 2cp \sum_{n=1}^{\infty} \frac{1}{c^2 p^2 + (n\pi)^2} (1 - e^{-cp}(-1)^n) \cos \left( \frac{n\pi x}{p} \right).
\]
13. Take \( p = 1 \) in Exercise 1, call the function in Exercise 1 \( f(x) \) and the function in this exercise \( g(x) \). By comparing graphs, we see that

\[
g(x) = \frac{1}{2} (1 + f(x)).
\]

Thus the Fourier series of \( g \) is

\[
\frac{1}{2} \left( 1 + \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)} \sin(2k+1) \pi x \right) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)} \sin(2k+1) \pi x.
\]

17. (a) Take \( x = 0 \) in the Fourier series of Exercise 4 and get

\[
0 = \frac{p^2}{3} - \frac{4p^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.
\]

(b) Take \( x = p \) in the Fourier series of Exercise 4 and get

\[
p^2 = \frac{p^2}{3} - \frac{4p^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n^2} \Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

Summing over the even and odd integers separately, we get

\[
\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2}.
\]

But \( \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{4} \frac{\pi^2}{6} \). So

\[
\frac{\pi^2}{6} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \frac{\pi^2}{24} \Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.
\]
21. From the graph, we have
\[ f(x) = \begin{cases} 
-1 - x & \text{if } -1 < x < 0, \\
1 + x & \text{if } 0 < x < 1.
\end{cases} \]

So
\[ f(-x) = \begin{cases} 
1 - x & \text{if } -1 < x < 0, \\
-1 + x & \text{if } 0 < x < 1;
\end{cases} \]

hence
\[ f_e(x) = \frac{f(x) + f(-x)}{2} = \begin{cases} 
-x & \text{if } -1 < x < 0, \\
x & \text{if } 0 < x < 1,
\end{cases} \]

and
\[ f_o(x) = \frac{f(x) - f(-x)}{2} = \begin{cases} 
1 & \text{if } -1 < x < 0, \\
1 & \text{if } 0 < x < 1.
\end{cases} \]

Note that, \( f_e(x) = |x| \) for \(-1 < x < 1\). The Fourier series of \( f \) is the sum of the Fourier series of \( f_e \) and \( f_o \). From Example 1 with \( p = 1 \),

\[ f_e(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^2} \cos[(2k + 1)\pi x]. \]

From Exercise 1 with \( p = 1 \),

\[ f_o(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k + 1} \sin[(2k + 1)\pi x]. \]

Hence
\[ f(x) = \frac{1}{2} + \frac{4}{\pi} \sum_{k=0}^{\infty} \left[ \frac{-\cos[(2k + 1)\pi x]}{\pi(2k + 1)^2} + \frac{\sin[(2k + 1)\pi x]}{2k + 1} \right]. \]

25. Since \( f \) is \( 2p \)-periodic and continuous, we have \( f(-p) = f(-p + 2p) = f(p) \). Now

\[ a'_n = \frac{1}{2p} \int_{-p}^{p} f'(x) dx = \frac{1}{2p} \int_{-p}^{p} f(x) |_{-p}^{p} = \frac{1}{2p} (f(p) - f(-p)) = 0. \]

Integrating by parts, we get

\[ a'_n = \frac{1}{p} \int_{-p}^{p} f'(x) \cos \frac{n\pi x}{p} \ dx \]

\[ = \frac{1}{p} \left[ f(x) \cos \frac{n\pi x}{p} \right]_{-p}^{b_n} - \frac{n\pi}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p} \ dx \]

\[ = \frac{n\pi}{p} b_n. \]

Similarly,

\[ b'_n = \frac{1}{p} \int_{-p}^{p} f'(x) \sin \frac{n\pi x}{p} \ dx \]

\[ = \frac{1}{p} \left[ f(x) \sin \frac{n\pi x}{p} \right]_{-p}^{a_n} - \frac{n\pi}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p} \ dx \]

\[ = -\frac{n\pi}{p} a_n. \]
29. The function in Exercise 8 is piecewise smooth and continuous, with a piecewise smooth derivative. We have

\[ f'(x) = \begin{cases} \frac{-d}{d} & \text{if } 0 < x < d, \\ 0 & \text{if } d < |x| < p, \\ \frac{c}{\pi} & \text{if } -d < x < 0. \end{cases} \]

The Fourier series of \( f' \) is obtained by differentiating term by term the Fourier series of \( f \) (by Exercise 26). Now the function in this exercise is obtained by multiplying \( f \) by \(-\frac{d}{c}\). So the desired Fourier series is

\[
\frac{d}{c} f'(x) = -\frac{d}{c} \frac{2c}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - \cos \frac{dn\pi}{p}}{n^2} \left( \frac{n\pi}{p} \right) \sin \frac{n\pi}{p} x = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos \frac{dn\pi}{p}}{n} \sin \frac{n\pi}{p} x.
\]

33. The function \( F(x) \) is continuous and piecewise smooth with \( F'(x) = f(x) \) at all the points where \( f \) is continuous (see Exercise 25, Section 2.1). So, by Exercise 26, if we differentiate the Fourier series of \( F \), we get the Fourier series of \( f \). Write

\[ F(x) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi}{p} x + B_n \sin \frac{n\pi}{p} x \right) \]

and

\[ f(x) = \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right). \]

Note that the \( a_0 \) term of the Fourier series of \( f \) is 0 because by assumption \( \int_0^{2p} f(x) \, dx = 0 \). Differentiate the series for \( F \) and equate it to the series for \( f \) and get

\[
\sum_{n=1}^{\infty} \left( -A_n \frac{n\pi}{p} \sin \frac{n\pi}{p} x + B_n \frac{n\pi}{p} \right) = \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right).
\]

Equate the \( n \)th Fourier coefficients and get

\[-A_n \frac{n\pi}{p} = b_n \quad \Rightarrow \quad A_n = -\frac{p}{n\pi} b_n; \]

\[B_n \frac{n\pi}{p} = a_n \quad \Rightarrow \quad B_n = \frac{p}{n\pi} a_n. \]

This derives the \( n \)th Fourier coefficients of \( F \) for \( n \geq 1 \). To get \( A_0 \), note that \( F(0) = 0 \) because of the definition of \( F(x) = \int_0^x f(t) \, dt \). So

\[
0 = F(0) = A_0 + \sum_{n=1}^{\infty} A_n = A_0 + \sum_{n=1}^{\infty} -\frac{p}{n\pi} b_n;
\]

and so \( A_0 = \sum_{n=1}^{\infty} \frac{p}{n\pi} b_n \). We thus obtained the Fourier series of \( F \) in terms of the Fourier coefficients of \( f \); more precisely,

\[
F(x) = \frac{p}{\pi} \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \left( -\frac{p}{n\pi} b_n \cos \frac{n\pi}{p} x + \frac{p}{n\pi} a_n \sin \frac{n\pi}{p} x \right).
\]

The point of this result is to tell you that, in order to derive the Fourier series of \( F \), you can integrate the Fourier series of \( f \) term by term. Furthermore, the only assumption on \( f \) is that it is piecewise
smooth and integrates to 0 over one period (to guarantee the periodicity of \( F \)). Indeed, if you start with the Fourier series of \( f \),

\[
f(t) = \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} t + b_n \sin \frac{n\pi}{p} t \right),
\]

and integrate term by term, you get

\[
F(x) = \int_0^x f(t) \, dt = \sum_{n=1}^{\infty} \left( a_n \int_0^x \cos \frac{n\pi}{p} t \, dt + b_n \int_0^x \sin \frac{n\pi}{p} t \, dt \right)
\]

\[
= \sum_{n=1}^{\infty} \left( a_n \left( \frac{p}{n\pi} \right) \sin \frac{n\pi}{p} t \bigg|_0^x + b_n \left( -\frac{p}{n\pi} \right) \cos \frac{n\pi}{p} t \bigg|_0^x \right)
\]

\[
= \frac{p}{\pi} \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} \left( -\frac{p}{n\pi} b_n \cos \frac{n\pi}{p} x + \frac{p}{n\pi} a_n \sin \frac{n\pi}{p} x \right),
\]

as derived earlier. See the following exercise for an illustration.
Solutions to Exercises 7.4

1. The even extension is the function that is identically 1. So the cosine Fourier series is just the constant 1. The odd extension yields the function in Exercise 1, Section 2.3, with \( p = 1 \). So the sine series is

\[
\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{2k+1}.
\]

This is also obtained by evaluating the integral in (4), which gives

\[
b_n = 2 \int_0^1 \sin(n\pi x) \, dx = -\frac{2}{n\pi} \cos(n\pi) \bigg|_0^1 = \frac{2}{n\pi}(1 - (-1)^n).
\]

9. We have

\[
b_n = 2 \int_0^1 x(1-x) \sin(n\pi x) \, dx.
\]

To evaluate this integral, we will use integration by parts to derive the following two formulas: for \( a \neq 0 \),

\[
\int x \sin(ax) \, dx = -\frac{x \cos(ax)}{a} + \frac{\sin(ax)}{a^2} + C,
\]

and

\[
\int x^2 \sin(ax) \, dx = \frac{2 \cos(ax)}{a^3} - \frac{x^2 \cos(ax)}{a} + \frac{2x \sin(ax)}{a^2} + C.
\]

So

\[
\int x(1-x) \sin(ax) \, dx = -\frac{2 \cos(ax)}{a^3} - \frac{x \cos(ax)}{a} + \frac{x^2 \cos(ax)}{a} + \frac{\sin(ax)}{a^2} - \frac{2x \sin(ax)}{a^2} + C.
\]

Applying the formula with \( a = n\pi \), we get

\[
\int_0^1 x(1-x) \sin(n\pi x) \, dx = \left. -\frac{2 \cos(n\pi x)}{(n\pi)^3} - \frac{x \cos(n\pi x)}{n\pi} + \frac{x^2 \cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{(n\pi)^2} \right|_0^1
\]

\[
= \left\{ \begin{array}{ll}
\frac{4}{(n\pi)^3} & \text{if } n \text{ is odd}, \\
0 & \text{if } n \text{ is even}.
\end{array} \right.
\]

Thus

\[
b_n = \left\{ \begin{array}{ll}
\frac{8}{(n\pi)^3} & \text{if } n \text{ is odd}, \\
0 & \text{if } n \text{ is even}.
\end{array} \right.
\]

Hence the sine series in

\[
\frac{8}{\pi^3} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{(2k+1)^3}.
\]
13. We have
\[ \sin \pi x \cos \pi x = \frac{1}{2} \sin 2\pi x. \]
This yields the desired 2-periodic sine series expansion.

17. (b) Sine series expansion:

\[
\begin{align*}
b_n &= \frac{2}{p} \int_0^a \frac{h}{a} x \sin \frac{n\pi x}{p} \, dx + \frac{2}{p} \int_0^p \frac{h}{a-p} (x-p) \sin \frac{n\pi x}{p} \, dx \\
&= \frac{2h}{ap} \left[ \frac{p}{n\pi} \cos \frac{n\pi x}{p} \bigg|_0^a + \frac{p}{n\pi} \int_0^a \cos \frac{n\pi x}{p} \, dx \right] \\
&\quad + \frac{2h}{(a-p)p} \left[ (x-p) \frac{(-p)}{n\pi} \cos \frac{n\pi x}{p} \bigg|_0^p + \int_a^p \frac{p}{n\pi} \cos \frac{n\pi x}{p} \, dx \right] \\
&= \frac{2h}{pa} \left[ \frac{ap}{n\pi} \cos \frac{n\pi a}{p} + \frac{p^2}{(n\pi)^2} \sin \frac{n\pi a}{p} \right] \\
&\quad + \frac{2h}{(a-p)p} \left[ \frac{p}{n\pi} (a-p) \cos \frac{n\pi a}{p} - \frac{p^2}{(n\pi)^2} \sin \frac{n\pi a}{p} \right] \\
&= \frac{2hp}{(n\pi)^2} \sin \frac{n\pi a}{p} \left[ \frac{1}{p} - \frac{1}{a-p} \right] \\
&= \frac{2hp^2}{(n\pi)^2 (p-a)} \sin \frac{n\pi a}{p}.
\end{align*}
\]

Hence, we obtain the given Fourier series.
Solutions to Exercises 7.5

1. From Example 1, for \( a \neq 0, \pm i, \pm 2i, \pm 3i, \ldots \),

\[
e^{ax} = \frac{\sinh \frac{\pi a}{\pi}}{n} \sum_{n=-\infty}^{\infty} \left( -1 \right)^n \frac{1}{a-in} e^{inx} \quad (-\pi < x < \pi);
\]

consequently,

\[
e^{-ax} = \frac{\sinh \frac{\pi a}{\pi}}{n} \sum_{n=-\infty}^{\infty} \left( -1 \right)^n \frac{1}{a+in} e^{inx} \quad (-\pi < x < \pi),
\]

and so, for \(-\pi < x < \pi\),

\[
\cosh ax = \frac{e^{ax} + e^{-ax}}{2}
\]

\[
= \frac{\sinh \frac{\pi a}{\pi}}{2} \sum_{n=-\infty}^{\infty} \left( -1 \right)^n \left( \frac{1}{a+in} + \frac{1}{a-in} \right) e^{inx}
\]

\[
= \frac{a \sinh \frac{\pi a}{2} \pi}{\pi} \sum_{n=-\infty}^{\infty} \left( -1 \right)^n \frac{n}{n^2 + a^2} e^{inx}.
\]

2. From Example 1, for \( a \neq 0, \pm i, \pm 2i, \pm 3i, \ldots \),

\[
e^{ax} = \frac{\sinh \frac{\pi a}{\pi}}{n} \sum_{n=-\infty}^{\infty} \left( -1 \right)^n \frac{1}{a-in} e^{inx} \quad (-\pi < x < \pi);
\]

consequently,

\[
e^{-ax} = \frac{\sinh \frac{\pi a}{\pi}}{n} \sum_{n=-\infty}^{\infty} \left( -1 \right)^n \frac{1}{a+in} e^{inx} \quad (-\pi < x < \pi),
\]

and so, for \(-\pi < x < \pi\),

\[
\sinh ax = \frac{e^{ax} - e^{-ax}}{2}
\]

\[
= \frac{\sinh \frac{\pi a}{\pi}}{2} \sum_{n=-\infty}^{\infty} \left( -1 \right)^n \left( \frac{1}{a-in} - \frac{1}{a+in} \right) e^{inx}
\]

\[
= i \frac{\sinh \frac{\pi a}{\pi}}{\pi} \sum_{n=-\infty}^{\infty} \left( -1 \right)^n \frac{n}{n^2 + a^2} e^{inx}.
\]

5. Use identities (1); then

\[
\cos 2x + 2 \cos 3x = \frac{e^{2ix} + e^{-2ix}}{2} + \frac{2}{2} e^{3ix} + e^{-3ix}
\]

\[
= e^{-3ix} + \frac{e^{-2ix}}{2} + \frac{e^{2ix}}{2} + e^{3ix}.
\]

9. If \( m = n \) then

\[
c_m = c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} f(x) e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = 1.
\]
If \( m \neq n \), then
\[
c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)x} dx
\]
\[
= \frac{-i}{2(n-m)\pi} e^{i(n-m)x} \Big|_{-\pi}^{\pi}
\]
\[
= \frac{-i}{2(n-m)\pi} (e^{i(n-m)\pi} - e^{-i(n-m)\pi})
\]
\[
= \frac{-i}{2(n-m)\pi} (\cos[(n-m)\pi] - \cos[-(n-m)\pi]) = 0.
\]

Thus all the Fourier coefficients are 0 except \( c_n = n \). Hence \( e^{inx} \) is its own Fourier series.

(b) Because of the linearity of the Fourier coefficients and by part (a), the function is its own Fourier series.

13. (a) This is straightforward. Start with the Fourier series in Exercise 1: For \( a \neq 0 \), \( \pm i, \pm 2i, \pm 3i, \ldots \), and \( -\pi < x < \pi \), we have
\[
\cosh ax = \frac{a \sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{n}{n^2 + a^2} e^{inx}.
\]

On the left side, we have
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \cosh^2(ax) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cosh(2ax) + 1 dx
\]
\[
= \frac{1}{2\pi} \left[ x + \frac{1}{2a} \sinh(2ax) \right]_{-\pi}^{\pi} = \frac{1}{2\pi} \left[ \pi + \frac{1}{2a} \sinh(2a\pi) \right].
\]

On the right side of Parseval’s identity, we have
\[
\frac{(a \sinh \pi a)^2}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + a^2)^2}.
\]

Hence
\[
\frac{1}{2\pi} \left[ \pi + \frac{1}{2a} \sinh(2a\pi) \right] = \frac{(a \sinh \pi a)^2}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + a^2)^2}.
\]

Simplifying, we get
\[
\sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + a^2)^2} = \frac{\pi}{2(a \sinh \pi a)^2} \left[ \pi + \frac{1}{2a} \sinh(2a\pi) \right].
\]

(b) This part is similar to part (a). Start with the Fourier series of Exercise 2: For \( a \neq 0 \), \( \pm i, \pm 2i, \pm 3i, \ldots \), and \( -\pi < x < \pi \), we have
\[
\sinh ax = \frac{i \sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{n}{n^2 + a^2} e^{inx}.
\]

On the left side, we have
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \sinh^2(ax) dx = \frac{1}{\pi} \int_{0}^{\pi} \cosh(2ax) - 1 dx
\]
\[
= \frac{1}{2\pi} \left[ -x + \frac{1}{2a} \sinh(2ax) \right]_{0}^{\pi} = \frac{1}{2\pi} \left[ -\pi + \frac{1}{2a} \sinh(2a\pi) \right].
\]
On the right side of Parseval’s identity, we have
\[ \frac{\sinh^2 \pi a}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{n^2}{(n^2 + a^2)^2}. \]

Hence
\[ \frac{1}{2\pi} \left[ -\pi + \frac{1}{2a} \sinh(2a\pi) \right] = \frac{\sinh^2 (\pi a)}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{n^2}{(n^2 + a^2)^2}. \]

Simplifying, we get
\[ \sum_{n=-\infty}^{\infty} \frac{n^2}{(n^2 + a^2)^2} = \frac{\pi}{2\sinh^2 (\pi a)} \left[ -\pi + \frac{1}{2a} \sinh(2a\pi) \right]. \]

17. In the Taylor series
\[ e^z = \sum_{m=1}^{\infty} \frac{z^m}{m!} \] (all \( z \)),

take \( z = e^{i\theta} \). Then, for all \( \theta \),
\[ e^{i\theta} = \sum_{m=1}^{\infty} \frac{e^{im\theta}}{m!}, \]

which is the complex form of the Fourier series of \( e^{i\theta} = e^{\cos \theta + i\sin \theta} \).

21. To find the complex Fourier coefficients of \( f \), we use (6) and the fact that the (usual) Fourier coefficients of \( f \) are \( a_n = 0 \) for \( n \) and \( b_n = \frac{2(-1)^{n+1}}{n} \) (see Exercise 13, Section 7.2). Thus
\[ c_n = -\frac{i}{2} b_n = -\frac{i(-1)^{n+1}}{n} \]

if \( n > 0 \) and
\[ c_n = \frac{i}{2} b_n = \frac{i(-1)^{n+1}}{n} \]

if \( n < 0 \). Thus the Fourier coefficients of \( f \ast f \) are
\[ -\frac{1}{n^2} \] for all \( n \neq 0 \).

These coefficients are 4 times the coefficients of the convolution that we have in Example 4. Thus the convolution in this exercise is 4 times the function in Example 6. Thus, for \( x \) in \((0, 2\pi)\),
\[ f \ast f(x) = \frac{1}{2} \left( \frac{\pi^2}{3} - (x - \pi)^2 \right). \]

25. (a) At \( x = \pi \), the Fourier series converges to the average of the function,
\[ \frac{(f(\pi+)+f(\pi-))/2}{2} = \frac{e^{i\pi} + e^{-i\pi}}{2} = \cosh a\pi \]
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(this is clear on Figure 1). Thus

\[
\cosh a\pi = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2} (a + in) e^{in\pi} = (-1)^n
\]

\[
= \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{-1} \frac{(a + in)}{a^2 + n^2} + \frac{1}{a} + \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} (a + in)e^{in\pi}
\]

\[
= \frac{1}{a} + \frac{\sinh a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(a - in)}{a^2 + n^2} + \sum_{n=1}^{\infty} \frac{(a + in)}{a^2 + n^2}
\]

\[
= \frac{1}{a} + \frac{\sinh a\pi}{\pi} \sum_{n=1}^{\infty} \frac{2a}{a^2 + n^2}.
\]

(b) Multiply both sides by \(\frac{a\pi}{\sinh a\pi}\) and the desired identity follows.