Solutions to Exercises 5.1

1. Write

\[ f(z) = \frac{1+z}{z} = \frac{1}{z} + 1. \]

\( f \) has one simple pole at \( z_0 = 0 \). The Laurent series expansion of \( f(z) \) at \( z_0 = 0 \) is already given. The residue at 0 is the coefficient of \( \frac{1}{z} \) in the Laurent series \( a_{-1} \). Thus \( a_{-1} = \text{Res}(f, 0) = 1 \).

5. We have one pole of order 3 at \( z_0 = -3i \).

Laurent series of \( f \) around \( z_0 \):

\[
\left(\frac{z - 1}{z + 3i}\right)^3 = \frac{1}{(z + 3i)^3} [(z + 3i) + (-3i - 1)]^3
\]

\[
= \frac{1}{(z + 3i)^3} [ (z + 3i)^3 + 3(z + 3i)^2(-1 - 3i) + 3(z + 3i)(-1 - 3i) + (-3i - 1)^3 ]
\]

\[
= 1 + 3\frac{(-1 - 3i)}{z + 3i} + 3\frac{(-1 - 3i)}{z + 3i}^2 + \frac{(-3i - 1)^3}{(z + 3i)^3}
\]

Thus \( a_{-1} = 3(-1 - 3i) = \text{Res}(-3i) \).

9. Write

\[ f(z) = \csc(\pi z) \frac{z + 1}{z - 1} = \frac{1}{\sin \pi z} \frac{z + 1}{z - 1}. \]

Simple poles at the integers, \( z = k, z \neq 1 \). For \( k \neq 1 \),

\[
\text{Res}(f, k) = \lim_{z \to k} (z - k) \frac{1}{\sin \pi z} \frac{z + 1}{z - 1}
\]

\[
= \lim_{z \to k} \frac{z + 1}{z - 1} \lim_{z \to k} \sin \pi z
\]

\[
= \frac{k + 1}{k - 1} \lim_{z \to k} \frac{1}{\pi \cos \pi z} = (-1)^k \frac{k + 1}{k - 1}.
\]

At \( z_0 = 1 \), we have a pole of order 2. To simplify the computation of the residue, let’s rewrite \( f(z) \) as follows:

\[
\frac{1}{\sin \pi z} \frac{z + 1}{z - 1} = \frac{1}{\sin \pi z} \frac{(z - 1) + 2}{z - 1}
\]

\[
= \frac{1}{\sin \pi z} + \frac{2}{(z - 1) \sin \pi z}
\]

We have

\[ \text{Res}(f, 1) = \text{Res}(\frac{1}{\sin \pi z}, 1) + \text{Res}(\frac{2}{(z - 1) \sin \pi z}, 1); \]

\[ \text{Res}(\frac{1}{\sin \pi z}, 1) = \lim_{z \to 1} \frac{1}{\sin \pi z} = -\frac{1}{\pi} \quad \text{(Use l’Hospital’s rule).} \]

\[ \text{Res}(\frac{2}{(z - 1) \sin \pi z}, 1) = \lim_{z \to 1} \frac{d}{dz} \frac{2(z - 1)}{\sin \pi z}
\]

\[
= 2 \lim_{z \to 1} \frac{\pi \cos \pi z - (z - 1) \pi \cos \pi z}{(\sin \pi z)^2}
\]

\[
= 2 \lim_{z \to 1} \frac{\pi \cos \pi z - (z - 1) \pi \cos \pi z + (z - 1) \pi \sin \pi z}{2 \pi \sin \pi z \cos \pi z} = 0
\]
So \( \text{Res} (f, 1) = -\frac{1}{\pi} \).

13. The easiest way to compute the integral is to apply Cauchy’s generalized formula with

\[ f(z) = \frac{z^2 + 3z - 1}{z^2 - 3}, \]

which is analytic inside and on \( C_1(0) \). Hence

\[ \int_{C_1(0)} \frac{z^2 + 3z - 1}{z(z^2 - 3)} \, dz = 2\pi i f(0) = 2\pi i \left( \frac{1}{3} \right) = \frac{2\pi i}{3}. \]

Note that from this value, we conclude that

\[ \text{Res} \left( \frac{z^2 + 3z - 1}{z(z^2 - 3)} \right), 0) = \frac{1}{3} \]

because the integral is equal to \( 2\pi i \) times the residue at 0.

17. The function

\[ f(z) = \frac{1}{z(z-1)(z-2)\cdots(z-10)} \]

has simple poles at 0 and 1 inside \( C_{\frac{1}{2}}(0) \). We have

\[ \text{Res} (f(z), 0) = \frac{1}{(0-1)(0-2)\cdots(0-10)} = \frac{1}{10!} \]

\[ \text{Res} (f(z), 1) = \frac{1}{1(1-2)\cdots(1-10)} = -\frac{1}{9!}. \]

Hence

\[ \int_{C_{\frac{1}{2}}(0)} \frac{dz}{z(z-1)(z-2)\cdots(z-10)} = 2\pi i \left( \text{Res} (f(z), 0) + \text{Res} (f(z), 1) \right) \]

\[ = 2\pi i \left( \frac{1}{10!} - \frac{1}{9!} \right) = -\frac{18}{10!} \pi i \]

21. The function

\[ f(z) = \frac{e^{z^2}}{z^6} \]

has a pole of order 6 at 0. To compute the residue at 0, we find the coefficient \( a_1 \) in the Laurent serie expansion about 0. We have

\[ \frac{1}{z^6} e^{z^2} = \frac{1}{z^6} \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!}. \]

It is clear that this expansion has no terms with odd powers of \( z \), positive or negative. Hence \( a_{-1} = 0 \) and so

\[ \int_{C_{\frac{1}{2}}(0)} \frac{e^{z^2}}{z^6} \, dz = 2\pi i \text{Res} (0) = 0. \]

25. Same approach as in Exercise 21:

\[ \frac{\sin z}{z^6} = \frac{1}{z^6} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \]

\[ = \frac{1}{z^6} \left[ z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right] \]
Coefficient of $\frac{1}{z}$: $a_{-1} = \frac{1}{5!}$, so
\[
\int_{C_1(0)} \frac{\sin z}{z^6} \, dz = 2\pi i \text{Res } (0) = \frac{2\pi i}{5!}.
\]

29. (a) The Order of a pole of $\csc(\pi z) = \frac{1}{\sin \pi z}$ is the order of the zero of
\[
\frac{1}{\csc(\pi z)} = \sin \pi z.
\]
Since the zeros of $\sin \pi z$ occur at the integers and are all simple zeros (see Example 1, Section 4.6), it follows that $\csc \pi z$ has simple poles at the integers.

(b) For an integer $k$,
\[
\text{Res } (\csc \pi z, k) = \lim_{z \to k} (z-k) \csc \pi z = \lim_{z \to k} \frac{z-k}{\sin \pi z} = \lim_{z \to k} \frac{1}{\frac{1}{\pi} \cos \pi z} \quad (\text{l'Hospital's rule})
\]
\[
= \frac{(-1)^k}{\pi}.
\]

(c) Suppose that $f$ is analytic at an integer $k$. Apply Proposition 1(iii), then
\[
\text{Res } (f(z) \csc(\pi z), k) = \frac{(-1)^k}{\pi} f(k).
\]

33. A Laurent series converges absolutely in its annulus of convergence. Thus to multiply two
Laurent series, we can use Cauchy products and sum the terms in any order. Write
\[
f(z)g(z) = \sum_{n=\infty}^{\infty} a_n (z-z_0)^n \sum_{n=\infty}^{\infty} b_n (z-z_0)^n
\]
\[
= \sum_{n=\infty}^{\infty} c_n (z-z_0)^n,
\]
where $c_n$ is obtained by collecting all the terms in $(z-z_0)^n$, after expanding the product. Thus
\[
c_n = \sum_{j=\infty}^{\infty} a_j b_{n-j};
\]
in particular
\[
c_{-1} = \sum_{j=\infty}^{\infty} a_j b_{-1-j},
\]
and hence
\[
\text{Res } (f(z)g(z), z_0) = \sum_{j=\infty}^{\infty} a_j b_{-j-1}.
\]

37. (a) We have, Exercise 35(a), Section 4.5,
\[
J_0(z) = \frac{1}{2\pi i} \int_{C_1(0)} e^{\frac{z}{\zeta} - \frac{\zeta}{z}} \frac{d\zeta}{\zeta}.
\]
Thus
\[ \int_0^\infty J_0(t)e^{-st} dt = \frac{1}{2\pi i} \int_0^\infty \int_{C_1(0)} e^{-t(s - \frac{1}{2}(\zeta - \frac{1}{\zeta}))} \frac{d\zeta}{\zeta} dt \]

(b) For \( \zeta \) on \( C_1(0) \), we have
\[ \zeta - \frac{1}{\zeta} = \zeta - \bar{\zeta} = 2i \text{Im}(\zeta), \]
which is 0 if \( \text{Im}(\zeta) = 0 \) (i.e., \( \zeta = \pm 1 \)) or is purely imaginary. In any case, for all \( \zeta \in C_1(0) \), and all real \( s > 0 \) and \( t \), we have
\[ |e^{-t(s - \frac{1}{2}(\zeta - \frac{1}{\zeta}))}| = |e^{-ts}| |e^{\frac{1}{2}(\zeta - \frac{1}{\zeta})}| = e^{-ts}. \]

So, by the inequality on integrals (Th.2, Sec. 3.2),
\[ \left| \frac{1}{2\pi i} \int_{C_1(0)} e^{-t(s - \frac{1}{2}(\zeta - \frac{1}{\zeta}))} \frac{d\zeta}{\zeta} \right| \leq \frac{2\pi}{2\pi \zeta} \max_{\zeta \in C_1(0)} \left| e^{-t(s - \frac{1}{2}(\zeta - \frac{1}{\zeta}))} \right| \frac{1}{\zeta} \]
\[ = \max_{\zeta \in C_1(0)} \left| e^{-t(s - \frac{1}{2}(\zeta - \frac{1}{\zeta}))} \right| (|\zeta| = 1) \]
\[ \leq e^{-ts} \]

Thus the iterated integral in (a) is absolutely convergent because
\[ \left| \frac{1}{2\pi i} \int_0^\infty \int_{C_1(0)} e^{-t(s - \frac{1}{2}(\zeta - \frac{1}{\zeta}))} \frac{d\zeta}{\zeta} \right| \leq \int_0^\infty \left| \int_{C_1(0)} e^{-t(s - \frac{1}{2}(\zeta - \frac{1}{\zeta}))} \frac{d\zeta}{\zeta} \right| dt \]
\[ \leq \int_0^\infty e^{-ts} dt = \frac{1}{s} < \infty \]

(c) Interchange the order of integration, and evaluate the integral in \( t \), and get
\[ \int_0^\infty J_0(t)e^{-st} dt = \frac{1}{2\pi i} \int_{C_1(0)} \int_0^\infty e^{-t(s - \frac{1}{2}(\zeta - \frac{1}{\zeta}))} dt \frac{d\zeta}{\zeta} \]
\[ = \frac{1}{2\pi i} \int_{C_1(0)} \frac{1}{(s - \frac{1}{2}(\zeta - \frac{1}{\zeta}))} e^{-t(s - \frac{1}{2}(\zeta - \frac{1}{\zeta}))} \left| \frac{d\zeta}{\zeta} \right| \]
\[ = \frac{1}{\pi i} \int_{C_1(0)} \frac{1}{-\zeta^2 + 2s\zeta + 1} d\zeta \]
because, as \( t \to \infty \),
\[ |e^{-t(s - \frac{1}{2}(\zeta - \frac{1}{\zeta}))}| = e^{-ts} \to 0. \]

(d) We evaluate the integral using the residue theorem. We have simple poles at
\[ \zeta = \frac{-s \pm \sqrt{s^2 + 1}}{-1} = s \pm \sqrt{s^2 + 1}. \]
Only \( s - \sqrt{s^2 + 1} \) is inside \( C_0(1) \). To see this, note that because \( s > 0 \) and \( \sqrt{s^2 + 1} > 1 \), we have \( s + \sqrt{s^2 + 1} > 1 \). Also
\[ s < \sqrt{s^2 + 1} < 1 + s \Rightarrow -1 - s < -\sqrt{s^2 + 1} < -s \Rightarrow -1 < s - \sqrt{s^2 + 1} < 0. \]
By Proposition 1(ii)

\[
\text{Res} \left( \frac{1}{-\zeta^2 + 2s\zeta + 1}, s - \sqrt{s^2 + 1} \right) = \frac{1}{-2s + 2s} \bigg|_{\zeta = s - \sqrt{s^2 + 1}} = \frac{1}{2\sqrt{s^2 + 1}}
\]

Thus, for \( s > 0 \),

\[
\int_{0}^{\infty} J_0(t)e^{-st} \, dt = \frac{2\pi i}{\pi i} \cdot \frac{1}{2\sqrt{s^2 + 1}} = \frac{1}{\sqrt{s^2 + 1}}.
\]

(e) Repeat the above steps making the appropriate changes.

Step (a'): Use the integral representation of \( J_n \) and get

\[
I_n = \int_{0}^{\infty} J_n(t)e^{-st} \, dt = \frac{1}{2\pi i} \int_{0}^{\infty} \int_{C_1(0)} e^{-i(s+\frac{1}{2n}(\zeta - \frac{1}{\zeta}))} \frac{d\zeta}{\zeta^{n+1}} \, dt
\]

Step (b') is exactly like (b) because, for \( \zeta \) on \( C_1(0) \), we have \(|\zeta^{n+1}| = |\zeta| = 1\).

Step (c'): As in (c), we obtain

\[
I_n = \frac{1}{\pi i} \int_{C_1(0)} \frac{1}{(-\zeta^2 + 2s\zeta + 1)\zeta^n} d\zeta.
\]

Let \( \eta = \frac{1}{\zeta} = C_1 \). Then \( d\zeta = -\frac{1}{\eta^2} d\eta \). As \( \zeta \) runs through \( C_1(0) \) in the positive direction, \( \eta \) runs through \( C_1(0) \) in the negative direction. Hence

\[
I_n = \frac{1}{\pi i} \int_{-C_1(0)} \frac{-\frac{1}{\eta^2} d\eta}{(-\frac{1}{\eta^2} + 2s\frac{1}{\eta} + 1)\frac{1}{\eta^n}} = \frac{1}{\pi i} \int_{C_1(0)} \frac{\eta^n d\eta}{\eta^2 + 2sn - 1}
\]

Step (d'): We evaluate the integral using the residue theorem. We have simple poles at

\[
\zeta = \frac{-s \pm \sqrt{s^2 + 1}}{-1} = -s \pm \sqrt{s^2 + 1}.
\]

Only \( -s + \sqrt{s^2 + 1} \) is inside \( C_0(1) \). (Just arge as we did in (d).) By Proposition 1(ii)

\[
\text{Res} \left( \frac{\eta^n}{\eta^2 + 2s\eta - 1}, -s + \sqrt{s^2 + 1} \right) = \frac{\eta^n}{2\eta + 2s} \bigg|_{\eta = -s + \sqrt{s^2 + 1}} = \frac{(\sqrt{s^2 + 1} - s)^n}{2\sqrt{s^2 + 1}}
\]

Thus, for \( s > 0 \),

\[
\int_{0}^{\infty} J_n(t)e^{-st} \, dt = \frac{1}{\sqrt{s^2 + 1}} \left( \sqrt{s^2 + 1} - s \right)^n.
\]
Solutions to Exercises 5.2

1. Let \( z = e^{i\theta} \), \( dz = ie^{i\theta} d\theta \), \( d\theta = -\frac{i}{z} dz \), \( \cos \theta = \frac{z + 1/z}{2} \). Then

\[
\int_0^{2\pi} \frac{d\theta}{2 - \cos \theta} = \int_{C_1(0)} \frac{-\frac{i}{z} dz}{2 - \frac{(z + 1/z)}{2}}
\]

\[
= -i \int_{C_1(0)} \frac{dz}{2z - \frac{z}{2} - \frac{1}{2}}
\]

\[
= -i \int_{C_1(0)} \frac{dz}{-\frac{z^2}{2} + 2z - \frac{1}{2}}
\]

\[
= 2\pi \sum_j \text{Res} \left( -\frac{z^2}{2} + 2z - \frac{1}{2}, z_j \right),
\]

where the sum of the residues extends over all the poles of \( -\frac{z^2}{2} + 2z - \frac{1}{2} \) inside the unit disk. We have

\[-\frac{z^2}{2} + 2z - \frac{1}{2} = 0 \iff z^2 - 4z + 1 = 0.\]

The roots are \( z = 2 \pm \sqrt{3} \), and only \( z_1 = 2 - \sqrt{3} \) is inside \( C_1(0) \). We compute the residue using Proposition 1(ii), Sec. 5.1:

\[
\text{Res} \left( -\frac{z^2}{2} + 2z - \frac{1}{2}, z_1 \right) = \frac{1}{-z_1 + 2} = \frac{1}{\sqrt{3}}.
\]

Hence

\[
\int_0^{2\pi} \frac{d\theta}{2 - \cos \theta} = \frac{2\pi}{\sqrt{3}}.
\]
5. Let \( z = e^{i\theta} \), \( dz = ie^{i\theta} d\theta \), \( d\theta = \frac{-idz}{z} \), \( \cos \theta = \frac{z + 1/2}{2} \), and \( \cos 2\theta = \frac{e^{2i\theta} + e^{-2i\theta}}{2} = \frac{z^2 + 1}{2} \). Then

\[
\int_0^{2\pi} \cos 2\theta \frac{5 + 4 \cos \theta}{5 + 4 \cos \theta} d\theta = -i \int_{C_1(0)} \frac{\frac{1}{2}(z^2 + 1/2)}{5 + 4(\frac{z + 1/2}{2})} \frac{dz}{z}
\]

\[
= -\frac{i}{2} \int_{C_1(0)} \frac{z^4 + 1}{z^2 + 2z + 2z} \frac{dz}{z}
\]

\[
= -\frac{i}{2} \int_{C_1(0)} \frac{z^4 + 1}{z^2(2z^2 + 5z + 2)} \frac{dz}{z}
\]

\[
= -\frac{i}{2} \sum_j 2\pi i \text{Res} \left( \frac{z^4 + 1}{z^2(2z^2 + 5z + 2)}, z_j \right)
\]

where the sum of the residues extends over all the poles of \( \frac{z^4 + 1}{z^2(2z^2 + 5z + 2)} \) inside the unit disk. We have a pole of order 2 at 0 and possible more poles at the roots of \( 2z^2 + 5z + 2 = 0 \). Let’s compute the residue at 0.

\[
\text{Res} \left( \frac{z^4 + 1}{z^2(2z^2 + 5z + 2)} \right) = 0 \Rightarrow \frac{d}{dz} \frac{z^4 + 1}{2z^2 + 5z + 2} = -\frac{5}{4}.
\]

For the nonzero poles, solve

\[
2z^2 + 5z + 2 = 0.
\]

The roots are \( z = \frac{-5 \pm 3}{4} \). Only \( z_1 = -\frac{1}{2} \) is inside \( C_1(0) \). We compute the residue using Proposition 1(ii), Sec. 5.1:

\[
\text{Res} \left( \frac{z^4 + 1}{z^2(2z^2 + 5z + 2)}, z_1 \right) = \frac{z_1^4 + 1}{z_1^2} \left. \frac{d}{dz} \frac{1}{(2z^2 + 5z + 2)} \right|_{z_1} = \frac{(1/2)^4 + 1}{(1/2)(2(1/2) + 5)} = \frac{17}{12}.
\]

Hence

\[
\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4 \cos \theta} d\theta = \pi \left( \frac{17}{12} - \frac{5}{4} \right) = \frac{\pi}{6}.
\]
9. Let \( z = e^{i\theta} \), \( dz = ie^{i\theta}d\theta \), \( d\theta = \frac{-i\,dz}{z} \), \( \cos \theta = \frac{z + 1/z}{2} \), \( \sin \theta = \frac{z - 1/z}{2i} \). Then

\[
\int_{0}^{2\pi} \frac{d\theta}{7 + 2\cos \theta + 3\sin \theta} = -i \int_{C_{1}(0)} \frac{dz}{z(7 + (z + 1/z) + \frac{1}{2!}z^2 - \frac{1}{3!}z^3)}
\]

\[
= -i \int_{C_{1}(0)} \frac{dz}{(1 - \frac{3}{2}i)z^2 + 7z + (1 + \frac{3}{2}i)}
\]

\[
= 2\pi \sum_{j} \text{Res} \left( \frac{1}{(1 - \frac{3}{2}i)z^2 + 7z + (1 + \frac{3}{2}i)} , z_{j} \right),
\]

where the sum of the residues extends over all the poles of \( \frac{1}{(1 - \frac{3}{2}i)z^2 + 7z + (1 + \frac{3}{2}i)} \) inside the unit disk. Solve

\[
(1 - \frac{3}{2}i)z^2 + 7z + (1 + \frac{3}{2}i) = 0.
\]

You’ll find

\[
z = \frac{-7 \pm \sqrt{49 - 4(1 - \frac{3}{2}i)(1 + \frac{3}{2}i)}}{2(1 - \frac{3}{2}i)} = \frac{-7 \pm \sqrt{49 - 4(1 + \frac{3}{4})}}{2(1 - \frac{3}{2}i)}
\]

\[
= \frac{-7 \pm \sqrt{36}}{2 - 3i} = \frac{-7 \pm 6}{2 - 3i}
\]

\[
= \frac{-13}{2 - 3i} \text{ or } \frac{-1}{2 - 3i}
\]

\[
= \frac{-13(2 + 3i)}{13} = -(2 + 3i) \text{ or } \frac{-1}{2 - 3i} = \frac{2 + 3i}{13}.
\]

We have

\[
|2 + 3i| = \sqrt{13} > 1 \text{ and } \left| \frac{-2 + 3i}{13} \right| = \frac{\sqrt{13}}{13} < 1.
\]

So only \( z_{1} = \frac{-2 + 3i}{13} \) is inside \( C_{1}(0) \). We compute the residue using Proposition 1(ii), Sec. 5.1:

\[
\text{Res} \left( z_{1} \right) = \frac{1}{2(1 - \frac{3}{2}i)z_{1} + 7}
\]

\[
= \frac{1}{-2(1 - \frac{3}{2}i)\frac{-2 + 3i}{13} + 7} = \frac{1}{6}.
\]

Hence

\[
\int_{0}^{2\pi} \frac{d\theta}{7 + 2\cos \theta + 3\sin \theta} = 2\pi \frac{1}{6} = \frac{\pi}{3}.
\]
13. The solution will vary a little from what is in the text. Note the trick based on periodicity.

**Step 1:** Double angle formula

\[ a + b \cos^2 \theta = a + b \left( \frac{1 + \cos 2\theta}{2} \right) = \frac{2a + b + b \cos 2\theta}{2}, \]

so

\[ \frac{1}{a + b \cos^2 \theta} = \frac{2}{2a + b + b \cos 2\theta}. \]

**Step 2.** Change variables in the integral: \( t = 2\theta, \ dt = 2d\theta \). Then

\[ I = \int_0^{2\pi} \frac{d\theta}{a + b \cos^2 \theta} = \int_0^{4\pi} \frac{dt}{2a + b + b \cos t}. \]

The function \( f(t) = \frac{1}{2a + b + b \cos t} \) is \( 2\pi \)-periodic. Hence its integral over intervals of length \( 2\pi \) are equal. So

\[ I = \int_0^{2\pi} \frac{dt}{2a + b + b \cos t} + \int_{2\pi}^{4\pi} \frac{dt}{2a + b + b \cos t} = 2 \int_0^{2\pi} \frac{dt}{2a + b + b \cos t}. \]

**Step 4.** Now use the method of Section 5.2 to evaluate the last integral. Let \( z = e^{it}, \ dz = ie^{it}dt, \ dt = \frac{-i\ dz}{z} \). Then

\[ 2 \int_0^{2\pi} \frac{dt}{2a + b + b \cos t} = -4i \int_{C_1(0)} \frac{dz}{bz^2 + (4a + 2b)z + b} = 8\pi \sum_j \text{Res} \left( \frac{1}{bz^2 + (4a + 2b)z + b}, z_j \right), \]

where the sum of the residues extends over all the poles of \( \frac{1}{bz^2 + (4a + 2b)z + b} \) inside the unit disk. Solve

\[ bz^2 + (4a + 2b)z + b = 0, \]

and get

\[ z = \frac{-4a + 2b \pm \sqrt{(4a + 2b)^2 - 4b^2}}{2b} = \frac{-(2a + b) \pm 2\sqrt{a(a + b)}}{b} \]

\[ = z_1 = \frac{-(2a + b) + 2\sqrt{a(a + b)}}{b} \quad \text{or} \quad z_2 = \frac{-(2a + b) - 2\sqrt{a(a + b)}}{b}. \]

It is not hard to prove that \(|z_1| < 1\) and \(|z_2| > 1\). Indeed, for \( z_2 \), we have

\[ |z_2| = \frac{2a + b}{b} + \frac{2\sqrt{a(a + b)}}{b} = 1 + \frac{2a + 2\sqrt{a(a + b)}}{b} > 1 \]

because \( a, b \) are > 0. Now the product of the roots of a quadratic equation \( \alpha z^2 + \beta z + \gamma = 0 \) \( (\alpha \neq 0) \) is always equal to \( \frac{\gamma}{\alpha} \). Applying this in our case, we find that \( z_1 \cdot z_2 = 1 \), and since \(|z_2| > 1\), we must have \(|z_1| < 1\).

We compute the residue at \( z_1 \) using Proposition 1(ii), Sec. 5.1:

\[ \text{Res} \left( \frac{1}{bz^2 + (4a + 2b)z + b}, z_1 \right) = \frac{1}{2bz_1 + (4a + 2b)} \]

\[ = \frac{1}{-(2a + b) + 2\sqrt{a(a + b)} + (4a + 2b)} \]

\[ = \frac{1}{4\sqrt{a(a + b)}}. \]
Hence

\[ \int_{0}^{2\pi} \frac{d\theta}{a + \cos^2 \theta} = 2\pi \frac{1}{4\sqrt{a(a + b)}} = \frac{2\pi}{\sqrt{a(a + b)}} \]
Solutions to Exercises 5.3

1. Use the same contour as Example 1. Several steps in the solution are very similar to those in Example 1; in particular, Steps 1, 2, and 4. Following the notation of Example 1, we have

$$I_{\gamma_R} = I_{[-R, R]} + I_{\sigma_R}.$$  

Also, $\lim_{R \to \infty} I_{\sigma_R} = 0$, and

$$\lim_{R \to \infty} I_{[-R, R]} = I = \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}.$$  

These assertions are proved in Example 1 and will not be repeated here. So all we need to do is evaluate $I_{\gamma_R}$ for large values of $R$ and then let $R \to \infty$. We have

$$I_{\gamma_R} = 2\pi i \sum_{j} \text{Res} \left( \frac{1}{z^4 + 1}, z_j \right),$$  

where the sum ranges over all the residues of $\frac{1}{z^4 + 1}$ in the upper half-plane. The function $\frac{1}{z^4 + 1}$ have four (simple) poles. These are the roots of $z^4 + 1 = 0$ or $z^4 = -1$. Using the result of Example 1, we find the roots to be

$$z_1 = \frac{1 + i}{\sqrt{2}}, \quad z_2 = \frac{-1 + i}{\sqrt{2}}, \quad z_3 = \frac{-1 - i}{\sqrt{2}}, \quad z_4 = \frac{1 - i}{\sqrt{2}}.$$  

In exponential form,

$$z_1 = e^{i\frac{\pi}{4}}, \quad z_2 = e^{i\frac{3\pi}{4}}, \quad z_3 = e^{i\frac{5\pi}{4}}, \quad z_4 = e^{i\frac{7\pi}{4}}.$$  

Only $z_1$ and $z_2$ are in the upper half-plane, and so inside $\gamma_R$ for large $R > 0$. Using Prop.1(ii), Section 5.1,

$$\text{Res} \left( \frac{1}{z^4 + 1}, z_1 \right) = \frac{1}{\frac{d}{dz} z^4 + 1 \bigg|_{z=z_1}} = \frac{1}{4z_1^3} = \frac{1}{4} e^{-i\frac{\pi}{4}},$$

$$\text{Res} \left( \frac{1}{z^4 + 1}, z_2 \right) = \frac{1}{\frac{d}{dz} z^4 + 1 \bigg|_{z=z_2}} = \frac{1}{4z_2^3} = \frac{1}{4} e^{i\frac{3\pi}{4}} = \frac{1}{4} e^{-i\frac{\pi}{4}}.$$  

So

$$I_{\gamma_R} = 2\pi i \left( \frac{1}{4} e^{-i\frac{\pi}{4}} + \frac{1}{4} e^{-i\frac{3\pi}{4}} \right)$$

$$= \frac{\pi i}{2} \left( \cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} + \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right)$$

$$= \frac{\pi i}{2} \left(-i \sqrt{2} \right) = \frac{\pi}{\sqrt{2}}.$$  

Letting $R \to \infty$, we obtain $I = \frac{\pi}{\sqrt{2}}$. 
5. The integral converges absolutely, as in Step 1 of Example 1. We will reason as in this example, and omit some of the details. Here $I_{\gamma R} = I_{[-R, R]} + I_{\sigma R}$; $\lim_{R \to \infty} I_{\sigma R} = 0$, and

$$\lim_{R \to \infty} I_{[-R, R]} = I = \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^3}.$$

All we need to do is evaluate $I_{\gamma R}$ for large values of $R$ and then let $R \to \infty$. The function $f(z) = \frac{1}{(z^2 + 1)^3}$ has poles of order 3 at $z_1 = i$ and $z_2 = -i$, but only $z_1$ is in the upper half-plane. You can evaluate the integral $I_{\gamma R}$ using the residue theorem; however, an equal good and perhaps faster way in this case is to use Cauchy’s generalized integral formula (Sec. 3.6). Write

$$I_{\gamma R} = \int_{\gamma R} \frac{dz}{(z^2 + 1)^3} = \int_{\gamma R} \frac{1}{(z + i)(z - i)^3} dz = \int_{\gamma R} \frac{dz}{(z + i)^3 (z - i)^3}.$$

Let $g(z) = \frac{1}{(z + i)^3}$. According to Theorem 2, Sec. 3.6,

$$I_{\gamma R} = 2\pi i \frac{g''(i)}{2!}.$$

Compute:

$$g'(z) = -3(z + i)^{-4}, \quad g''(z) = 12(z + i)^{-5},$$

so

$$g''(i) = 12(2i)^{-5} = \frac{12}{25}(-i).$$

Finally,

$$I_{\gamma R} = \frac{2\pi}{25} = \frac{3\pi}{8}.$$

Letting $R \to \infty$, we obtain $I = \frac{3\pi}{8}$.

9. Proceed as in Example 3, as follows. Let $x = e^t, \ dx = e^t \ dt$. Then

$$\int_{0}^{\infty} \frac{dx}{x^3 + 1} = \int_{-\infty}^{\infty} \frac{e^t}{e^{3t} + 1} dt = \int_{-\infty}^{\infty} \frac{e^x}{e^{3x} + 1} dx,$$

where in the last integral we reverted to the variable $x$ for convenience. Use a contour as in Fig. 6. Refer to Example 3 for notation. As in Example 3, $\lim_{R \to \infty} |I_2| = 0$ and $\lim_{R \to \infty} |I_4| = 0$. Let us repeat the proof for $I_4$ here. For $I_4$, $z$ is on $\gamma_4$, so $z = -R + iy$, where $0 \leq y \leq \frac{2\pi}{3}$. Hence

$$|I_4| = \left| \int_{\gamma_4} \frac{e^z}{e^{3z} + 1} \ dz \right| \leq l(I_4) \cdot M,$$

where $l(I_4) = \frac{2\pi}{3} = \text{length of vertical side } \gamma_4$, and $M$ is the maximum value of $\left| \frac{e^z}{e^{3z} + 1} \right|$ on $\gamma_4$. For $z$ on $\gamma_4$,

$$\left| \frac{e^z}{e^{3z} + 1} \right| = \left| \frac{e^{R+iy}}{e^{3(-R+iy)} + 1} \right| \leq \frac{e^{-R}}{1 - e^{-3R}},$$

which tends to 0 as $R \to \infty$. To justify the last inequality, note that by the reverse triangle inequality:

$$\left| e^{3(-R+iy)} + 1 \right| \geq 1 - \left| e^{-3R} \right| \left| e^{3iy} \right| \overset{R=1}{=} 1 - e^{-3R},$$
so
\[ \left| \frac{1}{e^{3(-R+y)} + 1} \right| \leq \frac{1}{1 - e^{-3R}} \]
as desired.

As in Example 3, let \( I_1 \to I \) as \( R \to \infty \). For \( I_3 \), we have \( z = x + \frac{2\pi}{3} \), so \( x \) varies from \( R \) to \( -R \), \( dz = dx \), so
\[
I_3 = \int_{-R}^{R} \frac{e^{x + \frac{2\pi}{3}}}{e^{3x + 2\pi} + 1} \, dx = -e^{-\frac{2\pi}{3}} \int_{-R}^{R} \frac{e^{x}}{e^{3x} + 1} \, dx = -e^{-\frac{2\pi}{3}} I_1.
\]

Now
\[
e^{3z} + 1 = 0 \quad \Rightarrow \quad e^{3z} = -1 = e^{\pi i}
\]
\[
\Rightarrow \quad 3z = i\pi + 2k\pi i, \quad (k = 0, 1, \pm 2, \ldots)
\]
\[
\Rightarrow \quad z = \frac{i\pi}{3} (2k + 1).
\]

Only one root, \( z = \frac{i\pi}{3} \), is inside \( \gamma_R \), and so the function \( \frac{e^{z}}{e^{3z} + 1} \) has one pole at \( z = \frac{i\pi}{3} \) inside \( \gamma_R \). Hence (see Example 3 for a justification)
\[
\left(1 - e^{-\frac{2\pi}{3}}\right) I = 2\pi i \text{Res} \left( \frac{e^{z}}{e^{3z} + 1}, \frac{\pi i}{3} \right)
\]
\[
= 2\pi i \frac{e^{\frac{\pi i}{3}}}{3e^{\frac{2\pi i}{3}} + 1} = \frac{2\pi i}{3} \frac{e^{\frac{i\pi}{3}}}{e^{\frac{2\pi i}{3}}}
\]
\[
= \frac{2\pi i}{3} \frac{e^{\pi i}}{\sin \frac{\pi}{3}} = -2\pi i \frac{1}{\sin \frac{\pi}{3}}.
\]

Solving for \( I \),
\[
I = \frac{2\pi}{3} \frac{e^{\pi i}}{\sin \frac{\pi}{3}} = \frac{\pi}{3 \sin \frac{\pi}{3}} = \frac{2\pi}{3 \sqrt{3}}.
\]

13. Use the same reasoning as in Example 3. Let \( x = e^t \), \( dx = e^t \, dt \). Then
\[
I = \int_{0}^{\infty} \frac{\sqrt{x}}{x^3 + 1} \, dx = \int_{-\infty}^{\infty} \frac{e^\frac{1}{2}}{e^{3t} + 1} \, e^t \, dt = \int_{-\infty}^{\infty} \frac{e^{\frac{3t}{2}}}{e^{3x} + 1} \, dx.
\]

Use a contour as in Fig.6. Refer to Example 3 for notation: \( \lim_{R \to \infty} |I_2| = 0 \), \( \lim_{R \to \infty} |I_4| = 0 \), and \( \lim_{R \to \infty} |I_1| = I \), the desired integral. For \( I_3 \), we have \( z = x + \frac{2\pi}{3} \), so \( x \) varies from \( R \) to \( -R \), \( dz = dx \), so
\[
I_3 = \int_{R}^{-R} \frac{e^{\frac{3x}{2} + \frac{2\pi i}{3}}}{e^{3x} + 1} \, dx = \int_{R}^{-R} \frac{e^{\frac{3x}{2} \pi i}}{e^{3x} + 1} \, dx = \int_{-R}^{R} \frac{e^{\frac{3x}{2} \pi i}}{e^{3x} + 1} \, dx = I_1.
\]
Same poles as in Exercise 9. So

$$2I = 2\pi i \text{Res} \left( \frac{e^{\frac{3}{2}z}}{e^{3z} + 1} \right)$$

$$= 2\pi i \frac{e^{\frac{3}{2}z}}{\pi (e^{3z} + 1)|_{z = \frac{i\pi}{3}}}$$

$$= 2\pi i \frac{e^{\frac{3}{2}i\pi}}{3e^{i\pi}} = \frac{2\pi i(i)}{-3} = \frac{2\pi}{3};$$

$$I = \frac{\pi}{3}$$

17. As in Example 4: Let $2x = e^t$, $2dx = e^t \, dt$. Then

$$I = \int_0^\infty \frac{\ln(2x)}{x^2 + 4} \, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{t}{t^2 + 4} \, e^{-t} \, dt = 2 \int_{-\infty}^\infty \frac{xe^x}{e^{2x} + 16} \, dx.$$ 

Use a rectangular contour as in Fig. 6, whose vertical sides have length $\pi$. Refer to Example 3 for notation: $\lim_{R \to \infty} |I_2| = 0$, $\lim_{R \to \infty} |I_4| = 0$, and $\lim_{R \to \infty} |I_1| = I$, the desired integral. For $I_3$, $z = x + i\pi$, $x$ varies from $R$ to $-R$, $dz = dx$, so

$$I_3 = 2 \int_R^{-R} \frac{(x + i\pi)e^{x+i\pi}}{e^{2x+2\pi i} + 16} \, dx$$

$$= -2 \int_R^{-R} \frac{(x + i\pi)e^t(-1)}{e^{2x} + 16} \, dx$$

$$= 2 \int_R^{-R} \frac{xe^x}{e^{2x} + 16} \, dx + 2\pi i \int_R^{-R} \frac{e^x}{e^{2x} + 16} \, dx = I_1 + B_Ri,$$

where $B_R$ is a real constant, because the integrand is real-valued. We have

$$I_{\gamma_R} = I_1 + I_2 + I_3 + I_4 = 2I_1 + I_2 + I_4 + iB_R.$$ 

Letting $R \to \infty$, we get

$$\lim_{R \to \infty} I_{\gamma_R} = 2I + iB,$$

where

$$B = \lim_{R \to \infty} B_R = \int_{-\infty}^\infty \frac{e^x}{e^{2x} + 16} \, dx.$$ 

At the same time

$$I_{\gamma_R} = 2\pi i \text{Res} \left( \frac{2ze^z}{e^{2z} + 16}, z_0 \right),$$

where $z_0$ is the root of $e^{2z} + 16 = 0$ that lies inside $\gamma_R$ (there is only one root, as you will see):

$$e^{2z} + 16 = 0 \Rightarrow e^{2z} = 16e^{i\pi} = e^{i\ln(16)+i\pi}$$

$$\Rightarrow 2z = 4\ln 2 + i\pi + 2k\pi i$$

$$\Rightarrow z = 2\ln 2 + i\frac{\pi}{2}(2k + 1).$$
Only $z = 2 \ln 2 + i \frac{\pi}{2}$ is inside the contour. So

$$I_{\gamma_R} = 2\pi i \text{Res} \left( \frac{2ze^z}{e^{2z} + 16}, 2\ln 2 + i \frac{\pi}{2} \right)$$

$$= 2(2 \ln 2 + i \frac{\pi}{2}) e^{2 \ln 2 + i \frac{\pi}{2}}$$

$$= \frac{\pi}{4} (4 \ln 2 + i \pi)$$

Thus

$$2I + 2\pi i B = \pi \ln 2 + i \frac{\pi^2}{4}.$$  

Taking real and imaginary parts, we find

$$I = \frac{\pi \ln 2}{2} \quad \text{and} \quad B = \frac{\pi}{8}.$$  

This gives the value of the desired integral $I$ and also of the integral

$$\frac{\pi}{8} = B = \int_{-\infty}^{\infty} \frac{e^x}{e^{2x} + 16} \, dx.$$  

21. We use the contour $\gamma_R$ in Figure 9 and follow the hint. Write $I_{\gamma_R} = I_1 + I_2 + I_3$. On $\gamma_1$, $z = x$,

$$dz = dx,$$

$$\int_{\gamma_1} \frac{1}{z^3 + 1} \, dz = \int_0^R \frac{1}{x^3 + 1} \, dx \to I = \int_0^\infty \frac{1}{x^3 + 1} \, dx, \quad \text{as} \ R \to \infty.$$  

On $\gamma_3$, $z = e^{ \frac{2\pi}{3} i } x$, $dz = e^{ \frac{2\pi}{3} i } \, dx$, $x$ varies from $R$ to $0$:

$$\int_{\gamma_3} \frac{1}{z^3 + 1} \, dz = e^{ \frac{2\pi}{3} i } \int_0^R \frac{1}{e^{3\frac{2\pi}{3} i } x^3 + 1} \, dx$$

$$= -e^{ \frac{2\pi}{3} i } \int_0^R \frac{1}{x^3 + 1} \, dx \to -e^{ \frac{2\pi}{3} i } I \quad \text{as} \ R \to \infty.$$  

For $I_2$, we have $z = Re^{it}$, $0 \leq t \leq \frac{2\pi}{3}$:

$$\left| \int_{I_{\gamma_2}} \frac{1}{z^3 + 1} \, dz \right| \leq l(\gamma_2) \max_{z \text{ on } \gamma_2} \left| \frac{1}{z^3 + 1} \right|$$

$$\leq 2\pi \cdot \frac{1}{R^3 - 1} \to 0 \quad \text{as} \ R \to \infty.$$  

Now

$$I_1 + I_2 + I_3 = I_{\gamma_R} = 2\pi i \text{Res} \left( \frac{1}{z^3 + 1}, z_0 \right),$$

where $z_0 = e^{i \frac{\pi}{2}}$ is the only pole of $\frac{1}{z^3 + 1}$ inside $\gamma_R$. By Proposition 1(ii), Sec. 5.1,

$$\text{Res} \left( \frac{1}{z^3 + 1}, e^{i \frac{\pi}{2}} \right) = \frac{1}{3e^{i \frac{\pi}{2}}} = \frac{e^{-i \frac{\pi}{3}}}{3},$$

So

$$I_1 + I_2 + I_3 = 2\pi i \frac{e^{-i \frac{\pi}{3}}}{3}.$$
Letting $R \to \infty$, we get
\[
I - e^{-i\pi}I = 2\pi i \frac{e^{-i\frac{2\pi}{3}}}{3} \Rightarrow (1 - e^{\frac{2\pi}{3}i})I = 2\pi i \frac{e^{-i\frac{2\pi}{3}}}{3}
\]
\[
\Rightarrow I = 2\pi i \frac{e^{-i\frac{2\pi}{3}}}{3} \times \frac{1}{1 - e^{\frac{2\pi}{3}i}}
\]
\[
\Rightarrow I = 2\pi i \frac{e^{-i\pi}}{3 (e^{-\frac{\pi}{3}i} - e^{\frac{\pi}{3}i})} = \frac{\pi}{3 \sin \frac{\pi}{3}}
\]
\[
\Rightarrow I = \frac{2\pi}{3\sqrt{3}}
\]

**Solutions to Exercises 5.4**

1. Use a contour as in Fig. 2 and proceed exactly as in Example 1, replacing $s$ by 4. No need to repeat the details.

5. The degree of the denominator is 2 more than the degree of the numerator; so we can use the contour in Fig. 2 and proceed as in Example 1.

**Step 1:** The integral is absolutely convergent.
\[
\left| \frac{x^2 \cos 2x}{(x^2+1)^2} \right| \leq \frac{x^2}{(x^2+1)^2} \leq \frac{1}{x^2+1},
\]
because
\[
\frac{x^2}{(x^2+1)^2} = \frac{(x^2+1)-1}{(x^2+1)^2} = \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2} \leq \frac{1}{x^2+1}.
\]
Since $\int_{-\infty}^{\infty} \frac{1}{x^2+1} \, dx < \infty$ (you can actually compute the integral $= \pi$), we conclude that our integral is absolutely convergent.

**Step 2:**
\[
\int_{-\infty}^{\infty} \frac{x^2 \cos 2x}{(x^2+1)^2} \, dx = \int_{-\infty}^{\infty} \frac{x^2 \cos 2x}{(x^2+1)^2} \, dx + i \int_{-\infty}^{\infty} \frac{x^2 \sin 2x}{(x^2+1)^2} \, dx = \int_{-\infty}^{\infty} \frac{x^2 e^{2ix}}{(x^2+1)^2} \, dx
\]
because
\[
\int_{-\infty}^{\infty} \frac{x^2 \sin 2x}{(x^2+1)^2} \, dx = 0,
\]
being the integral of an odd function over a symmetric interval.

**Step 3:** Let $\gamma_R$ and $\sigma_R$ be as in Fig. 2. We will show that
\[
\int_{\sigma_R} \frac{z^2 e^{2iz}}{(z^2+1)^2} \, dz \to 0 \text{ as } R \to \infty.
\]
\[
|I_{\sigma_R}| = \left| \int_{\sigma_R} \frac{z^2 e^{2iz}}{(z^2+1)^2} \, dz \right| \leq l(\sigma_R) \max_{z \in \sigma_R} \left| \frac{z^2 e^{2iz}}{(z^2+1)^2} \right| = \pi R \cdot M.
\]
As in (7) in Sec. 5.4, for $z$ on $\sigma_R$,
\[
|e^{2iz}| \leq e^{-2R \sin \theta} \leq 1.
\]
So
\[
\left| \frac{z^2 e^{2iz}}{(z^2+1)^2} \right| \leq \left| \frac{z^2}{(z^2+1)^2} \right| \leq \left| \frac{z^2 + 1}{(z^2+1)^2} \right|
\]
\[
\leq \left| \frac{1}{z^2 + 1} \right| + \left| \frac{1}{(z^2+1)^2} \right|
\]
\[
\leq \frac{1}{R^2 - 1 + (R^2 - 1)^2}.
\]
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So

\[ |I_{\sigma_R}| = \frac{\pi R}{R^2 - 1} + \frac{\pi R}{(R^2 - 1)^2}, \]

and this goes to zero as \( R \to \infty \).

**Step 4:** We have

\[ \int_{\gamma_R} \frac{z^2 e^{2iz}}{(z^2 + 1)^2} \, dz = 2\pi \text{Res} \left( \frac{z^2 e^{2iz}}{(z^2 + 1)^2}, i \right), \]

because we have only one pole of order 2 at \( i \) in the upper half-plane.

\[ \text{Res} \left( \frac{z^2 e^{2iz}}{(z^2 + 1)^2}, i \right) = \lim_{z \to i} \frac{d}{dz} \left[ (z - i)^2 \frac{z^2 e^{2iz}}{(z^2 + 1)^2} \right]_{z=i} = \frac{1}{e^{-2}}, \]

after many (hard-to-type-but-easy-to-compute) steps that we omit. So \( I_{\gamma_R} = 2\pi i \left( \frac{e^{-2}}{4} \right) = -\pi \frac{e^{-2}}{2} \). Letting \( R \to \infty \) and using the fact that \( I_{\gamma_R} \to I \), the desired integral, we find \( I = -\pi \frac{e^{-2}}{2} \).

9. In the integral the degree of the denominator is only one more than the degree of the numerator. So the integral converges in the principal value sense as in Examples 2 or 4. Let us check if the denominator has roots on the real axis:

\[ x^2 + x + 9 = 0 \Rightarrow x = -\frac{1}{2} \pm i \frac{\sqrt{35}}{2}. \]

We have no roots on the real axis, so we will proceed as in Example 2, and use Jordan’s Lemma. Refer to Example 2 for further details of the solution. Consider

\[ \int_{\gamma_R} \frac{z}{z^2 + z + 9} e^{i\pi z} \, dz = \int_{\gamma_R} f(z) e^{i\pi z} \, dz, \]

where \( \gamma_R \) is as in Fig. 5. By Corollary 1,

\[ \left| \int_{\sigma_R} f(z) e^{i\pi z} \, dz \right| \to 0, \text{ as } R \to \infty. \]

Apply the residue theorem:

\[ \int_{\pi_R} f(z) e^{i\pi z} \, dz = 2\pi i \text{Res} \left( f(z) e^{i\pi z}, z_1 \right), \]

where \( z_1 = -\frac{1}{2} + i \frac{\sqrt{35}}{2} \) is the only (simple) pole of \( f(z) e^{i\pi z} \) in the upper half-plane. By Proposition 1(ii), Sec. 5.1, we have

\[ \text{Res} \left( f(z) e^{i\pi z}, z_1 \right) = \frac{ze^{i\pi z}}{\frac{1}{z_1^2}(z_1^2 + z + 9)} \bigg|_{z = z_1} = \frac{z_1 e^{i\pi z_1}}{2z_1 + 1} = \frac{-\left( \frac{1}{2} + \frac{\sqrt{35}}{2} \right) e^{-\pi \frac{\sqrt{35}}{2}}}{2 \sqrt{35}} = \frac{e^{-\pi \frac{\sqrt{35}}{2}}}{2 \sqrt{35}} - i \frac{e^{-\pi \frac{\sqrt{35}}{2}}}{2}. \]
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So

\[
\int_{\gamma_R} f(z)e^{i\pi z} \, dz = 2\pi i \left( \frac{e^{-\pi \sqrt{35}}}{2\sqrt{35}} - \frac{e^{-\sqrt{35}}}{2} \right) = \pi e^{-\pi \sqrt{35}} + i\pi \sqrt{35}
\]

But

\[
\lim_{R \to \infty} \int_{\gamma_R} f(z)e^{i\pi z} \, dz = \int_{-\infty}^{\infty} \frac{x \cos \pi x}{x^2 + x + 9} \, dx + i \int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + x + 9} \, dx.
\]

Taking real and imaginary parts, we get

\[
\int_{-\infty}^{\infty} \frac{x \cos \pi x}{x^2 + x + 9} \, dx = \pi e^{-\pi \sqrt{35}} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + x + 9} \, dx = \pi e^{-\pi \sqrt{35}}.
\]

13. Use an indented contour \( \gamma_{r,R} \) as in Fig. 9, with an indentation around 0, and consider the integral

\[
I_{r,R} = \int_{\gamma_{r,R}} \frac{1 - e^{iz}}{z^2} \, dz = \int_{\gamma_{r,R}} g(z) \, dz,
\]

where \( g(z) = \frac{1 - e^{iz}}{z^2} \). Note that \( g(z) \) has a simple pole at 0. To see this, consider its Laurent series expansion around 0:

\[
g(z) = \frac{1}{z^2} \left( 1 - (1 + iz) + (iz)^2/2! + (iz)^3/3! + \cdots \right)
\]

\[
= -\frac{i}{z} + \frac{1}{2} + \frac{z}{3!} - \cdots
\]

Moreover, \( \text{Res}(g(z), 0) = -i \). By Corollary 2,

\[
\lim_{r \to 0^+} \int_{\sigma_r} g(z) = i\pi(-i) = \pi.
\]

(Keep in mind that \( \sigma_r \) has a positive orientation, so it is traversed in the opposite direction on \( \gamma_{r,R} \). See Fig. 9.) On the outer semi-circle, we have

\[
\left| \int_{\sigma_R} g(z) \, dz \right| \leq \frac{\pi R}{R^2} \max_{z \in \sigma_R} |1 - e^{iz}| = \frac{\pi}{R} \max_{z \in \sigma_R} |1 - e^{iz}|.
\]

Using an estimate as in Example 1, Step 3, we find that for \( z \) on \( \sigma_R \),

\[
|e^{iz}| \leq e^{-R \sin \theta} \leq 1.
\]

Hence \( \max_{z \in \sigma_R} |1 - e^{iz}| \leq 1 + 1 = 2 \). So \( |\sigma_R| \leq \frac{2\pi}{R} \to 0 \), as \( R \to \infty \). Now \( g(z) \) is analytic inside and on the simple path \( \gamma_{r,R} \). So by Cauchy’s theorem,

\[
\int_{\gamma_{r,R}} g(z) \, dz = 0.
\]

So

\[
0 = \int_{\gamma_{r,R}} g(z) \, dz = \int_{\gamma_{r,R}} g(z) \, dz - \int_{\sigma_r} g(z) \, dz + \int_{[-R, -r]} g(z) \, dz + \int_{[r, R]} g(z) \, dz.
\]

As \( r \to 0^+ \) and \( R \to \infty \), we obtain

\[
P.V. \int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} \, dx = \text{Re}(I) = \text{Re}(\pi) = \pi.
\]
17. Use an indented contour as in Fig. 9 with an indentation around 0. We have
\[
P.V. \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 + 1)} \, dx = \text{Im} \left( P.V. \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2 + 1)} \, dx \right) = \text{Im}(I) .
\]
\[
I_{r,R} = \int_{\gamma_{r,R}} \frac{e^{iz}}{z(z^2 + 1)} \, dz = \int_{\gamma_{r,R}} g(z) \, dz,
\]
where \( g(z) = \frac{e^{iz}}{z(z^2 + 1)} \). Note that \( g(z) \) has a simple pole at 0, and \( \text{Res}(g(z), 0) = 1 \). By Corollary 2,
\[
\lim_{r \to 0^+} \int_{\sigma_r} g(z) = i\pi(1) = i\pi.
\]
(As in Exercise 13, \( \sigma_r \) has a positive orientation, so it is traversed in the opposite direction on \( \gamma_{r,R} \). See Fig. 9.) As in Example 1, \( |I_{\sigma_R}| \to 0 \), as \( R \to \infty \). Now \( g(z) \) has a simple pole at \( i \) inside \( \gamma_{r,R} \). So by the residue theorem,
\[
\int_{\gamma_{r,R}} g(z) \, dz = 2\pi i \text{Res}(g(z), i) = 2\pi i \frac{e^{i(i)}}{i(2i)} = -\pi e^{-1}.
\]
So
\[
-\pi e^{-1} = \int_{\gamma_{r,R}} g(z) \, dz = \int_{\sigma_R} g(z) \, dz - \int_{\sigma_r} g(z) \, dz = \int_{[-R, -r]} g(z) \, dz + \int_{[r, R]} g(z) \, dz .
\]
As \( r \to 0^+ \) and \( R \to \infty \), we obtain
\[
-\pi e^{-1} = I - i\pi .
\]
Solving for \( I \) and taking imaginary parts, we find
\[
I = i(\pi - \pi e^{-1}) \quad \text{Im}(I) = \pi - \pi e^{-1} ,
\]
which is the value of the desired integral.
25. (a) For $w > 0$, consider

$$J = \int_C \frac{e^{iwz}}{e^{2\pi z} - 1} \, dz,$$

where $C$ is the indented contour in Figure 14. The integrand is analytic inside and on $C$. So

$$0 = J = \int_C \frac{e^{iwz}}{e^{2\pi z} - 1} \, dz = I_1 + I_2 + I_3 + I_4 + I_6 + I_6,$$

where $I_j$ is the integral over the $j$th component of $C$, starting with the line segment $[\epsilon, R]$ and moving around $C$ counterclockwise. As $\epsilon \to 0^+$ and $R \to \infty$, $I_1 \to I$, the desired integral.

For $I_3$, $z = x + i$, where $x$ varies from $R$ to $\epsilon$:

$$I_3 = \int_{\epsilon}^{R} \frac{e^{iwx}}{e^{2\pi(x+i)} - 1} \, dx = -e^{-w} \int_{\epsilon}^{R} \frac{e^{iwx}}{e^{2\pi x} - 1} \, dx.$$

As $\epsilon \to 0^+$ and $R \to \infty$, $I_3 \to -e^{-w} I$.

For $I_2$, $z = R + iy$ where $y$ varies from $0$ to $1$:

$$|I_2| \leq |\max_{0 \leq y \leq 1} \left( \frac{e^{iw \cdot (R+iy)}}{e^{2\pi(R+iy)} - 1} \right)|.$$

$$\left| \frac{e^{iwz}}{e^{2\pi z} - 1} \right| = \left| \frac{e^{iw(R+iy)}}{e^{2\pi(R+iy)} - 1} \right| = \left| \frac{e^{iwR}}{e^{2\pi R} - 1} \right| \to 0,$$

as $R \to \infty$.

So $I_2 \to 0$, as $R \to \infty$.

For $I_5$, $z = iy$, where $y$ varies from $1 - \epsilon$ to $\epsilon$:

$$I_5 = \int_{1-\epsilon}^{\epsilon} \frac{e^{iwy}}{e^{2\pi iy} - 1} \, dy = -i \int_{\epsilon}^{1-\epsilon} \frac{e^{-wy}}{e^{2\pi iy} - 1} \, dy.$$

For $I_4$, the integral over the quarter circle from $(\epsilon, \epsilon + i)$ to $(0, i - \epsilon)$, we apply Corollary 2, to compute the limit

$$\lim_{\epsilon \to 0} \int_{\gamma_4} \frac{e^{iwz}}{e^{2\pi z} - 1} \, dz = -\frac{i}{2} \text{Res} \left( \frac{e^{iwz}}{e^{2\pi z} - 1}, i \right) = -\frac{i}{2} \frac{e^{iw(i)}}{2\pi e^{2\pi i}} = -\frac{i}{4} e^{-w}.$$

For $I_6$, the integral over the quarter circle from $(0, i\epsilon)$ to $(\epsilon, 0)$, we apply Corollary 2, to compute the limit

$$\lim_{\epsilon \to 0} \int_{\gamma_6} \frac{e^{iwz}}{e^{2\pi z} - 1} \, dz = \frac{\pi}{2} \text{Res} \left( \frac{e^{iwz}}{e^{2\pi z} - 1}, 0 \right) = \frac{\pi}{2} \frac{e^{iw(0)}}{2\pi e^{2\pi (0)}} = -\frac{i}{4} e^{-w}.$$

Plug these findings in (1) and take the limit as $R \to \infty$ then as $\epsilon \to 0$, and get

$$\lim_{\epsilon \to 0} \left[ \int_{\epsilon}^{\infty} \frac{e^{iwz}}{e^{2\pi z} - 1} \, dz - e^{-w} \int_{\epsilon}^{\infty} \frac{e^{iwz}}{e^{2\pi z} - 1} \, dz - i \int_{\epsilon}^{1-\epsilon} \frac{e^{-wy}}{e^{2\pi iy} - 1} \, dy \right] - i e^{-w} \frac{i}{4} - \frac{i}{4} = 0.$$
or
\[
\lim_{\epsilon \to 0} \left[ \int_{\epsilon}^{\infty} \frac{e^{iwx}}{e^{2\pi x} - 1} \, dx - e^{-w} \int_{\epsilon}^{\infty} \frac{e^{iwx}}{e^{2\pi x} - 1} \, dx - i \int_{\epsilon}^{1-\epsilon} \frac{e^{-wy}}{e^{2\pi iy} - 1} \, dy \right] = i(e^{-w} + \frac{1}{4}).
\]

(Note: The limits of each individual improper integral does not exist. But the limit of the sum, as shown above does exist. So, we must work with the limit of the three terms together.) Take imaginary parts on both sides:

(2) \[ \lim_{\epsilon \to 0} \text{Im} \left[ \int_{\epsilon}^{\infty} \frac{e^{iwx}}{e^{2\pi x} - 1} \, dx - e^{-w} \int_{\epsilon}^{\infty} \frac{e^{iwx}}{e^{2\pi x} - 1} \, dx - i \int_{\epsilon}^{1-\epsilon} \frac{e^{-wy}}{e^{2\pi iy} - 1} \, dy \right] = \frac{e^{-w} + 1}{4}. \]

Now
\[
\text{Im} \left[ \int_{\epsilon}^{\infty} \frac{e^{iwx}}{e^{2\pi x} - 1} \, dx - e^{-w} \int_{\epsilon}^{\infty} \frac{e^{iwx}}{e^{2\pi x} - 1} \, dx - i \int_{\epsilon}^{1-\epsilon} \frac{e^{-wy}}{e^{2\pi iy} - 1} \, dy \right] = \int_{\epsilon}^{\infty} \sin w x \left( 1 - e^{-w} \right) \, dx + \int_{\epsilon}^{1-\epsilon} \text{Im} \left( \frac{(-i)e^{-wy}}{e^{2\pi iy} - 1} \right) \, dy;
\]

\[
\text{Im} \left( \frac{(-i)e^{-wy}}{e^{2\pi iy} - 1} \right) = -\text{Re} \left( \frac{(-i)e^{-wy}}{e^{2\pi iy} - 1} \right) = -\text{Re} \left( \frac{e^{-wy}}{e^{2\pi iy} - 1} \right) = -e^{-wy} \text{Re} \left( \frac{1}{e^{2\pi iy} - 1} \right) = -e^{-wy} \text{Re} \left( \frac{e^{-i\pi y}}{e^{i\pi y} - e^{-i\pi y}} \right) = -e^{-wy} \text{Re} \left( \frac{\cos \pi y - i \sin \pi y}{2i \sin \pi y} \right) = \frac{e^{-wy}}{2}.
\]

Plugging this in (3) and using (2), we get
\[
\lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \sin w x \left( 1 - e^{-w} \right) \, dx + \int_{\epsilon}^{1-\epsilon} \frac{e^{-wy}}{2} \, dy = \frac{e^{-w} + 1}{4}.
\]

Evaluate the second integral and get
\[
\lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{\sin w x}{e^{2\pi x} - 1} (1 - e^{-w}) \, dx + \lim_{\epsilon \to 0} \frac{e^{-w(1-\epsilon)} - e^{-w\epsilon}}{2(-w)} = \frac{e^{-w} + 1}{4}.
\]

\[
(1 - e^{-w}) \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{\sin w x}{e^{2\pi x} - 1} \, dx + \frac{1 - e^{-w}}{2w} = \frac{e^{-w} + 1}{4}.
\]

So
\[
(1 - e^{-w}) \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{\sin w x}{e^{2\pi x} - 1} \, dx = \frac{e^{-w} + 1}{4} - \frac{1 - e^{-w}}{2w};
\]
\[
\lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{\sin w x}{e^{2\pi x} - 1} \, dx = \frac{1}{1 - e^{-w}} \left[ \frac{e^{-w} + 1}{4} - \frac{1 - e^{-w}}{2w} \right]
\]
\[
\int_{0}^{\infty} \frac{\sin w x}{e^{2\pi x} - 1} \, dx = \frac{-1}{2w} + \frac{1}{4} \frac{e^w + 1}{e^w - 1}.
\]
(b) From (10), Section 4.4 (recall $B_0 = 1$):

$$z \coth z = \sum_{n=0}^{\infty} 2^{2n} B_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < \pi$$

$$\frac{z}{2} \coth \frac{z}{2} = \sum_{n=0}^{\infty} 2^{2n} B_{2n} \frac{\left(\frac{z}{2}\right)^{2n}}{(2n)!}, \quad \left|\frac{z}{2}\right| < \pi$$

$$\frac{z}{2} e^{\frac{z}{2}} + e^{-\frac{z}{2}} = \sum_{n=0}^{\infty} B_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < 2\pi$$

$$\frac{1}{2} e^{\frac{z}{2}} + e^{-\frac{z}{2}} - \frac{B_0}{z} = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n-1}, \quad |z| < 2\pi$$

$$\frac{1}{4} e^{\frac{z}{2}} + \frac{1}{4} e^{-\frac{z}{2}} - \frac{1}{2z} = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n-1}, \quad |z| < 2\pi$$

(c) Replace $\sin wx$ in the integral in (a) by its Taylor series

$$\sin wx = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{w^{2k-2} x^{2k-1}}{(2k-1)!},$$

ue (b), and interchange order of integration, and get

$$\int_0^{\infty} \frac{1}{e^{\pi x} - 1} \sum_{n=0}^{\infty} (-1)^n \frac{(wx)^{2n+1}}{(2n+1)!} dx = \frac{1}{2} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} w^{2n-1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^{\infty} \frac{x^{2n+1}}{e^{2\pi x} - 1} dx = \frac{1}{2} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} w^{2n-1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} \int_0^{\infty} \frac{x^{2n-1}}{e^{2\pi x} - 1} dx = \frac{1}{2} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} w^{2n-1}.$$

Comparing the coefficients of $w$, we obtain

$$\int_0^{\infty} \frac{x^{2n-1}}{e^{2\pi x} - 1} dx = \frac{(-1)^{n-1}}{4n} B_{2n} \quad (n = 1, 2, \ldots).$$
Solutions to Exercises 5.5

1. As in Example 1, take \( w > 0 \) (the integral is an even function of \( w \)),

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos wx \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{iwx} \, dx \quad \text{(because} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \sin wx \, dx = 0) \]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - iwx + iw^2)} \, dx
\]

\[
= e^{-\frac{1}{2}w^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-iw)^2} \, dx = \frac{e^{-\frac{1}{2}w^2}}{\sqrt{2\pi}} J.
\]

To evaluate \( J \), consider the integral

\[
I = \int_{\gamma_R} e^{-\frac{1}{2}(z-iw)^2} \, dz,
\]

where \( \gamma_R \) is a rectangular contour as in Fig. 1, with length of the vertical sides equal to \( w \). By Cauchy’s theorem, \( I = 0 \) for all \( R \).

Let \( I_j \) denote the integral on \( \gamma_j \) (see Example 1). Using the estimate in Example 1, we see that \( I_2 \) and \( I_4 \) tend to 0 as \( R \to \infty \). On \( \gamma_3 \), \( z = z + iw \), where \( x \) varies from \( R \) to \( -R \):

\[
\int_{-R}^{R} e^{-\frac{1}{2}(x+iw-iw)^2} \, dx = -\int_{-R}^{R} e^{-\frac{1}{2}x^2} \, dx,
\]

and this tends to \(-\sqrt{2\pi}\) as \( R \to \infty \), by (1), Sec. 5.4. Since \( I = 0 \) for all \( R \), it follows that

\[
J = \lim_{R \to \infty} I_{\gamma_1} = -\lim_{R \to \infty} I_{\gamma_3} = \sqrt{2\pi}.
\]

Consequently,

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos wx \, dx = e^{-\frac{w^2}{2}}
\]

for all \( w > 0 \). Since the integral is even in \( w \), the formula holds for \( w < 0 \). For \( w = 0 \) the formula follows from (1), Sec. 5.5.
Let \( f(x) = \cos 2x \), then \( f'(x) = -2 \sin 2x \) and \( f''(x) = -4 \cos 2x \). If \( x \in (0, \frac{\pi}{4}) \), then \( f''(x) < 0 \) and the graph concaves down. So any chord joining two points on the graph of \( y = \cos 2x \) above the interval \((0, \frac{\pi}{4})\) lies under the graph of \( y = \cos 2x \). Take the two points on the graph, \((0, 1)\) and \((\frac{\pi}{4}, 0)\). The equation of the line joining them is \( y = -\frac{4}{\pi} x + 1 \). Since it is under the graph of \( y = \cos 2x \) for \( x \in (0, \frac{\pi}{4}) \), we obtain
\[
-\frac{4}{\pi} x + 1 \leq \cos 2x \quad \text{for } 0 \leq x \leq \frac{\pi}{4}
\]

(b) Let \( I_j \) denote the integral of \( e^{-z^2} \) over the path \( \gamma_j \) in Fig. 13. Since \( e^{-z^2} \) is entre, by Cauchy's theorem, \( I_1 + I_2 + I_3 = 0 \).

(c) For \( I_1 \), \( z = x \),
\[
I_1 = \int_0^R e^{-z^2} \, dx \to \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}, \text{as } R \to \infty,
\]
by (1), Sec. 5.5. On \( \gamma_2 \), \( z = Re^{i\theta} \), \( z^2 = R^2(\cos 2\theta + i \sin 2\theta) \),
\[
|e^{-z^2}| = |e^{-R^2(\cos 2\theta + i \sin 2\theta)}| = e^{-R^2 \cos 2\theta} \leq e^{-R^2(1 - \frac{1}{4})}.
\]

Parametrize the integral \( I_2 \) and estimate:
\[
|I_2| = \left| \int_0^{\pi} e^{-R^2 e^{2i\theta}} Rie^{i\theta} \, d\theta \right|
\leq R \int_0^{\pi} |e^{-R^2 e^{2i\theta}} e^{i\theta}| \, d\theta \leq R \int_0^{\pi} e^{-R^2 (1 - \frac{1}{4})} \, d\theta
\leq Re^{-R^2} \int_0^{\pi} e^{R^2 \frac{1}{4} \theta} \, d\theta
= Re^{-R^2} \left[ \frac{\pi}{4R} e^{R^2 \frac{1}{4}} \right]_0 = \frac{\pi}{4R} e^{-R^2} \left[ e^{R^2} - 1 \right],
\]
which tends to 0 as \( R \to \infty \).

(d) On \( \gamma_3 \), \( z = xe^{i\frac{x}{2}} \), where \( x \) varies from \( R \) to 0, \( dz = e^{i\frac{x}{2}} \, dx \). So
\[
I_3 = -e^{i\frac{x}{2}} \int_0^R e^{-x^2 e^{i\frac{x}{2}}} \, dx = -e^{i\frac{x}{2}} \int_0^R e^{-x^2} \, dx
= -e^{i\frac{x}{2}} \int_0^\infty (\cos x^2 - i \sin x^2) \, dx.
\]
As \( R \to \infty \), \( I_3 \) converges to
\[
-e^{i\frac{x}{2}} \int_0^\infty (\cos x^2 - i \sin x^2) \, dx.
\]

(e) Let \( R \to \infty \) in the sum \( I_1 + I_2 + I_3 = 0 \) and get
\[
\frac{\sqrt{\pi}}{2} - e^{i\frac{x}{2}} \int_0^\infty (\cos x^2 - i \sin x^2) \, dx = 0;
\]
\[
e^{i\frac{x}{2}} \int_0^\infty (\cos x^2 - i \sin x^2) \, dx = \frac{\sqrt{\pi}}{2};
\]
\[
\int_0^\infty (\cos x^2 - i \sin x^2) \, dx = \frac{\sqrt{\pi}}{2} e^{-i\frac{x}{2}};
\]
\[
\int_0^\infty (\cos x^2 - i \sin x^2) \, dx = \frac{\sqrt{\pi}}{2} \left( \frac{\sqrt{\pi}}{2} - i \frac{\sqrt{\pi}}{2} \right).
\]
The desired result follows upon taking real and imaginary parts.
9. We will integrate the function \( f(z) = \frac{1}{z(z-1)\sqrt{z-2}} \) around the contour in Figure 16. Here \( \sqrt{z-2} \) is defined with the branch of the logarithm with a branch cut on the positive real axis. It is multple valued on the semi-axis \( x > 2 \). Approaching the real axis from above and to right of 2, we have \( \lim_{z \to x} \sqrt{z-2} = \sqrt{x-2} \). Approaching the real axis from below and to right of 2, we have \( \lim_{z \to x} \sqrt{z-2} = -\sqrt{x-2} \). Write

\[
I = I_1 + I_2 + I_3 + I_4,
\]

where \( I_1 \) is the integral over the small circular part; \( I_2 \) is the integral over the interval above the \( x \)-axis to the right of 2; \( I_3 \) is the integral over the larger circular path; \( I_4 \) is the integral over the interval (neg. orientation) below the \( x \)-axis to the right of 2. We have \( I_1 \to 0 \) as \( r \to 0 \) and \( I_3 \to 0 \) as \( R \to \infty \). (See Example 3 for similar details.) We have \( I_2 \to I \) as \( R \to \infty \) and \( I_4 \to I \) as \( R \to \infty \), where \( I \) is the desired integral. So

\[
2I = 2\pi i [\text{Res}(0) + \text{Res}(1)];
\]

\[
\text{Res}(0) = \lim_{z \to 0} \frac{1}{(z-1)\sqrt{z-2}} = -\frac{1}{\sqrt{2}}
\]

\[
= -\frac{1}{e^{\frac{1}{\log(\sqrt{2})}}} = -\frac{1}{e^{\frac{1}{\ln(2+i\pi)}}}
\]

\[
= \frac{1}{\sqrt{-1}} = i;
\]

\[
\text{Res}(1) = \frac{1}{\sqrt{-1}} = -i;
\]

\[
I = i\pi \left(\frac{i}{\sqrt{2}} - i\right) = \pi \left(1 - \frac{1}{\sqrt{2}}\right).
\]

13. (a) In Figure 19, let

\( \gamma_1 \) denote the small circular path around 0 (negative direction);

\( \gamma_2 \) the line segment from \( r \) to \( 1 - r \), above the \( x \)-axis (positive direction);

\( \gamma_3 \) the small semi-circular path around 1 above the \( x \)-axis (negative direction);

\( \gamma_4 \) the line segment from \( 1 + r \) to \( R - 1 \), above the \( x \)-axis (positive direction);

\( \gamma_5 \) the large circular path around 0 (positive direction);

\( \gamma_6 \) the line segment from \( R \) to \( 1 + r \), below the \( x \)-axis (negative direction);

\( \gamma_7 \) the small semi-circular path around 1 below the \( x \)-axis (negative direction);

\( \gamma_8 \) the line segment from \( 1 - r \) to \( r \), below the \( x \)-axis (negative direction).

We integrate the function

\[
f(z) = \frac{z^p}{z(1-z)}
\]

on the contour \( \gamma \), where \( z^p = e^{p\log z} \) (branch cut along positive \( x \)-axis). By Cauchy’s theorem,

\[
\oint_{\gamma} f(z) \, dz = 0 \quad \Rightarrow \quad \sum_{j=1}^{8} I_j = 0.
\]

Review the integrals \( I_3, I_4, I_7, \) and \( I_8 \) from Example 3, then you can show in a similar way that \( I_1, I_3, \) and \( I_7 \) tend to 0 as \( r \to 0 \). Also \( I_5 \to 0 \) as \( R \to \infty \). We will give some details. For \( I_1 \),

\[
|I_1| = \left| \int_{\gamma_1} \frac{z^p}{z(1-z)} \, dz \right|
\]

\[
= 2\pi r \frac{r^p}{r(1-r)} = 2\pi \frac{r^p}{(1-r)} \to 0 \quad \text{as} \quad r \to 0.
\]
Chapter 5  Residue Theory

For $I_2$ and $I_5$, we have

$$I_2 + I_5 \rightarrow \int_0^\infty \frac{x^p}{x(1-x)} \, dx, \text{ as } r \to 0 \text{ and } R \to \infty.$$  

For $I_3$ and $I_6$, we have

$$I_3 + I_6 \rightarrow -\int_0^\infty \frac{e^{p \log x}}{x(1-x)} \, dx = -e^{2\pi i} \int_0^\infty \frac{x^p}{x(1-x)} \, dx, \text{ as } r \to 0 \text{ and } R \to \infty.$$  

To evaluate $I_4$ and $I_7$, we use a trick that will allow us to apply Lemma 2, Sec. 5.4. Note that on $\gamma_4$, $\log_0 z = \text{Log } z$, and on $\gamma_7$, $\log_0 z = \log \pi z$. This allows us to replace $\log_0$ by a branch of the log which is analytic in a neighborhood of the contour of integration and this allows us to apply Lemma 2 of Sec. 5.4. According to this lemma, as $r \to 0$,

$$I_4 = \int_{\gamma_4} \frac{e^{p \text{Log } z}}{z(1-z)} \, dz \to i\pi e^{p \text{Log } (1)} = i\pi;$$

and

$$I_7 = \int_{\gamma_7} \frac{e^{p \log z}}{z(1-z)} \, dz \to i\pi e^{2\pi i}.$$  

So as $r \to 0$,

$$I_4 + I_7 = \int_{\gamma_4} \frac{e^{p \text{Log } z}}{z(1-z)} \, dz \to i\pi (1 + e^{2\pi i}).$$  

Adding the integrals together and then taking limits, we get

$$(1 - e^{2\pi i})I + i\pi (1 + e^{2\pi i}) = 0$$

$$I = -i\pi \frac{1 + e^{2\pi i}}{1 - e^{2\pi i}} = \pi \cot p\pi.$$  

(b) Use $x = e^t$, do the substitution, then replace $t$ by $x$, and get, from (a),

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{px}}{1 - e^x} \, dx = \pi \cot p\pi \quad (0 < p < 1).$$  

(c) Change variables $x = 2u$, $dx = 2 \, du$, then

$$\pi \cot p\pi = 2 \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{2pu}}{1 - e^{2u}} \, du$$

$$= 2 \text{P.V.} \int_{-\infty}^{\infty} e^{2pu}e^{-u} \, du = -\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{(2p-1)u}}{\sinh u} \, du$$

$$-\pi \cot \left(\frac{w + 1}{2}\right) = \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{wu}}{\sinh u} \, du \quad (w = 2p - 1).$$  

But

$$-\cot \left(\frac{w + 1}{2}\right) = \tan \frac{\pi w}{2},$$

so

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{wu}}{\sinh u} \, du = \tan \frac{\pi w}{2}.$$  

Replace $u$ by $x$ and get

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{wx}}{\sinh x} \, dx = \pi \tan \frac{\pi w}{2}.$$
(d) If \( |a| < b \), take \( t = bx \) and \( w = a \), then

\[
P.V. \int_{-\infty}^{\infty} \frac{e^{ax}}{\sinh bx} \, dx = \frac{\pi}{b} \tan \frac{\pi a}{2b}.
\]

(e) Replace \( a \) by \(-a\) in (d) and get

\[
P.V. \int_{-\infty}^{\infty} \frac{e^{-ax}}{\sinh bx} \, dx = -\frac{\pi}{b} \tan \frac{\pi a}{2b}.
\]

Subtract from (d) and divide by 2:

\[
\int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh bx} \, dx = \frac{\pi}{b} \tan \frac{\pi a}{2b} \quad (b > |a|).
\]

Note that the integral is convergent so there is no need to use the principal value.

17. We use a contour like the one in Fig. 17: Let

\( \gamma_1 \) denote the small circular path around 0 (negative direction);
\( \gamma_2 \) the line segment from \( r \) to \( R \), above the \( x \)-axis (positive direction);
\( \gamma_3 \) the large circular path around 0 (positive direction);
\( \gamma_4 \) the line segment from \( R \) to \( r \), below the \( x \)-axis (negative direction). We integrate the function

\[ f(z) = \frac{\sqrt{z}}{z^2 + z + 1} \]

on the contour \( \gamma \), where \( \sqrt{z} = e^{\frac{1}{2} \log z} \) (branch cut along positive \( x \)-axis). By the residue theorem,

\[
\int_{\gamma} f(z) \, dz = 2\pi i \sum_{j} \text{Res} (f, z_j) \quad \Rightarrow \quad \sum_{j=1}^{4} I_j = 2\pi i \sum_{j} \text{Res} (f, z_j),
\]

where the sum is over all the residues of \( f \) in the region inside \( \gamma \). The poles of \( f \) in this region are at the roots of \( z^2 + z + 1 = 0 \) or

\[ z = \frac{-1 \pm \sqrt{3}}{2}; \quad z_1 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}, \quad z_2 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}. \]

We have

\[
|z_1| = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1, \quad z_1 = e^{i\frac{2\pi}{3}}
\]

\[
\log_0(z_1) = \ln |z_1| + i \arg_0(z_1) = 0 + i \frac{2\pi}{3} = i \frac{2\pi}{3}
\]

\[
\text{Res} \left( z_1 \right) = \frac{\sqrt{z}}{2z_1 + 1} = e^{\frac{i}{2} \log_0(z_1)} = \frac{z_1}{2z_1 + 1}.
\]

Similarly,

\[
|z_2| = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2} = 1, \quad z_2 = e^{i\frac{4\pi}{3}}
\]

\[
\log_0(z_2) = \ln |z_2| + i \arg_0(z_2) = 0 + i \frac{4\pi}{3} = i \frac{4\pi}{3}
\]

\[
\text{Res} \left( z_2 \right) = \frac{\sqrt{z}}{2z_2 + 1} = e^{\frac{i}{2} \log_0(z_2)} = \frac{z_2}{2z_2 + 1}.
\]
Let us now compute the integrals. For $I_1$,

$$|I_1| = \left| \int_{\gamma_1} \frac{\sqrt{z}}{z^2 + z + 1} \, dz \right|$$

$$= 2\pi r \frac{\sqrt{r}}{1 - r^2 - r} \to 0 \text{ as } r \to 0.$$ 

For $I_2$, we have

$$I_2 \to \int_0^\infty \frac{\sqrt{x}}{x^2 + x + 1} \, dx = I, \text{ as } r \to 0 \text{ and } R \to \infty.$$ 

A simple estimate shows that $I_3 \to 0$ as $R \to \infty$. For $I_4$, $\sqrt{z} = e^{\frac{1}{2}(\ln|z| + 2\pi i)} = \sqrt{x}e^{i\pi}$. So

$$I_4 \to - \int_0^\infty \frac{e^{\frac{1}{2}\log x}}{x^2 + x + 1} \, dx = -e^{\pi i} \int_0^\infty \frac{\sqrt{x}}{x^2 + x + 1} \, dx = I, \text{ as } r \to 0 \text{ and } R \to \infty.$$ 

Adding the integrals together and then taking limits, we get

$$2I = 2\pi i \left( \frac{e^{i\pi/2}}{2e^{i\pi/2} + 1} + \frac{e^{3i\pi/2}}{2e^{3i\pi/2} + 1} \right)$$

$$I = \pi i \left( \frac{e^{i\pi/2}}{2e^{i\pi/2} + 1} + \frac{e^{3i\pi/2}}{2e^{3i\pi/2} + 1} \right)$$

$$= \pi i \left( \frac{2e^{i\pi/2} + e^{i3\pi/2} + 2e^{3i\pi/2} + e^{5i\pi/2}}{4e^{i\pi/2} + 2e^{i3\pi/2} + 2e^{3i\pi/2} + 1} \right) = \frac{\pi}{\sqrt{3}}.$$ 

Use

$$e^{i\pi/2} = \frac{1}{2} + \frac{i}{2}\sqrt{3}, \quad e^{3i\pi/2} = \frac{1}{2} - \frac{i}{2}\sqrt{3},$$

$$e^{i3\pi/2} = -1, \quad e^{4i\pi/2} = -\frac{1}{2} - \frac{i}{2}\sqrt{3},$$

$$e^{5i\pi/2} = \frac{1}{2} - \frac{i}{2}\sqrt{3}, \quad e^{6i\pi/2} = 1.$$
Solutions to Exercises 5.6

1. Note that 

\[ f(z) = \frac{1}{z^2 + 9} \]

has two simples poles at \( \pm 3i \). Since \( f(z) \) does not have poles on the integers, we may apply Proposition 1 and get

\[
\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + 9} = -\pi [\text{Res} (f(z) \cot \pi z, 3i) + \text{Res} (f(z) \cot \pi z, -3i)]
\]

because \( \cot(iz) = -i \coth(z) \). You can prove the last identity by using (25) and (26) of Section 1.6.

5. Reason as in Exercise 1 with 

\[ f(z) = \frac{1}{(z-\frac{1}{2})(z-1)+\pi} \]

which has two simples poles at \( \pm \frac{1}{2} \). Since \( f(z) \) does not have poles on the integers, we may apply Proposition 1 and get

\[
\sum_{k=-\infty}^{\infty} \frac{1}{4k^2 - 1} = -\pi \left[ \text{Res} (f(z) \cot \pi z, \frac{1}{2}) + \text{Res} (f(z) \cot \pi z, -\frac{1}{2}) \right]
\]

because \( \cot(\frac{1}{2}) = 0 \).

9. Reason as in Exercise 1 with 

\[ f(z) = \frac{1}{(z-\frac{3+\sqrt{3}}{2})(z-\frac{3-i\sqrt{3}}{2})+\pi} \]

which has two simples poles where \( z^2 - 3z + 3 = 0 \) or

\[ z = \frac{3 \pm \sqrt{3}}{2}; \quad z_1 = \frac{3 + i\sqrt{3}}{2}, \quad z_2 = \frac{3 - i\sqrt{3}}{2}. \]

Since \( f(z) \) does not have poles on the integers, we may apply Proposition 1 and get

\[
\sum_{k=-\infty}^{\infty} \frac{1}{(k-\frac{3-\sqrt{3}}{2})(k-\frac{3+\sqrt{3}}{2})+1} = -\pi [\text{Res} (f(z) \cot \pi z, z_1) + \text{Res} (f(z) \cot \pi z, z_2)]
\]

Let’s compute:

\[ \text{Res} (z_1) = \lim_{z \to z_1} \frac{\cot \pi z}{z - z_2} = \frac{\cot \pi z_1}{z_1 - z_2} \]

\[ z_1 - z_2 = i\sqrt{3}, \]

\[ \cot(\pi z_1) = \cot \left( \frac{3 + i\sqrt{3}}{2} \right) = -\tan \left( i\pi \frac{\sqrt{3}}{2} \right) = -i \tanh \left( \pi \frac{\sqrt{3}}{2} \right). \]

(Prove and use the identities \( \cot(z + \frac{3\pi}{2}) = -\tan(z) \) and \( \tan(iz) = i \tanh z \).)

So

\[ \text{Res} (z_1) = -i \tanh \left( \frac{\sqrt{3}}{2} \pi \right) = -\frac{\tanh \left( \frac{\sqrt{3}}{2} \pi \right)}{\sqrt{3}}. \]
A similar computation shows that \( \text{Res}(z_2) = \text{Res}(z_1) \), hence

\[
\sum_{k=\infty}^{\infty} \frac{1}{(k-2)(k-1)+1} = \frac{2\pi}{\sqrt{3}} \tanh \left( \frac{\sqrt{3}}{2} \pi \right)
\]

**13.** Just repeat the proof of Proposition 1. The only difference is that here we no longer have \( \text{Res}(f(z) \cot \pi z, 0) = f(z) \). This residue has to be computed anew.

**17.** (a) Apply the result of Exercise 13 with \( f(z) = \frac{1}{z^{2n}} \):

\[
\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = -\pi \text{Res}\left( \frac{\cot(\pi z)}{z^{2n}}, 0 \right),
\]

because \( f \) has only one pole of order \( 2n \) at 0. The sum on the left is even, so

\[
\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = -\frac{\pi}{2} \text{Res}\left( \frac{\cot(\pi z)}{z^{2n}}, 0 \right),
\]

(b) Recall the Taylor series expansion of \( z \cot z \) from Exercise 31, Section 4.4,

\[
z \cot z = \sum_{k=0}^{\infty} (-1)^k \frac{2^k B_{2k} z^{2k}}{(2k)!}
\]

\[
(\pi z) \cot(\pi z) = \sum_{k=0}^{\infty} (-1)^k \frac{2^k B_{2k} \pi^{2k} z^{2k}}{(2k)!}
\]

\[
cot(\pi z) = \sum_{k=0}^{\infty} (-1)^k \frac{2^k B_{2k} \pi^{2k-1} z^{2k-1}}{(2k)!}
\]

So

\[
\frac{\cot(\pi z)}{z^{2n}} = \sum_{k=0}^{\infty} (-1)^k \frac{2^k B_{2k} \pi^{2k-1} z^{2k-1}}{(2k)!} z^{2k-2n-1}
\]

The residue at 0 is \( a_{-1} \), the coefficient of \( \frac{1}{z} \), which is obtained from the series above when \( 2k-2n-1 = -1 \) or \( n = k \). Hence

\[
\text{Res}\left( \frac{\cot(\pi z)}{z^{2n}}, 0 \right) = (-1)^n \frac{2^{2n} B_{2n} \pi^{2n-1}}{(2n)!}
\]

Using (a), we get

\[
\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1} B_{2n} \pi^{2n}}{(2n)!}
\].
Solutions to Exercises 5.7

1. The roots of the polynomial are easy to find using the quadratic formula:

\[ z^2 + 2z + 2 = 0 \quad \Rightarrow \quad z = -1 \pm i. \]

Thus no roots are in the first quadrant. This, of course, does not answer the exercise. We must arrive at this answer using the method of Example 1, with the help of the argument principle.

First, we must argue that \( f \) has no roots of the positive \( x - axis \). This is clear, because if \( x > 0 \) then \( x^2 + x + 2 > 2 \) and so it cannot possibly be equal to 0. Second, we must argue that there are no roots on the upper imaginary axis. \( f(iy) = -y^2 + 2y + 2 = -y^2 + 2y \). If \( y = 0 \), \( f(0) = 2 \neq 0 \).

The number of zeros of the polynomial \( f(z) = z^2 + 2z + 2 \) is easy to find using the quadratic formula:

Thus no roots are in the first quadrant. This, of course, does not answer the exercise. We must consider \( x \) and \( x \) separately.

5. Follow the steps in the solution of Exercise 1, but here the roots of \( z^4 + 8z^2 + 16z + 20 = 0 \) are not so easy to find, so we will not give them.

Argue that \( f \) has no roots of the positive \( x - axis \). This is clear, because if \( x > 0 \) then \( x^4 + 8x^2 + 16x + 20 > 20 \) and so it cannot possibly be equal to 0. Second, we must argue that there are no roots on the upper imaginary axis. \( f(iy) = y^4 - 8y^2 + 16iy + 20 = y^4 - 8y^2 + 20 - 8iy \).

If \( y = 0 \), \( f(0) = 20 \neq 0 \). If \( y > 0 \), then \( \text{Im} (f(y)) = -8y \neq 0 \). In all cases, \( f(iy) \neq 0 \) if \( y \geq 0 \).

The number of zeros of the polynomial \( f(z) = z^4 + 8z^2 + 16z + 20 \) is equal to the number of times the image of \( \sigma_R \) wraps around the origin, where \( \gamma_R \) is the circular path in the first quadrant, in Fig. 4. This path consists of the interval \([0, R] \), the circular arc \( \sigma_R \), and the interval on the imaginary axis from \( iR \) to 0. To find the image on \( \gamma_R \), we consider the image of each component separately.

Since \( f(x) \) is real for real \( x \), we conclude that the image of the interval \([0, R] \) is also an interval, and it is easy to see that this interval is \([2, R^2 + 2R + 2] \). So its initial point is \( w_0 = 2 \) and its terminal point is \( w_1 = R^2 + 2R + 2 \).

The image of the arc \( \sigma_R \) starts at the point \( w_1 = R^2 + 2R + 2 \) and ends at \( f(R) = -R^2 + 2iR + 2 = w_2 \), which is the image of the terminal point of \( \sigma_R \). We have \( \text{Im} (w_2) = 2R \) and \( \text{Re} (w_2) = 2 - R^2 < 0 \) if \( R \) is very large. Hence the point \( f(w_2) \) is in the second quadrant. Also, for very large \( R \), and \( |z| = R \), the mapping \( z \mapsto f(z) \) is approximately like \( z \mapsto z^2 \). So \( f(z) \) takes \( \sigma_R \) and maps it "approximately" to the semi-circle (the map \( w = z^2 \) doubles the angles), with initial point \( w_1 \) and terminal point \( w_2 \).

We now come to the third part of the image of \( \gamma_R \). We know that it starts at \( w_2 \) and end at \( w_0 \). As this image path go from \( w_2 \) to \( w_0 \), does it wrap around zero or not? To answer this question, we consider \( f(iy) = 2 - R^2 + 2iy \). Since \( \text{Im} (f(iy)) > 0 \) if \( y > 0 \), we conclude that the image point of \( iy \) remains in the upper half-plane as it moves from \( w_2 \) to \( w_0 \). Consequently, the image curve does not wrap around 0; and hence the polynomial has no roots in the first quadrant-as expected.

Since \( f(x) \) is real for real \( x \), we conclude that the image of the interval \([0, R] \) is also an interval, and it is easy to see that this interval is \([20, R^4 + 8R^2 + 16R + 20] \). So its initial point is \( w_0 = 2 \) and its terminal point is \( w_1 = R^4 + 8R^2 + 16R + 20 \).

The image of the arc \( \sigma_R \) starts at the point \( w_1 = R^4 + 8R^2 + 16R + 20 \) on the real axis and ends at \( f(iR) = R^4 - 8R^2 + 20 + 16iR = w_2 \), which is the image of the terminal point of \( \sigma_R \). We have \( \text{Im} (w_2) = 16R > 0 \) and \( \text{Re} (w_2) = R^4 - 8R^2 + 20 > 0 \) if \( R \) is very large. Hence the point \( f(w_2) \) is in the first quadrant. Also, for very large \( R \), and \( |z| = R \), the mapping \( z \mapsto f(z) \) is approximately like \( z \mapsto z^2 \). So \( f(z) \) takes \( \sigma_R \) and maps it "approximately" to a circle (the map \( w = z^4 \) multiplies angles by 4), with initial point \( w_1 \) and terminal point \( w_2 \). So far, the image of \([0, R] \) and \( \sigma_R \) wraps one around the origin.
Chapter 5 Residue Theory

9. Apply Rouché’s theorem with \( f(z) = 11, \ g(z) = 7z^3 + 3z^2 \). On \( |z| = 1, \ |f(z)| = 11 \) and \( |g(z)| \leq 7 + 3 = 10 \). Since \( |f| > |g| \) on \( |z| = 1 \), we conclude that \( N(f) = N(f + g) \) inside \( C_1(0) \). Since \( N(f) = 0 \) we conclude that the polynomial \( 7z^3 + 3z^2 + 11 \) has no roots in the unit disk.

13. Apply Rouché’s theorem with \( f(z) = -3z, \ g(z) = e^z \). On \( |z| = 1, \ |f(z)| = 3 \) and

\[ |g(z)| = |e^{cost+i\sin t}| = e^{cost} \leq e. \]

Since \( |f| > |g| \) on \( |z| = 1 \), we conclude that \( N(f) = N(f + g) \) inside \( C_1(0) \). Since \( N(f) = 1 \) we conclude that the function \( e^{z} - 3z \) has one root in the unit disk.

17. Apply Rouché’s theorem with \( f(z) = 5, \ g(z) = z^5 + 3z \). On \( |z| = 1, \ |f(z)| = 5 \) and \( |g(z)| \leq 4 \). Since \( |f| > |g| \) on \( |z| = 1 \), we conclude that \( N(f) = N(f + g) \) inside \( C_1(0) \). Since \( N(f) = 0 \) we conclude that the function \( z^5 + 3z + 5 \) has no roots in the unit disk. Hence \( \frac{1}{z^5 + 3z + 5} \) is analytic in the unit disk and by Cauchy’s theorem

\[ \int_{C_1(0)} \frac{dz}{z^n + 3z + 5} = 0. \]

21. (a) Let \( g(z) = z \) and \( f(z) = z^n - 1 \). Then

\[ \frac{1}{2\pi i} \int_{C_R(0)} \frac{n z^{n-1}}{z^n - 1} dz = \frac{1}{2\pi i} \int_{C_R(0)} \frac{g(z)}{f(z)} \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{n} g(z_j), \]

where \( z_j \) are the roots of \( f(z) \). These are precisely the \( n \) roots of unity. Hence \( \sum_{j=1}^{n} g(z_j) = \sum_{j=1}^{n} z_j = S \). Let \( \zeta = e^{\frac{2\pi i}{n}}, \ d\zeta = -\frac{1}{n} dz \), as \( z \) runs through \( C_R(0) \) in the positive direction, \( \zeta \) runs through \( C_{1/R}(0) \) in the negative direction. So

\[ \frac{1}{2\pi i} \int_{C_R(0)} \frac{n z^{n-1}}{z^n - 1} dz = \frac{1}{2\pi i} \int_{C_{1/R}(0)} \frac{n}{\zeta^n (1 - \zeta^n)} d\zeta = \frac{n}{2\pi i} \int_{C_{1/R}(0)} \frac{d\zeta}{1 - \zeta^n}. \]

(b) Evaluate the second integral in (a) using Cauchy’s generalized integral formula and conclude that

\[ \frac{n}{2\pi i} \int_{C_{1/R}(0)} \frac{d\zeta}{1 - \zeta^n} = \frac{d}{d\zeta} \frac{1}{1 - \zeta^n} \bigg|_{\zeta=0} = \frac{-n \zeta^{n-1}}{(1 - \zeta^n)^2} \bigg|_{\zeta=0} = 0. \]

Using (a), we find \( S = 0 \).

A different way to evaluate \( S \) is as follows: From (a),

\[ S = \frac{n}{2\pi i} \int_{C_R(0)} \frac{z^n}{z^n - 1} dz \]

\[ = \frac{n}{2\pi i} \int_{C_R(0)} \frac{z^n - 1 + 1}{z^n - 1} dz = \frac{n}{2\pi i} \int_{C_R(0)} \frac{1}{z^n - 1} dz \]

\[ = 0 + 0 = 0. \]
The first integral is 0 by Cauchy’s theorem. The second integral is zero because $C_R(0)$ contains all the roots of $p(z) = z^n - 1$ (see Exercise 38, Sec. 3.4).

25. Suppose that $f$ is not identically 0 in $\Omega$, and let $B_r(z_0) \subset \Omega$ be a disk such that $f(z) \neq 0$ for all $z$ on $C_r(z_0)$. Since $|f|$ is continuous, it attains its maximum and minimum on $C_r(z_0)$. So there is a point $z_1$ on $C_r(z_0)$ such that $|f(z_1)| = m = \min |f|$ on $C_r(z_0)$. Since $|f(z_1)| \neq 0$, we find that $m > 0$.

Apply uniform convergence to get an index $N$ such that $n > N$ implies that $|f_n - f| < m \leq |f|$ on $C_r(z_0)$. Now, $|f| \geq m$ on $C_r(z_0)$, so we can apply Rouché’s theorem and conclude that $N(f) = N(f_n - f + f) = N(f_n)$ for all $n \geq N$. That is, the number of zeros of $f_n$ inside $C_r(z_0)$ is equal to the number of zeros of $f$ inside $C_r(z_0)$.

29. Follow the steps outlined in Exercise 28. Let $\Omega$ denote the open unit disk and let $f_n(z) = z^5 + z^4 + 6z^2 + 3z + 11 + \frac{1}{n}$. Then $f_n(z)$ converge uniformly to $p(z) = z^5 + z^4 + 6z^2 + 3z + 11$ on any closed disk in $\Omega$. Since $p(z)$ is clearly not identically zero, it follows from Hurwitz’s theorem that $N(f_n) = N(p)$ in $|z| < 1$ for all sufficiently large $n$. But, an application of Rouché’s theorem shows that $f_n(z)$ has no zeros in $N_1(0)$ or on $C_1(0)$. (Take $f(z) = 11 + \frac{1}{n}$ and $g(z) = z^5 + 6z^2 + 3z$.) Hence $p(z)$ has no roots in $N_1(0)$.

Note: Hurwitz’s theorem does not apply to $p(z)$ on the boundary of $N_1(0)$. On this boundary, $p(z)$ may have roots. Consider another simpler example: $p(z) = z^2 + 1$, $f_n(z) = z^2 + 1 + \frac{1}{n}$. Then $f_n$ converges to $p$ uniformly on any closed set. Clearly, $p(z)$ has two roots, $\pm i$, on the boundary of the unit disk, but $f_n(z)$ has no roots inside or on the boundary of the unit disk.