Solutions to Exercises 3.1

1. Given \( z_1 = 1 + i \) and \( z_2 = -1 - 2i \), apply (2), Sec. 3.1, to obtain the parametrization of the line segment \([z_1, z_2]\):

\[
\gamma(t) = (1-t)z_1 + tz_2 = (1-t)(1+i) + t(-1-2i), \quad 0 \leq t \leq 1,
\]
or

\[
\gamma(t) = t(-2-3i) + 1 + i, \quad 0 \leq t \leq 1.
\]

5. Use Example 1(a) with \( \alpha = -\frac{\pi}{4} \) and \( \beta = \frac{\pi}{4} \):

\[
\gamma(t) = e^{it}, -\frac{\pi}{4} \leq t \leq \frac{\pi}{4}.
\]

9. Use Example 1(a) and (c):

\[
\gamma(t) = z_0 + Re^{it} = -3 + 2i + 5e^{it}, \quad -\frac{\pi}{2} \leq t \leq 0.
\]

13. Write \( \gamma = (\gamma_1, \gamma_2) \), where \( \gamma_1 \) is the circular arc and \( \gamma_2 \) the line segment shown in Fig. 14, but in reverse direction. To parametrize \( \gamma_1 \), use Example 1(a):

\[
\gamma_1(t) = 2e^{-it}, \quad 0 \leq t \leq \frac{3\pi}{2}.
\]

Note the negative sign in the exponent, whose effect is to trace the arc in the negative direction. If we want to parametrize \( \gamma_1 \) using the interval \([0, 1]\), we can change \( t \) to \( \frac{3\pi}{2} - t \) and get

\[
\gamma_1(t) = 2e^{-i\frac{3\pi}{2}t}, \quad 0 \leq t \leq 1.
\]

To parametrize \( \gamma_2 \), use (2):

\[
\gamma_2(t) = (1-t)2i + t(5i) = 3it + 2i, \quad 0 \leq t \leq 1.
\]

To parametrize \( \gamma_2 \) by the interval \([1, 2]\), we can simply change \( t \) to \( t - 1 \) and get

\[
\gamma_2(t) = 3i(t-1) + 2i, \quad 1 \leq t \leq 2.
\]

So

\[
\gamma(t) = \begin{cases} 
2e^{-i\frac{3\pi}{2}t} & \text{if } 0 \leq t \leq 1 \\
3i(t-1) + 2i & \text{if } 1 \leq t \leq 2.
\end{cases}
\]

17. We have \( x(t) = t \) and \( y(t) = \sin \pi t \), where \( 0 \leq t \leq 1 \). Thus \( y = \sin \pi x \), where \( 0 \leq x \leq 1 \), and so the graph is the arch of \( \sin \pi x \) for \( 0 \leq x \leq 1 \).

21. We differentiate using the rules of calculus and treating complex constants as if they were real. We get

\[
\frac{d}{dt}(2 + i) \cos(3it) = (2 + i) \frac{d}{dt} \cos(3it) = -3i(2 + i) \sin(3it) = (3 - 6i) \sin(3it).
\]

25. To find a particular solution \( y_p \) of \( y'' + y' + y = \cos t \), we proceed as follows
1. Think of \( \cos t \) as being the real part of \( e^{it} \).

2. Find a particular solution of \( y'' + y' + y = e^{it} \) and let \( Y \) denote this particular solution.

3. Compute \( y_p = \text{Re}(Y) \).

To find \( Y \), try \( Y = Ae^{it} \), \( Y' = Aie^{it} \), \( Y'' = -Ae^{it} \). Plug into the equation and get

\[
-Ae^{it} + iAe^{it} + Ae^{it} = e^{it}
\]

\[
-A + iA + A = 1 \quad \text{(Divide by } e^{it}).
\]

\[
iA = 1 \Rightarrow A = -i.
\]

Hence \( Y = -ie^{it} \) and so \( y_p = \text{Re}(-ie^{it}) = \sin t \).

29. Proceed as in Exercise 21: Let \( Y \) denote a particular solution of \( y'' - 2y' - 3y = e^{4it} \). Then \( \text{Re}(Y) \) is a particular solution of \( y'' - 2y' - 3y = \cos 4t \), and \( \text{Im}(Y) \) is a particular solution of \( y'' - 2y' - 3y = \sin 4t \). So solving the equation with a complex exponential on the right side yields the solutions of two differential equations. Let us now solve \( y'' - 2y' - 3y = e^{4it} \). For this purpose, we let \( Y = Ae^{4it} \), \( Y' = 4iAe^{4it} \), and \( Y'' = -16Ae^{4it} \). Plugging into the equation and solving, we find

\[
-16Ae^{4it} - 8iAe^{4it} - 3Ae^{4it} = e^{4it}
\]

\[
-16A - 8iA - 3A = 1 \quad \text{(Divide by } e^{4it}).
\]

\[
A(-19 - 8i) = 1 \Rightarrow A = \frac{1}{-19 - 8i} = -19 - \frac{8}{425}i.
\]

Hence \( Y = (-\frac{19}{425} + \frac{8}{425}i)e^{4it} \) and so a particular solution of \( y'' - 2y' - 3y = \cos 4t \) is

\[
y_p = \text{Re}((-\frac{19}{425} + \frac{8}{425}i)e^{4it}) = -\frac{19}{425} \cos 4t - \frac{8}{425} \sin 4t.
\]

Also, a particular solution of \( y'' - 2y' - 3y = \sin 4t \) is

\[
y_p = \text{Im}((-\frac{19}{425} + \frac{8}{425}i)e^{4it}) = \frac{8}{425} \cos 4t - \frac{19}{425} \sin 4t.
\]

33. We have \( x(t) = a \cos t - b \cos \frac{at}{2} \) and \( y(t) = a \sin t - b \sin \frac{at}{2} \). So

\[
\gamma(t) = x(t) + y(t)
\]

\[
= a \cos t - b \cos \frac{at}{2} + ai \sin t - ib \sin \frac{at}{2}
\]

\[
= a(\cos t + i \sin t) - b(\cos \frac{at}{2} + i \sin \frac{at}{2})
\]

\[
= ae^{it} - be^{i\frac{at}{2}}
\]
Solutions to Exercises 3.2

1. Using (10),
\[
\int_0^{2\pi} e^{3ix} \, dx = \frac{1}{3i} e^{3ix} \bigg|_0^{2\pi} = \frac{1}{3i} (e^{6i\pi} - 1) = 0.
\]

5. Write
\[
\frac{x + i}{x - i} = \frac{x + i}{x - i} \times \frac{x + i}{x + i} = \frac{x^2 + 2ix - 1}{x^2 + 1} = \frac{(x^2 + 1) + 2ix - 2}{x^2 + 1}.
\]
\[
= 1 - \frac{2}{x^2 + 1} + \frac{2ix}{x^2 + 1}.
\]
So
\[
\int_{-1}^{1} \frac{x + i}{x - i} \, dx = \int_{-1}^{1} 1 \, dx - \int_{-1}^{1} \frac{2}{x^2 + 1} \, dx + 2i \int_{-1}^{1} \frac{x}{x^2 + 1} \, dx
\]
\[
= x - 2 \tan^{-1} x \bigg|_{-1}^{1} = 2 - \pi.
\]

9. Proceed as in Example 2:
\[
f(x) = \begin{cases} 
(3 + 2i)x & \text{if } -1 \leq x \leq 0, \\
ix^2 & \text{if } 0 \leq x \leq 1;
\end{cases}
\]
hence an antiderivative
\[
F(x) = \begin{cases} 
\frac{3+2i}{2} x^2 + C & \text{if } -1 \leq x \leq 0, \\
\frac{i}{3} x^3 & \text{if } 0 \leq x \leq 1.
\end{cases}
\]
Setting \( F(0+) = F(0-) \), we obtain \( 0 = \frac{3+2i}{2} 0 + C \) or \( C = 0 \). Hence a continuous antiderivative of \( f \) is
\[
F(x) = \begin{cases} 
\frac{3+2i}{2} x^2 & \text{if } -1 \leq x \leq 0, \\
\frac{i}{3} x^3 & \text{if } 0 \leq x \leq 1.
\end{cases}
\]

13. (a) Let \( p > 0 \) be an arbitrary real number, and \( m \) and \( n \) be arbitrary integers. Write \( f(x) = e^{\frac{inx}{p}} \). Then
\[
f(x + 2p) = e^{\frac{inx}{p} (x+2p)} = e^{\frac{inx}{p}} e^{\frac{2inx}{p}} = e^{\frac{inx}{p}} = f(x).
\]
Hence \( f \) is \( 2p \)-periodic.

(b) If \( m \neq n \), then

\[
\int_{-p}^{p} e^{i\frac{m\pi x}{p}} e^{i\frac{n\pi x}{p}} \, dx = \int_{-p}^{p} e^{i\frac{(m-n)\pi x}{p}} \, dx
\]

\[
= -\frac{i}{p(m-n)} e^{i\frac{(m-n)\pi x}{p}} \bigg|_{-p}^{p} = -\frac{i}{p(m-n)} \left( e^{i(m-n)\pi} - e^{-i(m-n)\pi} \right)
\]

\[
= -\frac{i}{p(m-n)} e^{-i(m-n)\pi} \left( e^{i2(m-n)\pi} - 1 \right) = 0
\]

If \( m = n \), then

\[
\int_{-p}^{p} e^{i\frac{m\pi x}{p}} e^{i\frac{n\pi x}{p}} \, dx = \int_{-p}^{p} e^{i\frac{m\pi x}{p}} e^{-i\frac{m\pi x}{p}} \, dx = \int_{-p}^{p} \, dx = 2p.
\]

We have thus established the orthogonality relations

\[
\int_{-p}^{p} e^{i\frac{m\pi x}{p}} e^{-i\frac{n\pi x}{p}} \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ 2p & \text{if } m = n. \end{cases}
\]

(c) Now suppose \( m \) and \( n \) are nonnegative integers. If \( m \neq n \), then the equality

\[
\int_{-p}^{p} e^{i\frac{m\pi x}{p}} e^{-i\frac{n\pi x}{p}} \, dx = 0
\]

implies that

\[
\int_{-p}^{p} \left( \cos \frac{m\pi}{p} x + i \sin \frac{m\pi}{p} x \right) \left( \cos \frac{n\pi}{p} x - i \sin \frac{n\pi}{p} x \right) \, dx = 0.
\]

Expanding and then taking real and imaginary parts, we get

(1) \[
\int_{-p}^{p} \left( \cos \frac{m\pi}{p} x \cos \frac{n\pi}{p} x + \sin \frac{m\pi}{p} x \sin \frac{n\pi}{p} x \right) \, dx = 0
\]

and

(2) \[
\int_{-p}^{p} \left( -\cos \frac{m\pi}{p} x \sin \frac{n\pi}{p} x + \sin \frac{m\pi}{p} x \cos \frac{n\pi}{p} x \right) \, dx = 0.
\]

Replacing \( n \) by \(-n\); that is, starting with the identity

\[
\int_{-p}^{p} e^{i\frac{m\pi x}{p}} e^{i\frac{n\pi x}{p}} \, dx = 0,
\]

we get

(3) \[
\int_{-p}^{p} \left( \cos \frac{m\pi}{p} x \cos \frac{n\pi}{p} x - \sin \frac{m\pi}{p} x \sin \frac{n\pi}{p} x \right) \, dx = 0
\]
and

\( \int_{-p}^{p} \left( \cos \frac{m\pi}{p} x \sin \frac{n\pi}{p} x + \sin \frac{m\pi}{p} x \cos \frac{n\pi}{p} x \right) \, dx = 0. \)

You can now obtain some of the desired integral identities by using linear combinations of (1)–(4). For example, adding (1) and (3) implies that

\( \int_{-p}^{p} \cos \frac{m\pi}{p} x \cos \frac{n\pi}{p} x \, dx = 0. \)

Adding (2) and (4) implies that

\( \int_{-p}^{p} \cos \frac{m\pi}{p} x \sin \frac{n\pi}{p} x \, dx = 0. \)

All other integral identities with \( m \neq n \) follow similarly. Consider the case \( m = -n \neq 0 \),

\( \int_{-p}^{p} e^{i \frac{m\pi}{p} x} e^{-i \frac{m\pi}{p} x} \, dx = 0. \)

Writing the integrand in terms of cosine and sine and then taking real and imaginary parts, we get

\( \int_{-p}^{p} \left( \cos^2 \frac{m\pi}{p} x - \sin^2 \frac{m\pi}{p} x \right) \, dx = 0 \)

Similarly, from the identity

\( \int_{-p}^{p} e^{i \frac{m\pi}{p} x} e^{i \frac{m\pi}{p} x} \, dx = 2p, \)

we obtain

\( \int_{-p}^{p} \left( \cos^2 \frac{m\pi}{p} x + \sin^2 \frac{m\pi}{p} x \right) \, dx = 2p. \)

Adding (5) and (6) shows that

\( \int_{-p}^{p} \cos^2 \left( \frac{m\pi}{p} x \right) \, dx = p. \)

The remaining identities follow similarly.

17. Write

\[ I = \int_{[z_1, z_2, z_3]} 2\pi \, dz = \int_{[z_1, z_2]} 2\pi \, dz + \int_{[z_2, z_3]} 2\pi \, dz = I_1 + I_2, \]

where \( z_1 = 1, z_2 = i, z_3 = 1+i \). To evaluate \( I_1 \), parametrize \([z_1, z_2]\) by \( z = \gamma_1(t) = (1-t)+it, \) \( 0 \leq t \leq 1, \, dz = (-1+i) \, dt \). So

\[ \int_{[z_1, z_2]} 2\pi \, dz = 2 \int_0^1 ((1-t) - it)(-1+i) \, dt = 2(-1+i) \int_0^1 (1-(1+i)t) \, dt \]

\[ = 2(-1+i)(t - \frac{1+i}{2}t^2) \bigg|_0^1 = 2i. \]
To evaluate $I_2$, parametrize $[z_2, z_3]$ by $z = \gamma_2(t) = (1 - t)i + t(1 + i) = i + t, \ 0 \leq t \leq 1$, $dz = dt$. So

$$\int_{[z_2, z_3]} 2\pi \, dz = 2 \int_0^1 (t - i) \, dt = 2\left(\frac{t}{2} - it\right)\big|_0^1 = 1 - 2i.$$ 

Thus $I = 1.$

21. Write

$$I = \int_{[z_1, z_2, z_3, z_4, z_1]} z \, dz = \int_{[z_1, z_2]} z \, dz + \int_{[z_2, z_3]} z \, dz + \int_{[z_3, z_4]} z \, dz + \int_{[z_4, z_1]} z \, dz = I_1 + I_2 + I_3 + I_4,$$

where $z_1 = 0, z_2 = 1, z_3 = 1 + i,$ and $z_4 = i$. To evaluate $I_1$, parametrize $[z_1, z_2]$ by $z = \gamma_1(t) = t, \ 0 \leq t \leq 1, dz = dt$. So

$$I_1 = \int_{[z_1, z_2]} z \, dz = \int_0^1 t \, dt = \frac{1}{2}.$$ 

To evaluate $I_2$, parametrize $[z_2, z_3]$ by $z = \gamma_2(t) = 1 + it, \ 0 \leq t \leq 1, dz = idt$. So

$$I_2 = \int_{[z_2, z_3]} z \, dz = \int_0^1 (1 + it) \, idt = i(t + \frac{i}{2}t^2)\big|_0^1 = i(1 + \frac{i}{2}) = \frac{-1}{2} + i.$$ 

To evaluate $I_3$, parametrize $[z_3, z_4]$ by $z = \gamma_3(t) = (1 - t) + i, \ 0 \leq t \leq 1, dz = -dt$. So

$$I_3 = \int_{[z_3, z_4]} z \, dz = - \int_0^1 ((1 - t) + i) \, dt = -(1 + i) + \frac{1}{2} = -\frac{1}{2} - i.$$ 

To evaluate $I_4$, parametrize $[z_4, z_1]$ by $z = \gamma_4(t) = (1 - t)i, \ 0 \leq t \leq 1, dz = -idt$. So

$$I_4 = \int_{[z_4, z_1]} z \, dz = -i \int_0^1 i(1 - t) \, dt = 1 - \frac{1}{2} = \frac{1}{2}.$$ 

Finally, adding the four integrals, we obtain $I = 0.$

**Note:** When you will learn about Cauchy’s Theorem in Section 3.4, or the main result of the next section, you will realize that all our work in this solution is unnecessary to conclude that the answer is 0.

25. In our computation, we will suppose that $a$ is an integer $\neq 0$. Then the curve is traced in its entirety if we take $0 \leq t \leq 4\pi$. We apply the definition of the path integral, with $\gamma(t) = ae^{it} + be^{-i\frac{\pi}{2}t}, \ 0 \leq t \leq 4\pi, dz = (ai e^{it} - i\frac{ab}{2}e^{-i\frac{\pi}{2}t})dt$:

$$\int_\gamma z \, dz = \int_0^{4\pi} (ae^{it} + be^{-i\frac{\pi}{2}t})(ai e^{it} - i\frac{ab}{2}e^{-i\frac{\pi}{2}t}) \, dt$$

$$= \int_0^{4\pi} (ia^2 e^{2it} - \frac{ab^2}{2}e^{-i\frac{\pi}{2}t} + iabe^{i(1-\frac{\pi}{2})t} - \frac{a^2b}{2}e^{i(1-\frac{\pi}{2})t}) \, dt = 0,$$
because the integrand is a linear combination of exponential functions with period 4π each. Thus the integral of each exponential function is 0.

29. We have \( f(z) = z^2 \), \( \gamma(t) = (1 - t)(2 + i) + t(-1 - i) = (-3 - 2i)t + 2 + i, 0 \leq t \leq 1, \)
\( dz = (-3 - 2i)dt. \) So
\[
\int_{[z_1, z_2]} (x^2 + y^2) \, dz = \int_0^1 ((-3 - 2i)t + 2 + i)((-3 + 2i)t + 2 - i)(-3 - 2i)dt
\]
\[
= (-3 - 2i) \int_0^1 (13t^2 + (-3 - 2i)(2 - i)t + (2 + i)(3 - 2i)t + 5)dt
\]
\[
= (-3 - 2i)(\frac{13}{3} + \frac{-3 - 2i(2 - i)}{2} + \frac{2 + i(-3 + 2i)}{2} + 5) = -4 - \frac{8}{3}i.
\]

33. We have \( \gamma(t) = (e^t - t) + 4ie^{\frac{t}{2}}, \gamma'(t) = (e^t - 1) + 2ie^{\frac{t}{2}}, \)
\[
|\gamma'(t)| = |(e^t - 1) + 2ie^{\frac{t}{2}}| = \sqrt{(e^t - 1)^2 + (2e^{\frac{t}{2}})^2}
\]
\[
= \sqrt{e^{2t} + 4e^t} = \sqrt{(e^t + 1)^2} = e^t + 1.
\]
So
\[
L = \int_0^1 |\gamma'(t)| \, dt = \int_0^1 (e^t + 1) \, dt = (e^t + t)|_0^1 = e.
\]

37. We apply the \( ML \)-inequality (29). We have \( L = l([z_1, z_2, z_3, z_1]) = 2 + 2\sqrt{5}, \) as can be easily verified by plotting the points \( z_1, z_2, \) and \( z_3. \) Now \( |z^{-5}| = |z|^{-5}, \) and for \( z \) on the path \([z_1, z_2, z_3, z_1], \) we have \( |z| \leq 2^\frac{3}{4}. \) So \( |z|^{-5} \leq 2^5 \) for \( z \) on the path \([z_1, z_2, z_3, z_1]. \) Thus, by the \( ML \)-inequality,
\[
\left| \int_{[z_1, z_2, z_3, z_1]} z^{-5} \, dz \right| \leq 2^{-\frac{5}{4}}(2 + 2\sqrt{5}).
\]

41. We have
\[
\int_{\Gamma} f(z) \, dz = \int_{(\gamma, -\gamma)} f(z) \, dz
\]
\[
= \int_{\gamma} f(z) \, dz + \int_{-\gamma} f(z) \, dz
\]
\[
= \int_{\gamma} f(z) \, dz - \int_{\gamma} f(z) \, dz = 0.
\]
Solutions to Exercises 3.3

1. An antiderivative of $f(z) = z^2 + z - 1$ is simply $F(z) = \frac{1}{2}z^3 + \frac{1}{2}z^2 - z + C$ where $C$ is an arbitrary complex constant. The function $F$ is entire and so we can take $\Omega = \mathbb{C}$.

5. To find an antiderivative of $\frac{1}{(z-1)(z+1)}$, we proceed as we would have done in calculus. Using partial fractions, write

\[
\frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}
\]

\[
\frac{1}{(z-1)(z+1)} = \frac{A(z+1) + B(z-1)}{(z-1)(z+1)}
\]

\[
1 = A(z+1) + B(z-1).
\]

Taking $z = -1$, it follows that $B = -\frac{1}{2}$. Taking $z = 1$, it follows that $A = \frac{1}{2}$. Hence

\[
\frac{1}{(z-1)(z+1)} = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}.
\]

An antiderivative of this function is

\[
F(z) = \frac{1}{2} \left( \log (z-1) - \log (z+1) \right) + C,
\]

where $C$ is an arbitrary complex constant. (You could also use a different branch of the logarithm.) The function $\log (z-1)$ is analytic in $\mathbb{C} \setminus (-\infty, 1]$, while the function $\log (z+1)$ is analytic in $\mathbb{C} \setminus (-\infty, -1]$. So the function $F(z) = \frac{1}{2} \left( \log (z-1) - \log (z+1) \right) + C$, is analytic in $\mathbb{C} \setminus (-\infty, 1]$ and we may take $\Omega = \mathbb{C} \setminus (-\infty, 1]$. In fact, the function $\log (z-1) - \log (z+1) = \log \frac{z-1}{z+1}$ is analytic in a larger region $\mathbb{C} \setminus [-1, 1]$. There are at least two possible ways to see this. One way is to note that the linear fractional transformation $w = \frac{z-1}{z+1}$ takes the interval $[-1, 1]$ onto the half-line $(-\infty, 0]$. All other values of $z$ outside the interval $[-1, 1]$ are mapped into $\mathbb{C} \setminus (-\infty, 0]$ and so the composition $\log \frac{z-1}{z+1}$ is analytic everywhere on $\mathbb{C} \setminus [-1, 1]$. Another way to show that $\log (z-1) - \log (z+1)$ is analytic in $\mathbb{C} \setminus [-1, 1]$ is to use Theorem 4, Sec. 2.3. Let $g(z) = \log (z-1) - \log (z+1)$ and $f(z) = e^z$. The function $g(z)$ is continuous on $\mathbb{C} \setminus [-1, 1]$, because the discontinuities of $\log (z-1)$ and $\log (z+1)$ cancel on $(-\infty, -1)$. The function $f(z) = e^z$ is obviously entire. The composition $f(g(z))$ is equal to the function $\frac{z-1}{z+1}$, which is analytic except at the points $z = \pm 1$. According to Theorem 4, Sec. 2.3, the function $g(z)$ is analytic in $\mathbb{C} \setminus [-1, 1]$.

9. An antiderivative of $z \sinh z^2$ is $\frac{1}{2} \cosh z^2 + C$, as you can verify by differentiation. The antiderivative is valid for all $z$.

13. All branches of the logarithm, $\log_a z$, have the same derivative $\frac{1}{z}$. The only difference is that the domain of analyticity differs in each case, since the branch cut is different. Again, taking a hint from calculus and the fact that an antiderivative of $\ln x$ is $-x + x \ln x + C$ (as you can verify directly or integrate $\ln x$ by parts), we find that an antiderivative of $f_1(z) = \log_0 z$ is $F_1(z) = -z + z \log_0 z + C$. The functions $f_1$ and $F_1$ are analytic in the region $\Omega_1 = \mathbb{C} \setminus [0, \infty)$. Similarly, an antiderivative of $f_2(z) = \log_\frac{1}{2} z$ is $F_2(z) = -z + z \log_\frac{1}{2} z + C$. The functions $f_2$ and $F_2$ are analytic in the region $\Omega_2 = \mathbb{C}$ minus the ray at angle $\frac{\pi}{2}$. So, an antiderivative of the function $\log_0 z + \log_\frac{1}{2} z + \frac{1}{2}$ is $-2z + z \log_0 z + z \log_\frac{1}{2} z + \log z + C$, and the antiderivative is good in the region $\mathbb{C}$ minus the rays at angle $0$, $\frac{\pi}{2}$, and $\pi$, or, simply, $\mathbb{C}$ minus the real line and the ray at angle $\frac{\pi}{2}$.

17. The integrand is entire and has an antiderivative in a region containing the path. So, by
Chapter 3  Complex Integration

Theorem 1,
\[
\int_\gamma z^2 \, dz = \left. \frac{1}{3} z^3 \right|^{\gamma(\pi)}_{\gamma(0)} = \frac{1}{3} \left( (e^{i\pi} + 3e^{2i\pi})^3 - (e^{i0} + 3e^{2i0})^3 \right)
\]
\[
= \frac{1}{3} \left( e^{i\pi} + 9e^{i\pi} + 27e^{i\pi} e^{2i\pi} + 27e^{i\pi} - (4)^3 \right)
\]
\[
= \frac{1}{3} \left( \frac{-\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} - 9 - 27(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}) - 27i - (4)^3 \right)
\]
\[
= \frac{1}{3} \left( (-14\sqrt{2} - 73 + i(-13\sqrt{2} - 27)) \right)
\]

21. Evaluate using Theorem 1:
\[
\int_\gamma \sin z \, dz = -\cos \left. \frac{\gamma(z)}{\gamma(0)} \right| = \cos(2e^{i0}) - \cos(2e^{i\pi})
\]
\[
= \cos 2 - \cos(2i) = \cos 2 - \cosh 2.
\]
(Use (25), Sec. 1.6.)

25. The function Log z has an analytic antiderivative in a region that contains the path [z₁, z₂, z₃, z₁]; for example, the function F(z) = z Log z - z. So by Theorem 1,
\[
\int_{[z₁, z₂, z₃, z₁]} z \, Log z \, dz = 0.
\]

29. Part (a) is clear. For (b), the path γ₁ lies completely in a region on which the function Log z is analytic. Since Log z is an antiderivative of \(\frac{1}{z}\), it follows from Theorem 1 that
\[
\int_{γ₁} \frac{1}{z} \, dz = \text{Log} (z₁) - \text{Log} (z₂).
\]
Similarly, the path γ₂ lies completely in a region on which the function log₀ z is analytic. Since log₀ z is an antiderivative of \(\frac{1}{z}\), it follows from Theorem 1 that and
\[
\int_{γ₂} \frac{1}{z} \, dz = \text{log₀} (z₂) - \text{log₀} (z₁).
\]
Now
\[
\int_{C₀(z₀)} \frac{1}{z} \, dz = \int_{γ₁} \frac{1}{z} \, dz + \int_{γ₂} \frac{1}{z} \, dz = \text{Log} z₁ - \text{Log} z₂ + \text{log₀} z₂ - \text{log₀} z₁.
\]
Recall that z₁ is on the positive y-axis, so Log z₁ = log₀ z₁. Also z₂ is on the negative y-axis, so arg₂₀ z₂ = \(\frac{3\pi}{2}\) and Arg z₂ = \(-\frac{\pi}{2}\). Hence log₀ z₂ - Log z₂ = 2\pi i. Putting this information together, we get
\[
\int_{C₀(z₀)} \frac{1}{z} \, dz = 2\pi i.
\]

33. If f is analytic in a region Ω, then obviously f is an antiderivative of f' in Ω. Let z₀ and z₁ be any two points in Ω. Since Ω is connected, we can join z₀ to z₁ by a polygonal path γ, with initial point z₀ and terminal point z₁. By Theorem 1,
\[
\int_γ f'(z) \, dz = f(z₁) - f(z₀).
\]
But f'(z) = 0, so f(z₁) - f(z₀) = 0 implying that f(z₀) = f(z₁). Since z₁ is arbitrary, it follows that f is constant equal to f(z₀).
Solutions to Exercises 3.4

1. Each path is continuously deformable to a point in Ω and Ω is connected. So the two paths are mutually continuously deformable.

5. Same reasoning as in Exercise 1.

9. The path in Figure 35 is an ellipse centered at $3i$ with length of major axis 6 and length of minor axis 3. The ellipse can be parametrized by

$$\gamma_0(t) = 3i + 3\cos t + i\sin t, \quad 0 \leq t \leq 2\pi.$$  

To parametrize $\gamma_0$ by the interval $[0, 1]$, we can use

$$\gamma_0(t) = 3i + 3\cos(2\pi t) + i\sin(2\pi t) = 3\cos(2\pi t) + i(3 + \sin(2\pi t)), \quad 0 \leq t \leq 1.$$  

It is clear from the figure that we can deform $\gamma_0$ continuously to $3i$, its center (there are, of course, many other possible points). Since the interior of the ellipse is convex, we can use (8) to construct the desired mapping of the unit square. We take $\gamma_0$ as described above and $\gamma_1(t) = 3i$. Then, for $0 \leq t \leq 1$ and $0 \leq s \leq 1$:

$$H(t, s) = (1 - s)\gamma_0(t) + s\gamma_1(t) = 3(1 - s)\cos(2\pi t) + i(1 - s)(3 + \sin(2\pi t)) + 3is.$$  

13. The function $f(z) = \frac{e^z}{z^2 + 2}$ is analytic inside an on the simple path $C_1(0)$. By Theorem 5,

$$\int_{C_1(0)} \frac{e^z}{z^2 + 2} dz = 0.$$  

17. The function $f(z) = \frac{e^z}{z^2 + i}$ has a problem at $z = -i$. Hence it is analytic inside an on the simple path $\gamma(t) = i + e^{it}, 0 \leq t \leq 2\pi$ (circle, centered at $i$ with radius 1). By Theorem 5,

$$\int_{\gamma} \frac{e^z}{z^2 + i} dz = 0.$$  

21. The path $[z_1, z_2, z_3, z_1]$ is contained in a region that does not intersect the branch cut of $\text{Log } z$. Hence the function $f(z) = z^2 \text{Log } z$ is analytic inside an on the simple path $[z_1, z_2, z_3, z_1]$, and so by Theorem 5,

$$\int_{[z_1, z_2, z_3, z_1]} z^2 \text{Log } z dz = 0.$$  

25. The path $C_2(0)$ contains both roots of the polynomial $z^2 - 1$. We will evaluate the integral by using the method of Example 5. We have

$$\frac{z}{z^2 - 1} = \frac{A}{z - 1} + \frac{B}{z + 1} \Rightarrow z = A(z + 1) + B(z - 1).$$  

Setting $z = 1$, we get $1 = 2A$ or $A = \frac{1}{2}$. Setting $z = -1$, we get $-1 = -2B$ or $B = \frac{1}{2}$. Hence

$$\frac{z}{z^2 - 1} = \frac{1}{2(z - 1)} + \frac{1}{2(z + 1)}.$$
and so
\[ \int_{C_2(0)} \frac{z}{z^2 - 1} \, dz = \frac{1}{2} \int_{C_2(0)} \frac{1}{z - 1} \, dz + \frac{1}{2} \int_{C_2(0)} \frac{1}{z + 1} \, dz = \frac{1}{2} \pi i + \frac{1}{2} \pi i = 2 \pi i, \]
where we have applied the result of Example 4 in evaluating the integrals.

29. Write \( \gamma = (\gamma_1, \gamma_2) \), where \( \gamma_1 \) is the circle centered at \( i \) and \( \gamma_2 \) is the circle centered at \(-1\). Then
\[ \int_{\gamma} \frac{1}{(z + 1)^2(z^2 + 1)} \, dz = \int_{\gamma_1} \frac{1}{(z + 1)^2(z^2 + 1)} \, dz + \int_{\gamma_2} \frac{1}{(z + 1)^2(z^2 + 1)} \, dz = I_1 + I_2. \]
The partial fraction decomposition of the integrand is
\[ \frac{1}{(z + 1)^2(z^2 + 1)} = \frac{1}{2(z + 1)} + \frac{1}{4(z + i)} - \frac{1}{4(z - i)}. \]
We have
\[ I_1 = \frac{1}{2} \int_{\gamma_1} \frac{dz}{z + 1} + \frac{1}{2} \int_{\gamma_1} \frac{dz}{(z + 1)^2} - \frac{1}{4} \int_{\gamma_1} \frac{dz}{z + i} - \frac{1}{4} \int_{\gamma_1} \frac{dz}{z - i} \]
\[ = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 - \frac{1}{4} \cdot 0 - \frac{1}{4} \cdot 2 \pi i = -\pi i, \]
where the first 3 integrals are 0 because the integrands are analytic inside an on \( \gamma_1 \), and the fourth integral follows from Example 4. Similarly,
\[ I_2 = \frac{1}{2} \int_{\gamma_2} \frac{dz}{z + 1} + \frac{1}{2} \int_{\gamma_2} \frac{dz}{(z + 1)^2} - \frac{1}{4} \int_{\gamma_2} \frac{dz}{z + i} - \frac{1}{4} \int_{\gamma_2} \frac{dz}{z - i} \]
\[ = \frac{1}{2} \cdot 2 \pi i + \frac{1}{2} \cdot 0 - \frac{1}{4} \cdot 0 - \frac{1}{4} \cdot 0 = \pi i, \]
where the first, third, and fourth integrals follow from Example 4, and the second integral follows from Example 4, Sec. 3.2. Thus, the desired integral is equal to
\[ I_1 + I_2 = \frac{\pi i}{2}. \]

33. (a) Let \( z_1, z_2, \ldots, z_n \) be distinct complex numbers \((n \geq 2)\). Reducing to the common denominator in the partial fraction decomposition
\[ \frac{1}{(z - z_1)(z - z_2) \cdots (z - z_n)} = \frac{A_1}{z - z_1} + \frac{A_2}{z - z_2} + \cdots + \frac{A_n}{z - z_n}, \]
we obtain
\[ \frac{1}{p(z)} = \frac{A_1(z - z_2) \cdots (z - z_n)}{p(z)} + \frac{A_2(z - z_1)(z - z_3) \cdots (z - z_n)}{p(z)} + \cdots + \frac{A_n(z - z_1)(z - z_2) \cdots (z - z_{n-1})}{p(z)}, \]
where \( p(z) = (z - z_1)(z - z_2) \cdots (z - z_n) \). From this expression, we see clearly that the coefficient of \( z^{n-1} \) in the numerator on the right side is \( A_1 + A_2 + \cdots + A_n \). Comparing with the left side, we see that \( A_1 + A_2 + \cdots + A_n = 0 \).

(b) Suppose that \( C \) is a simple closed path that contains the points \( z_1, z_2, \ldots, z_n \) in its interior. Using the partial fraction decomposition, we have
\[ \int_C \frac{1}{(z - z_1)(z - z_2) \cdots (z - z_n)} \, dz = A_1 \int_C \frac{dz}{z - z_1} + A_2 \int_C \frac{dz}{z - z_2} + \cdots + A_n \int_C \frac{dz}{z - z_n} \]
\[ = 2 \pi i (A_1 + A_2 + \cdots + A_n) = 0, \]
where we have used Example 4 to evaluate the integrals and part (a) to sum the \( A_j \)'s.
Solutions to Exercises 3.5

1. An example of a closed and bounded subset $K$ of the real line and a bounded subset $S$ of the real line such that $K$ and $S$ are disjoint but the distance from $K$ to $S$ is $0$: Let $K = [0, 1]$ and $S = (1, 2)$. Then $K \cap S = \emptyset$ but the distance from $K$ to $S$ is $0$.

5. The distance formula: for $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, $w_1 = c_1 + id_1$, and $w_2 = c_2 + id_2$:

$$|(z_1, w_1) - (z_2, w_2)| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2 + (d_1 - d_2)^2}.$$

(a) If $z = a + ib$ and $w = c + id$, then by definition

$$|z, w| = |(z, w) - (0, 0)| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

It is clear then that $|(z, w)| \geq 0$ and

$$(z, w) = (0, 0) \Leftrightarrow a^2 + b^2 + c^2 + d^2 = 0 \Leftrightarrow a = b = c = d = 0.$$

(b) For any complex number $\alpha$, write $\alpha = A + iB$, where $A$ and $B$ are real. Then

$$|\alpha(z, w)| = |(\alpha z, \alpha w)| = |(Az - Bb + i(Ab + Ba), Ac - Bd + i(Ad + Bc))|$$

$$= \sqrt{(Aa - Bb)^2 + (Ab + Ba)^2 + (Ac - Bd)^2 + (Ad + Bc)^2}$$

$$= \sqrt{A^2 + B^2}(a^2 + b^2 + c^2 + d^2)$$

$$= \sqrt{A^2 + B^2}\sqrt{a^2 + b^2 + c^2 + d^2}$$

$$= |\alpha||z, w|.$$

(c) We will prove the triangle inequality: $|(z_1, w_1) + (z_2, w_2)| \leq |(z_1, w_1)| + |(z_2, w_2)|$ by following the proof of the triangle inequality for the usual absolute value (see Sec. 1.2). First note that from the definition of the absolute value on $\mathbb{C} \times \mathbb{C}$, we have

$$|(z, w)|^2 = |z|^2 + |w|^2,$$

where on the right side we are using the usual absolute value on $\mathbb{C}$. Also note that for complex numbers $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, $w_1 = c_1 + id_1$, and $w_2 = c_2 + id_2$, we have

$$|z_1| |z_2| + |w_1| |w_2| \leq \sqrt{(|z_1|^2 + |w_1|^2)(|z_2|^2 + |w_2|^2)},$$

as can be verified by squaring both sides. We can now prove the desired identity:

$$|(z_1, w_1) + (z_2, w_2)|^2 = |(z_1 + z_2, w_1 + w_2)|^2 = |z_1 + z_2|^2 + |w_1 + w_2|^2$$

$$\leq (|z_1| + |z_2|)^2 + (|w_1| + |w_2|)^2$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|^2 + |w_1|^2 + |w_2|^2 + 2|w_1||w_2|^2$$

$$\leq |z_1|^2 + |w_1|^2 + |z_2|^2 + |w_2|^2 + 2\sqrt{(|z_1|^2 + |w_1|^2)(|z_2|^2 + |w_2|^2)}$$

$$\leq \frac{1}{2}(|z_1|^2 + |w_1|^2) + \frac{1}{2}(|z_2|^2 + |w_2|^2).$$

The desired inequality follows upon taking square roots on both sides.
9. Pick $y_0$ in the interval $(c, d)$. We have to show that

$$
\lim_{h \to 0} \frac{1}{h} \int_a^b (u(x, y_0 + h) - u(x, y_0)) \, dx = \int_a^b \frac{\partial}{\partial y} u(x, y_0) \, dx.
$$

Equivalently, we have to show that the difference

$$
\left| \frac{1}{h} \int_a^b (u(x, y_0 + h) - u(x, y_0)) \, dx - \int_a^b \frac{\partial}{\partial y} u(x, y_0) \, dx \right|
$$

can be made arbitrarily small for all small values of $h$. Since $\frac{\partial}{\partial y} u$ is continuous in $R$, it is uniformly continuous. So, given $\epsilon > 0$, we can find a $\delta > 0$ such that if $0 < h < \delta$, then

$$
\left| \frac{\partial}{\partial y} u(x, s) - \frac{\partial}{\partial y} u(x, y_0) \right| < \epsilon \quad \text{for all } y_0 < s < y_0 + h.
$$

Now using the fact that

$$
u(x, y_0 + h) - u(x, y_0) = \int_{y_0}^{y_0+h} \frac{\partial}{\partial y} u(x, s) \, ds,
$$

we find that for $0 < h < \delta$

$$
\left| \frac{1}{h} \int_a^b (u(x, y_0 + h) - u(x, y_0)) \, dx - \int_a^b \frac{\partial}{\partial y} u(x, y_0) \, dx \right|
= \left| \frac{1}{h} \int_a^b \int_{y_0}^{y_0+h} \frac{\partial}{\partial y} u(x, s) \, ds \, dx - \int_a^b \frac{\partial}{\partial y} u(x, y_0) \, dx \right|
\leq \int_a^b \frac{1}{h} \int_{y_0}^{y_0+h} \left| \frac{\partial}{\partial y} u(x, s) - \frac{\partial}{\partial y} u(x, y_0) \right| \, ds \, dx
\leq \frac{1}{h} \int_a^b \int_{y_0}^{y_0+h} \epsilon \, ds \, dx
= \epsilon(b - a).
$$

Since $\epsilon$ is arbitrary, the proof is complete.
Solutions to Exercises 3.6

1. Apply Cauchy’s formula with \( f(z) = \cos z \) at \( z = 0 \). Then
\[
\int_{C_1(0)} \frac{\cos z}{z} \, dz = \int_{C_1(0)} \frac{\cos z}{z - 0} \, dz = 2\pi i f(0) = 2\pi i.
\]

5. Apply Cauchy’s formula with \( f(z) = -\log z \) at \( z = i \). Then
\[
\int_{C_{\frac{1}{2}}(i)} \frac{\log z}{z+i} \, dz = \int_{C_{\frac{1}{2}}(i)} \frac{-\log z}{z - i} \, dz = -2\pi i \log i = -2\pi i (\ln 1 + \frac{\pi}{2}) = \pi^2.
\]

9. Apply the generalized Cauchy formula (6), with \( f(z) = \sin z \) at \( z = \pi \), with \( n = 2 \). Then
\[
\int_{\gamma} \frac{\sin z}{(z - \pi)^3} \, dz = \frac{2\pi i}{2!} f^{(2)}(\pi) = \pi i (-\sin \pi) = 0.
\]

13. Follow the solution in Example 2. Draw small nonintersecting negatively oriented circles inside \( \gamma, \gamma_1 \) centered at 0 and \( \gamma_2 \) centered at \( i \). Then
\[
\int_{\gamma} \frac{z + \cos(\pi z)}{z(z^2 + 1)} \, dz = \int_{\gamma_1} \frac{z + \cos(\pi z)}{z(z^2 + 1)} \, dz + \int_{\gamma_2} \frac{z + \cos(\pi z)}{z(z^2 + 1)} \, dz = I_1 + I_2.
\]
Applying Cauchy’s formula with \( f(z) = \frac{z + \cos(\pi z)}{z^2 + 1} \) at \( z = 0 \). Then (recall \( \gamma_1 \) is negatively oriented)
\[
I_1 = \int_{\gamma_1} \frac{z + \cos(\pi z)}{z(z^2 + 1)} \, dz = -2\pi i f(0) = -2\pi i \frac{0 + \cos 0}{0^2 + 1} = -2\pi i.
\]
Applying Cauchy’s formula with \( f(z) = \frac{z + \cos(\pi z)}{z + i} \) at \( z = i \). Then
\[
I_2 = \int_{\gamma_2} \frac{z + \cos(\pi z)}{z(z^2 + 1)} \, dz = \int_{\gamma_2} \frac{z + \cos(\pi z)}{z(z + i)(z - i)} \, dz
= -2\pi i f(i) = -2\pi i \frac{i + \cos \pi i}{i(2i)} = \pi i (i + \cosh \pi).
\]
So \( I_1 + I_2 = -\pi - i\pi(2 - \cosh \pi) \).

17. Factor the denominator as \( z^3 - 3z + 2 = (z + 2)(z - 1)^2 \). Apply the generalized Cauchy formula (6), with \( f(z) = \frac{1}{z^2} \) at \( z = 1 \), with \( n = 1 \). Then
\[
\int_{C_{\frac{1}{2}}(0)} \frac{dz}{(z + 2)(z - 1)^2} = 2\pi i f'(1) = 2\pi i \frac{-1}{3^2} = \frac{-2\pi i}{9}.
\]

21. (a) For \( |z| < 1 \), let \( F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{e^{it} - z} \, dt \) (we changed the notation of the exercise). We will apply Theorem 4, Sec. 3.5, with \( f(z, t) = \frac{e^{it}}{e^{it} - z} \), where \( t \) is on the path \([0, 2\pi]\) and \( z \) in the region \( |z| < 1 \). Because \( |e^{it}| = 1 \) and \( |z| < 1 \), it follows that \( f(z, t) \) is continuous for all \( z \) and \( t \) in their respective domains. Moreover, we have
\[
\frac{d}{dz} f(z, t) = \frac{e^{it}}{(e^{it} - z)^2},
\]
which is also continuous for all \( z \) and \( t \), in their respective domains, for the same reason. By Theorem 4, Sec. 3.5, the function \( F(z) \) is analytic for all \( |z| < 1 \).

(b) Let \( e^{it} = \zeta, \ i e^{it} dt = d\zeta \) or \( dt = \frac{d\zeta}{i} \). As \( t \) runs through \([0, 2\pi]\), \( \zeta \) runs through the unit circle in the positive direction. Hence

\[
F(z) = \frac{1}{2\pi i} \int_{C_1(0)} \frac{\zeta}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{C_1(0)} \frac{d\zeta}{\zeta - z} = 1.
\]

25. Define \( F(z) = \int_0^1 \cos(zt) \, dt \). Let \( f(z, t) = \cos(zt) \). Repeat the argument in Exercise 21(a) to see that \( F(z) \) is entire. If \( z = 0 \), \( F(z) = \int_0^1 dt = 1 \). For \( z \neq 0 \), we have

\[
F(z) = \int_0^1 \cos zt \, dt = \frac{1}{z} \sin z \bigg|_{t=0}^{t=1} = \frac{\sin z}{z}.
\]

Thus

\[
F(z) = \begin{cases} 
1 & \text{if } z = 0, \\
\frac{\sin z}{z} & \text{if } z \neq 0,
\end{cases}
\]

is an entire function.

29. For \( z \) inside \( C \), by Cauchy’s formula,

\[
f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta.
\]

But \( f(\zeta) = g(\zeta) \) for \( \zeta \) on \( C \). So, for all \( z \) inside \( C \),

\[
f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_C \frac{g(\zeta)}{\zeta - z} \, d\zeta = g(z),
\]

by Cauchy’s formula applied to \( g \).

33. (a) Since our proof depends heavily on Exercise 32, let us outline the proof of part (a) of that exercise. To prove that

\[
\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{(z - z_0)}{(\zeta - z)(\zeta - z_0)^2} f(\zeta) \, d\zeta,
\]

note that the integrand on the right side is equal to

\[
f(\zeta) \left[ \frac{1}{(z - z_0)(\zeta - z)} - \frac{1}{(\zeta - z_0)^2} - \frac{1}{(z - z_0)(\zeta - z_0)} \right]
\]

Integrating the partial fraction and using Cauchy’s formula, we obtain

\[
\frac{(z - z_0)}{(\zeta - z)(\zeta - z_0)^2} f(\zeta) = \frac{1}{2\pi i} \int_{C_R(z_0)} f(\zeta) \frac{1}{(z - z_0)(\zeta - z)} \, d\zeta + \frac{1}{2\pi i} \int_{C_R(z_0)} f(\zeta) \frac{1}{(\zeta - z_0)^2} \, d\zeta
\]

\[-\frac{1}{2\pi i} \int_{C_R(z_0)} f(\zeta) \frac{1}{z - z_0} \frac{1}{(\zeta - z_0)} \, d\zeta
\]

\[
= \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0).
\]
where $A$ or $B$ for every closed triangular path contained in $\gamma$ yields the desired identity.

In the preceding identity, replace $f(z)$ by $f(z, w)$, and get

$$\frac{f(z, w) - f(z_0, w)}{z - z_0} - \frac{d}{dz}(z_0, w) = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{(z - z_0)}{(\zeta - z)(\zeta - z_0)^2} f(\zeta, w) d\zeta.$$  

Integrate the identity over $\gamma$ and get

$$\int_{\gamma} f(z, w) dw - \int_{\gamma} f(z_0, w) dw - \int_{\gamma} \frac{d}{dz}(z_0, w) dw = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{(z - z_0)}{(\zeta - z)(\zeta - z_0)^2} f(\zeta, w) d\zeta,$$

or

$$\frac{F(z) - F(z_0)}{z - z_0} - \int_{\gamma} \frac{d}{dz}(z_0, w) dw = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{(z - z_0)}{(\zeta - z)(\zeta - z_0)^2} f(\zeta, w) d\zeta,$$

So

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - \int_{\gamma} \frac{d}{dz}(z_0, w) dw \right| = \left| \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{(z - z_0)}{(\zeta - z)(\zeta - z_0)^2} f(\zeta, w) d\zeta \right| \leq A \cdot l(\gamma),$$

where $A$ is the maximum value of

$$\left| \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{(z - z_0)}{(\zeta - z)(\zeta - z_0)^2} f(\zeta, w) d\zeta \right|.$$

According to Exercise 32(c), $A \leq \frac{2M}{R^2}|z - z_0|$, which implies

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - \int_{\gamma} \frac{d}{dz}f(z_0, \zeta) d\zeta \right| \leq \frac{2M}{R^2}|z - z_0| l(\gamma).$$

(b) Letting $z \to z_0$ in (a), the right side tends to 0, and the left side tends to $F'(z_0) - \int_{\gamma} \frac{d}{dz}f(z_0, \zeta) d\zeta$. Hence

$$F'(z_0) = \int_{\gamma} \frac{d}{dz}f(z_0, \zeta) d\zeta.$$

37. We will show the following version of Morera’s theorem. Suppose that $f$ is continuous and complex-valued in $\Omega$ such that for every point $z_0$ in $\Omega$, there is a disk $B_R(z_0)$ contained in $\Omega$ such that

$$\int_{\gamma} f(z) dz = 0$$

for every closed triangular path contained in $B_R(z_0)$. This is a stronger version than the one in the text, since we do assume that $\int_{\gamma} f(z) dz = 0$ for all closed paths $\gamma$ in $\Omega$.

(a) It is enough to show that, given any point $z_0$ in $\Omega$, there is a disk $B_R(z_0)$ contained in $\Omega$ such that $f$ has a analytic antiderivative $F(z)$ in $B_R(z_0)$. Indeed, if $F$ is analytic and
Chapter 3 Complex Integration

If \( F^m(z) = f(z) \) for all \( z \) in \( B_R(z_0) \), it follows from Corollary 1 that \( f \) is analytic in \( B_R(z_0) \), and since this holds for all \( z_0 \) in \( \Omega \), it follows that \( f \) is analytic in \( \Omega \).

(b) Given \( z_0 \), pick a disk \( B_R(z_0) \) such that integrals over triangular paths in \( B_R(z_0) \) are 0. To show that \( f \) has an analytic antiderivative in \( B_R(z_0) \), for \( z \) in \( B_R(z_0) \), define \( F(z) = \int_{[z_0,z]} f(\zeta) \, d\zeta \). Then \( F \) is analytic in \( B_R(z_0) \) by Theorem 4, Sec. 3.5. We now show that \( F'(z) = f(z) \). We have

\[
\frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z) = \frac{1}{\Delta z} \left[ \int_{[z_0,z+\Delta z]} f(\zeta) \, d\zeta - \int_{[z_0,z]} f(\zeta) \, d\zeta \right]
\]

\[
= \frac{1}{\Delta z} \left[ \int_{[z_0,z+\Delta z]} f(\zeta) \, d\zeta + \int_{[z,z_0]} f(\zeta) \, d\zeta \right]
\]

\[
= \frac{1}{\Delta z} \int_{[z,z+\Delta z]} f(\zeta) \, d\zeta.
\]

In the last equality, we used the fact that the integral over a triangular path in \( B_R(z_0) \) is 0.

c) Using Lemma 1, Section 3.3, we find that the limit in (b) as \( \Delta z \to 0 \) is \( f(z) \). Hence \( F'(z) = f(z) \), as desired.

41. If \( f \) and \( g \) are analytic at \( z_0 \), such that \( f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \) and \( g(z_0) = g'(z_0) = \cdots = g^{(m-1)}(z_0) = 0 \), but \( g^{(m)}(z_0) \neq 0 \), then according to Exercise 40, we can write \( f(z) = (z - z_0)^m \phi(z) \) and \( g(z) = (z - z_0)^m \psi(z) \) where \( \phi \) and \( \psi \) are analytic at \( z_0 \), \( \phi(z_0) = \frac{f^{(m)}(z_0)}{m!} \), and \( \psi(z_0) = \frac{g^{(m)}(z_0)}{m!} \). Then

\[
\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{(z - z_0)^m \phi(z)}{(z - z_0)^m \psi(z)} = \lim_{z \to z_0} \frac{\phi(z)}{\psi(z)} = \frac{\phi(z_0)}{\psi(z_0)} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)}.
\]
Solutions to Exercises 3.7

1. We have \(|f(z)| = |z|\). Obviously, if \(z\) belongs to the unit disk, \(|z| \leq 1\), then the largest value of \(|f(z)|\) is 1 and its smallest value is 0. Hence \(|f(z)|\) attains its maximum value at all points of the boundary, and it attains its minimum value at the point \(z = 0\), inside the region. The fact that the minimum is attained inside the region does not contradict Corollary 3, because \(f(z)\) vanishes at \(z = 0\) inside of \(\Omega_c\).

5. The function \(f(z) = \frac{z}{z^2 + 2}\) is continuous for all \(z\) such that \(2 \leq |z| \leq 3\) and does not vanish inside this annular region. Thus according to Corollary 3, \(|f(z)|\) attains its maximum and minimum at the boundary; that is at points where \(|z| = 2\) or \(|z| = 3\). Using the triangle inequality, we have

\[
|z^2 + 2| \leq |z|^2 + 2 \Rightarrow \frac{1}{|z|^2 + 2} \leq \frac{1}{|z^2 + 2|};
\]

\[
|z^2 + 2| \geq ||z|^2 - 2| \Rightarrow \frac{1}{|z^2 + 2|} \leq \frac{1}{||z|^2 - 2||}.
\]

On the part of the boundary \(|z| = 2\), we have

\[
\frac{1}{3} = \frac{2}{|2|^2 + 2} \leq |f(z)| = \left| \frac{z}{z^2 + 2} \right| \leq \frac{2}{|2|^2 - 2} = 1.
\]

On the part of the boundary \(|z| = 3\), we have

\[
\frac{3}{11} = \frac{3}{|3|^2 + 2} \leq |f(z)| = \left| \frac{z}{z^2 + 2} \right| \leq \frac{3}{|3|^2 - 2} = \frac{3}{7}.
\]

Thus the smallest value of \(|f(z)|\) is \(\frac{3}{11}\). It is attained at a point \(z\) with \(|z| = 3\). For this value of \(z\), we must have \(|z^2 + 2| = 11\). The only possibilities are \(z = \pm 3\).

The largest value of \(|f(z)|\) is 1. It is attained at a point \(z\) with \(|z| = 2\). For this value of \(z\), we must have \(|z^2 + 2| = 2\) or \(|z^2 = 3\). The only possibilities are \(z = \pm 2i\).

9. We have

\[
|f(z)| = |\ln |z| + i \arg z|;
\]

\[
|f(z)|^2 = (\ln |z|)^2 + (\arg z)^2.
\]

The largest value (respectively, minimum value) of \(|f(z)|\) is attained when \(|f(z)|^2\) attains its largest value (respectively, minimum value). The largest value of \(|f(z)|^2 = (\ln |z|)^2 + (\arg z)^2\) is clearly attained when \(|z| = 2\) and \(\arg z = \frac{\pi}{4}\). So \(|f(z)|\) attains its maximum value \(\sqrt{(\ln 2)^2 + (\frac{\pi}{4})^2}\) when \(z = 2e^{i\frac{\pi}{4}}\).

The smallest value of \(|f(z)|^2 = (\ln |z|)^2 + (\arg z)^2\) is clearly attained when \(|z| = 1\) and \(\arg z = 0\). So \(|f(z)|\) attains its smallest value 0 when \(z = 1\).

13. (a) To verify the identity

\[
z^n - w^n = (z - w)(z^{n-1} + z^{n-2}w + z^{n-3}w^2 + \cdots + w^{n-2} + w^{n-1}),
\]

expand the right side. All terms cancel except for \(z^n - w^n\).

(b) If \(p(z) = p_n z^n + p_{n-1} z^{n-1} + \cdots + p_1 z + p_0\) is a polynomial of degree \(n \geq 2\), and if \(p(z_0) = 0\), then \(p_n z_0^n + p_{n-1} z_0^{n-1} + \cdots + p_1 z_0 + p_0 = 0\). From (a),

\[
p(z) = p(z) - p(z_0) = p_n(z^n - z_0^n) + p_{n-1}(z^{n-1} - z_0^{n-1}) + \cdots + p_1(z - z_0)
\]

\[
= (z - z_0)q(z),
\]

where \(q(z) = (z^{n-1} + z^{n-2}z_0 + z^{n-3}z_0^2 + \cdots + z z_0^{n-2} + z_0^{n-1})\) is a polynomial of degree \(n - 1\) in \(z\).
17. Suppose that \( f \) is entire and that it omits an open nonempty set, say, there is an open disk \( B_R(w_0) \) with \( R > 0 \) in the \( w \)-plane such that \( f(z) \) is not in \( B_R(w_0) \) for all \( z \). Let \( g(z) = \frac{1}{f(z) - w_0} \). Then \( g(z) \) is also entire, because \( f(z) \neq w_0 \) for all \( z \). In fact, since \( f(z) \) is not in \( B_R(w_0) \), its distance to \( w_0 \) is always greater than \( R \). That is, \( |f(z) - w_0| \geq R \). But this implies that \( |g(z)| \leq \frac{1}{R} \), which in turn implies that \( g \) is constant, by Liouville’s theorem. Since \( g(z) \neq 0 \), this constant is obviously not 0. So \( C = \frac{1}{f(z) - w_0} \), hence \( f(z) = \frac{1}{C} + w_0 \) is constant.

21. Suppose that \( f(z) = f(x + iy) \) is periodic in \( x \) and \( y \), and let \( T_1 > 0 \) and \( T_2 > 0 \) be such that \( f((x+T_1) + i(y+T_2)) = f(x + iy) \) for all \( z = x + iy \). Because \( f(z) \) is periodic in both \( x \) and \( y \), its values repeat on every \( T_1 \times T_2 \)-rectangle. This means that, if we take the rectangle \( R = [0, T_1] \times [0, T_2] \) and consider the values of \( f \) on this rectangle, then these are all the values taken by \( f(z) \), for \( z \) in \( \mathbb{C} \). The reason is that the complex plane can be tiled by translates of \( R \) in the \( x \) and \( y \) direction, where in the \( x \) direction we translate by \( T_1 \) units at a time, and in the \( y \) direction we translate by \( T_2 \) units at a time. Now since \( f \) is continuous, it is bounded on \( R \); that is \( |f(z)| \leq M \) for some constant \( M \) and all \( z \) in \( R \). But since \( f \) takes on all its values in \( R \), we conclude that \( |f(z)| \leq M \) for all \( z \) in \( \mathbb{C} \), and thus \( f \) must be constant by Liouville’s theorem.

If \( f \) is constant in \( x \) or \( y \) alone, \( f \) need not be constant. As an example, take \( f(z) = \sin z \) then \( f \) is \( 2\pi \)-periodic in the \( x \) variable. By considering \( f(iz) = \sin(iz) \), we obtain a function that is constant in \( y \) alone. Both functions are entire and clearly not constant.

25. Suppose that \( f \) is analytic on \( |z| < 1 \) and continuous on \( |z| \leq 1 \). Suppose that \( f(z) \) is real-valued for all \( |z| = 1 \). Consider \( g(z) = e^{if(z)} \). Then \( g \) is analytic and nonvanishing in \( |z| \leq 1 \), because the complex exponential function is never 0. Moreover \( g \) is continuous on \( |z| = 1 \). By Corollary 3, \( g \) attains its maximum and minimum on the boundary \( |z| = 1 \). But since \( f(z) \) is real-valued if \( |z| = 1 \), it follows that \( |g(z)| = |e^{if(z)}| = 1 \) for all \( |z| = 1 \). Consequently, by Corollary 3, \( g \) is constant inside the unit disk. Hence, for all \( z \) inside the unit disk, \( f(z) = c + 2k\pi \), where \( c \) is a constant and \( k \) is an integer. Now it is easy to see that since \( f \) is continuous, its values cannot hop around a discrete set. (A proof of this fact may be constructed in a way similar to the proof of Lemma 2, Sec. 5.7.) So \( f(z) = c + i2k\pi \) for all \( z \), and so \( f \) is constant.

29. Suppose that \( f \) is analytic with \( |f(z)| \leq 1 \) for all \( |z| \leq 1 \). By the maximum principle, we have that \( |f(z)| \leq 1 \) for all \( |z| \leq 1 \). Given \( z \) in the open unit disk, we can form a circle centered at \( z \) with radius \( 1 - |z| \). This circle is contained in the unit disk. Since \( f \) is analytic on \( C_{1-|z|}(z) \), we have by Cauchy’s generalized formula,

\[
f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_{1-|z|}(z)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.
\]

For \( \zeta \) on \( C_{1-|z|}(z) \), we have \( |\zeta - z| = 1 - |z| \). So, since \( |f(\zeta)| \leq 1 \), using the familiar inequality for path integrals, we obtain

\[
|f^{(n)}(z)| = \left| \frac{n!}{2\pi i} \int_{C_{1-|z|}(z)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \right| \leq \frac{n!}{2\pi} \frac{2\pi (1 - |z|)}{(1 - |z|)^{n+1}} = \frac{n!}{(1 - |z|)^n}.
\]
Solutions to Exercises 3.8

1. (a) The function \( u(x, y) = xy \) is harmonic on \( \Omega = \mathbb{C} \), because

\[
  u_{xx} = 0, \quad u_{yy} = 0, \text{ and so } \Delta u = 0.
\]

(b) To find the conjugate gradient of \( u \), apply Lemma 1: for \( z = x + iy \),

\[
  \phi(z) = u_x - iu_y = y - ix = iz.
\]

Clearly, \( \phi \) is analytic in \( \Omega \) as asserted by Lemma 1.

5. In this problem we are told that \( u \) is the real part of an analytic function \( f \); that is \( f = u + iv \)
on a region \( \Omega \). To find the conjugate gradient of \( u \), we apply Lemma 1 and find \( \phi(z) = u_x - iu_y \).

But \( u_y = -v_x \), by the Cauchy-Riemann equations, so \( \phi(z) = u_x + iu_y = f'(z) \), by (8), Sec. 2.4.

9. Suppose that \( f \) is analytic on a region \( \Omega \). We want to prove that \( |f| \) is harmonic on \( \Omega \) if and only if \( f \) is constant. Note that one implication is immediate: If \( f \) is constant, then \( |f| \) is constant and hence it is harmonic. It is the other implication that is more interesting. We will offer two proofs! The following results are needed in the proofs.

**Lemma 1** Suppose \( f \) is analytic in a region \( \Omega \) and \( |f| \) is harmonic in \( \Omega \). If \( f(z_0) = 0 \) for some \( z_0 \) in \( \Omega \) and \( B_R(z_0) \) is any disk centered at \( z_0 \) and contained in \( \Omega \), then \( f \) is identically 0 in \( B_R(z_0) \).

**Proof** Suppose that \( f(z_0) = 0 \). Let \( B_R(z_0) \) be an open disk contained in \( \Omega \). Since \( |f| \) is harmonic on \( B_R(z_0) \), applying the mean value property, we find that for any \( 0 < r < R \),

\[
  0 = |f(z_0)| = \frac{1}{2\pi} \int_{0}^{2\pi} |f(z_0 + re^{it})| \, dt.
\]

Since the integrand is continuous and nonnegative, the only way for the integral to equal zero is for \( |f(z_0 + re^{it})| \) to be identically zero for all \( t \) in \([0, 2\pi]\), which implies that \( |f| \) and hence \( f \) is zero on the circle of radius \( r \) and center at \( z_0 \). Since this is true for all \( 0 < r < R \), it follows that \( f = 0 \) on \( B_R(z_0) \).

**Lemma 1** Suppose \( f \) is analytic in a region \( \Omega \) and \( |f| \) is harmonic in \( \Omega \). Either \( |f| \) is identically 0 in \( \Omega \) or \( |f| \) does not vanish in \( \Omega \).

**Proof** Let \( \Omega_1 = \{ z : z \in \Omega, |f(z)| = 0 \} \) and \( \Omega_2 = \{ z : z \in \Omega, |f(z)| \neq 0 \} \). Then \( \Omega = \Omega_1 \cup \Omega_2 \) and \( \Omega_1 \cap \Omega_2 = \emptyset \). If we can show that both \( \Omega_1 \) and \( \Omega_2 \) are open, then by the connectedness of \( \Omega \), either \( \Omega_1 = \Omega \) or \( \Omega_2 = \Omega \). Since \( |f| \) is continuous, it follows that \( \Omega_2 \) is open, because \( \Omega_2 \) is the inverse image of the open set \((0, \infty)\). The fact that \( \Omega_1 \) is open follows from Lemma 1, because every \( z_0 \) in \( \Omega_1 \) is contained in a disk on which \( |f| \) vanishes, i.e., the disk is contained in \( \Omega_1 \). This proves Lemma 2.

We now return to the proof of Exercise 9.

**First Proof.** This one is more direct, but requires more computations. In view of Lemma 2, we can suppose that \( |f| \neq 0 \) in \( \Omega \), otherwise \( |f| \) is identically 0 in \( \Omega \).

Write \( f = u + iv \); then \( \phi = |f| = (u^2 + v^2)^{1/2} \). We are given that \( \phi_{xx} + \phi_{yy} = 0 \). Compute the partial derivatives explicitly:

\[
  \phi_x = (u^2 + v^2)^{-1/2}(uw_x + vw_y);
\]

\[
  \phi_{xx} = -(u^2 + v^2)^{-3/2}(uw_x + vw_y)^2 + (u^2 + v^2)^{-1/2}((u_x)^2 + uw_{xx} + (v_x)^2 + vv_{xx})
\]

\[
  = (u^2 + v^2)^{-3/2}((-uw_x + vw_y)^2 + (u^2 + v^2)((u_x)^2 + uw_{xx} + (v_x)^2 + vv_{xx})).
\]

Changing \( x \) to \( y \), we obtain

\[
  \phi_{yy} = (u^2 + v^2)^{-3/2}([uw_y - vw_y]^2 + (u^2 + v^2)(uw_{yy} + vv_{yy})].
\]
Since $\phi$ is harmonic, we obtain

\[
0 = \phi_{xx} + \phi_{yy}
\]

\[
= (u^2 + v^2)^{-3/2} \left[ (uv_x - vu_x)^2 + (u^2 + v^2)(uu_{xx} + vv_{xx}) \right]
\]

\[
+ (u^2 + v^2)^{-3/2} \left[ (uv_y - vu_y)^2 + (u^2 + v^2)(uu_{yy} + vv_{yy}) \right]
\]

\[
= (u^2 + v^2)^{-3/2} \left[ (uv_x - vu_x)^2 + (uv_y - vu_y)^2 + (u^2 + v^2)(uu_{xx} + vv_{xx} + uu_{yy} + vv_{yy}) \right]
\]

\[
= (u^2 + v^2)^{-3/2} \left[ (uv_x - vu_x)^2 + (uv_y - vu_y)^2 \right];
\]

\[
\Rightarrow (uv_x - vu_x)^2 = 0 \text{ and } (uv_y - vu_y)^2 = 0
\]

\[
\Rightarrow u^2(v_x)^2 + v^2(u_x)^2 - 2uvu_xv_x = 0 \text{ and } u^2(v_y)^2 + v^2(u_y)^2 - 2uvu_yv_y = 0.
\]

Since $f = u + iv$ is analytic, the Cauchy-Riemann equations hold: $u_x = v_y$ and $u_y = -v_x$ and so

\[
u^2(u_y)^2 + v^2(u_x)^2 - 2uvu_xv_x = 0 \text{ and } u^2(u_x)^2 + v^2(u_y)^2 - 2uvu_yv_y = 0.
\]

Adding the two equations, we get

\[
u^2[(u_x)^2 + (u_y)^2] + v^2[(u_x)^2 + (u_y)^2] = 0 \Rightarrow (u^2 + v^2)[(u_x)^2 + (u_y)^2] = 0
\]

By assumption, $|f| \neq 0$, hence $u^2 + v^2 \neq 0$, and so $(u_x)^2 + (u_y)^2 = 0$, implying that $u_x = 0$ and $u_y = 0$, which in turn implies that $u$ is identically constant in $\Omega$. Consequently, $v$ is identically constant in $\Omega$ and so $f$ is identically constant in $\Omega$.

For the second proof, we need two lemmas that are interesting in their own right.

**Lemma 3** If $u$ and $e^u$ are harmonic in $\Omega$ then $u$ is identically constant.

**Proof** Let $\phi = e^u$. Then $\phi_x = e^u u_x$ and $\phi_{xx} = e^u (u_x)^2 + e^u u_{xx}$. Similarly, $\phi_y = e^u u_y$ and $\phi_{yy} = e^u (u_y)^2 + e^u u_{yy}$. Using $\Delta \phi = 0$ and $\Delta u = 0$, we get

\[
e^u[(u_x)^2 + (u_y)^2] = 0 \Rightarrow (u_x)^2 + (u_y)^2 = 0 \Rightarrow u_x = 0 \text{ and } u_y = 0.
\]

Hence $u$ is constant.

The following lemma is extremely useful.

**Lemma 4** If $f$ is analytic an nonvanishing on a region $\Omega$, then $\ln|f(z)|$ is harmonic on $\Omega$.

**Proof** One way to prove this is to notice that if $f(z)$ is analytic and nonvanishing, then $\ln|f(z)|$ is the real part of an analytic branch of the logarithm of $f(z)$, and so it is harmonic (see Exercise 36, Sec. 3.6). Another way is to simply verify Laplace’s equation. Let $\phi = 2 \ln|f(z)| = \ln(u^2 + v^2)$ and check that $\Delta \phi = 0$. This will imply that $\phi$ is harmonic and hence that $\ln|f(z)|$ is harmonic. To simplify notation, let $\psi(x, y) = u^2 + v^2$ so $\phi = \ln \psi$. Let’s compute:

\[
\phi_x = \frac{\partial \psi}{\partial x};
\]

\[
\phi_{xx} = \frac{\partial^2 \psi}{\partial x^2};
\]

\[
\phi_{yy} = \frac{\partial^2 \psi}{\partial y^2};
\]

\[
\phi_{xx} + \phi_{yy} = \frac{(\psi_{xx} + \psi_{yy}) \psi - (\psi_x^2 + \psi_y^2)}{\psi^2}.
\]
But

\[(\psi_x + \psi_y)\psi = (2u_x^2 + 2uu_{xx} + 2v_x^2 + 2vv_{xx} + 2u_y^2 + 2uu_{yy} + 2v_y^2 + 2vv_{yy})\psi = 0\]

\[= (2u_x^2 + 2u_y^2 + 2u(u_{xx} + u_{yy}) + 2v(v_{xx} + v_{yy}))\psi = 2(u_x^2 + u_y^2 + v_x^2 + v_y^2)(u^2 + v^2) = 4(u_x^2 + u_y^2)(u^2 + v^2),\]

because by the Cauchy-Riemann equation \(u_x = v_y\) and \(u_y = -v_x\). Also

\[\psi_x^2 + \psi_y^2 = (2uu_x + 2v_x)^2 + (2uu_y + 2v_y)^2 = 0\text{ by Cauchy-Riemann eq.}\]

\[= 4((uu_x)^2 + (vv_x)^2 + (uu_y)^2 + (vv_y)^2) + 8uu_xvv_x + 8uu_yvv_y = 0\]

\[= 4(u_x^2 + u_y^2)(u^2 + v^2).\]

Thus \(\Delta \phi = 0\), as desired.

**Second Proof.** In view of Lemma 2, we can suppose that \(f\) is nonvanishing on \(\Omega\). By Lemma 3, \(\ln |f|\) is harmonic. Also, \(e^{\ln |f|} = |f|\) is harmonic, by assumption. By Lemma 3, \(\ln |f|\) is constant, but this implies that \(|f|\) is constant.

13. Apply (7) and you get the solution

\[u(r, \theta) = 1 - r \cos \theta + r^2 \sin 2\theta \quad (0 \leq r \leq 1, \text{ all } \theta).\]

17. To find the isotherms in Exercise 13, it is easier to use Cartesian coordinates. With \(x = r \cos \theta\) and \(y = r \sin \theta\), we have

\[u(r, \theta) = 1 - r \cos \theta + r^2 \sin 2\theta = 1 - r \cos \theta + 2r \sin \theta \cos \theta = 1 - x + 2xy.\]

If \(T\) is a real number, the isotherm corresponding to \(T\) is determined by the equation

\[1 - x + 2xy = T \Rightarrow y = \frac{T - 1 + x}{2x}.\]

The isotherm is the portion of this hyperbola that lies in the unit disk. Note that the values of the boundary temperature vary between \(-2\) and \(3\). So if \(T < -2\) or \(T > 3\), the isotherm corresponding to \(T\) is empty.

21. The solution of the Dirichlet problem on the unit disk with boundary data

\[f(\theta) = \begin{cases} 
100 & \text{if } 0 \leq \theta \leq \pi, \\
0 & \text{if } \pi < \theta < 2\pi 
\end{cases}\]

is given by the Poisson formula (12):

\[u(r, \theta) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\phi)}{1 - 2r \cos(\theta - \phi) + r^2} d\phi = \frac{1 - r^2}{2\pi} \int_0^{\pi} \frac{100}{1 - 2r \cos(\phi - \theta) + r^2} d\phi.
\]

\[= 100 \frac{1 - r^2}{2\pi} \int_0^{\pi-\theta} \frac{dt}{1 - 2r \cos t + r^2},\]
where in the last integral we have changed variables: \( t = \phi - \theta, \, dt = d\phi \). By Exercise 20, we have

\[
\int \frac{dt}{1 - 2r \cos t + r^2} = \frac{2}{1 - r^2} \tan^{-1} \left( \frac{1 + r}{1 - r} \tan \left( \frac{t}{2} \right) \right) + C.
\]

This formula gives an antiderivative up to a constant \( C \). To evaluate a definite integral using it, we need to choose \( C \) in such a way that the antiderivative becomes continuous. One way to get a continuous antiderivative is to use the following formula:

\[
F(t) = \begin{cases} 
\frac{2}{1 - r^2} \tan^{-1} \left( \frac{1 + r}{1 - r} \tan \left( \frac{t}{2} \right) \right) & \text{if } -\pi < t < \pi, \\
F(t - 2\pi) + \pi & \text{otherwise.}
\end{cases}
\]

(A good way to see that this formula yields a continuous function, just plot it.) This formula says that \( F(t + 2\pi) = F(t) + \pi \). So between two consecutive intervals of length \( 2\pi \), the values of \( F \) increase by \( \pi \). Note that \( F \) is also odd: \( F(-\theta) = -F(\theta) \). If \( 0 < \theta < 2\pi \), then \( 0 < 2\pi - \theta < 2\pi \), and \(-\pi < -\pi - \theta < \pi \). We have

\[
u(r, \theta) = \frac{100}{\pi} F(t) \bigg|_{-\theta}^{\pi-\theta} = \frac{100}{\pi} (F(\pi - \theta) - F(-\theta)) = \frac{100}{\pi} (F(\pi - \theta) + F(\theta)).
\]

To write down the formula explicitly, we distinguish two cases: If \( 0 < \theta < \pi \), then \( 0 < \pi - \theta < \pi \), and

\[
u(r, \theta) = \frac{100}{\pi} \left[ \tan^{-1} \left( \frac{1 + r}{1 - r} \tan \left( \frac{\pi}{2} - \frac{\theta}{2} \right) \right) + \tan^{-1} \left( \frac{1 + r}{1 - r} \tan \left( \frac{\theta}{2} \right) \right) \right]
\]

where we have used the identities \( \tan \left( \frac{\theta}{2} - \alpha \right) = \cot \alpha \) and \( \tan^{-1}(-\alpha) = -\tan^{-1} \alpha \). If \( \pi < \theta < 2\pi \), then \(-\pi < -\pi - \theta < 0 \), and \(-\pi < \theta - 2\pi < 0 \), \( F(\theta) = F(\theta - 2\pi) + \pi \). So

\[
u(r, \theta) = \frac{100}{\pi} \left[ \tan^{-1} \left( \frac{1 + r}{1 - r} \cot \left( \frac{\theta}{2} \right) \right) + \tan^{-1} \left( \frac{1 + r}{1 - r} \tan \left( \frac{\theta}{2} \right) \right) + \pi \right].
\]

It is useful at this point to check that \( u \) verifies the boundary conditions. If \( 0 < \theta < \pi \), then \( 0 < \theta < \frac{\pi}{2} \), hence \( \tan \frac{\theta}{2} \) and \( \cot \frac{\theta}{2} \) are both positive, and so

\[
\lim_{r \to 1} \frac{1 + r}{1 - r} \tan \left( \frac{\theta}{2} \right) = +\infty \quad \text{and} \quad \lim_{r \to 1} \frac{1 + r}{1 - r} \cot \left( \frac{\theta}{2} \right) = +\infty.
\]

Consequently, if \( 0 < \theta < \pi \),

\[
\lim_{r \to 1} \nu(r, \theta) = \frac{100}{\pi} \left( \tan^{-1}(+) + \tan^{-1}(+) \right) = \frac{100}{\pi} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = 100,
\]

which is the correct boundary value for \( u \). If \( \pi < \theta < 2\pi \), then by a similar reasoning, we find that \( \lim_{r \to 1} u(r, \theta) = 0 \).

25. All the stated formulas are different ways to express a convolution of \( f \) with the Poisson kernel (see Sec. 7.5 for details about convolutions). Let

\[
P_r(\theta) = P(r, \theta) = \frac{R^2 - r^2}{R^2 - 2r R \cos \theta + r^2}.
\]

Then according to (12), the Poisson integral formula is

\[
\frac{1}{2\pi} \int_0^{2\pi} f(\phi) P_r(\theta - \phi) \, d\phi.
\]
This is precisely the definition of the convolution of the two $2\pi$-periodic functions $f(\phi)$ and $P_\tau(\phi)$. Other equivalent are derived by using properties of integrals of periodic functions. The stated formulas are special cases of formulas given in Sec. 7.5. In particular, see (13) and Corollary 2 of Sec. 7.5.
Solutions to Exercises 3.9

1. Consider calculus of a real variable and take the function \( f(x) = x^2 \sin(1/x) \) if \( x \neq 0 \) and \( f(0) = 0 \).

(a) For \( x \neq 0 \),
\[
f'(x) = 2x \sin \frac{1}{x} - x^2 \frac{1}{x^2} \cos \frac{1}{x} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.
\]

(b) If \( x = 0 \), then
\[
f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0,
\]
because \( h \to 0 \) and \( \frac{1}{h} \) is bounded between \(-1\) and \(1\), so the limit is 0 by the squeeze theorem.

(c) Even though \( f'(x) \) exists for all \( x \), it is not continuous at 0. First note that \( \lim_{x \to 0} \cos \frac{1}{x} \) does not exist. To see this, take \( x = x_n = \frac{2}{(2n+1)\pi} \), then
\[
\cos \frac{1}{x_n} = \cos \frac{(2n + 1)\pi}{2} = (-1)^n,
\]
and clearly the limit as \( n \to \infty \) does not exist. Now if \( \lim_{x \to 0} f'(x) \) does exist, then because \( f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \) and \( \lim_{x \to 0} 2x \sin \frac{1}{x} = 0 \) by the squeeze theorem, this would imply that \( \lim_{x \to 0} \cos \frac{1}{x} \) exists, which is a contradiction. So \( \lim_{x \to 0} f'(x) \) does not exist and hence \( f' \) is not continuous at \( x = 0 \).