Math 340 — Exam 2 solutions

1. $fgh = (1\,3\,5\,6)$. Then $o(fgh) = \text{lcm}(o(1\,3), o(5\,6)) = \text{lcm}(2, 2) = 2$. $f^{-1} = (2\,5\,1\,3)$ (write the cycle $f$ in the opposite order). $f$ is a 4-cycle, so $f$ can be written as a product of 3 two-cycles. Thus $f$ is odd.

2. $I \subseteq R$ is maximal if $I \neq R$ and whenever $I \subseteq J \subseteq R$ then either $I = J$ or $J = R$.

3. Let $N = \varphi^{-1}(N')$. Suppose that $n \in N$ (i.e., $\varphi(n) \in N'$) and $g \in G$. Then we want to show that $g^{-1}ng \in N$, i.e., that $\varphi(g^{-1}ng) \in N'$. Since $\varphi$ is a homomorphism we have $\varphi(g^{-1}ng) = \varphi(g^{-1})\varphi(n)\varphi(g) = \varphi(g)^{-1}\varphi(n)\varphi(g) \in N'$. The last inclusion follows since $N' \triangleleft G'$.

4. We need to show that for all $a, b \in G$, $NaNb = NbNa$. Let $a, b \in G$. By the hypothesis, $aba^{-1}b^{-1} \in N$, so $Naba^{-1}b^{-1} = N$. Thus in $G/N$, $NbNa = Nba = (Naba^{-1}b^{-1})ba = Naba^{-1}ea = Nabe = Nab$, as desired.

5. We define $\varphi : G \rightarrow G/M \times G/N$ by $\varphi(g) = (Mg, Ng)$. Then for any $g, h \in G$, $\varphi(gh) = (Mgh, Ng) = (MgMh, NgNh) = (Mg, Ng)(Mh, Nh) = \varphi(g)\varphi(h)$. Thus $\varphi$ is a homomorphism.

Since the identity in $G/M \times G/N$ is $(M, N)$ we see that $g \in \ker \varphi$ if and only if $(Mg, Ng) = (M, N)$, which happens if and only if $Mg = M$ and $Ng = N$. Thus $g \in \ker \varphi$ if and only if $g \in M$ and $g \in N$. Hence $\ker \varphi = M \cap N$.

Since whenever we have a group homomorphism the kernel is a normal subgroup we have just proved the theorem:

Let $M$ and $N$ be normal subgroups of $G$. Then $M \cap N$ is a normal subgroup of $G$. (Of course, this can be proved directly from the definition of normal too.)

6. Let $A, B \in GL_n(\mathbb{R})$. Using properties of determinants and absolute value we get $\varphi(AB) = |\det(AB)| = |\det(A)\det(B)| = |\det(A)| \cdot |\det(B)| = \varphi(A)\varphi(B)$. Thus $\varphi$ is a homomorphism.

Let $x \in \mathbb{R}_+$. If $A$ is the $n \times n$ matrix with an $x$ in the $(1, 1)$ position, 1’s along the rest of the main diagonal and 0’s elsewhere, then $\varphi(A) = |\det(A)| = |x| = x$. Thus $\varphi$ is onto.

Comment: Note that I gave a specific element of $GL_n(\mathbb{R})$ which works (once I had picked an arbitrary element of $\mathbb{R}_+$). Just restating the definition of onto isn’t sufficient.

In $(\mathbb{R}_+, \cdot)$, the identity element is 1. Thus $\ker \varphi = \{ A \in GL_n(\mathbb{R}) \mid \det(A) = \pm 1 \}$. By the First Homomorphism Theorem, $GL_n(\mathbb{R})/\ker \varphi \cong (\mathbb{R}_+, \cdot)$. 

7. Let \( A_1, A_2 \in I \). Then \( A_1 = \begin{bmatrix} a_1 & 0 \\ c_1 & 0 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} a_2 & 0 \\ c_2 & 0 \end{bmatrix} \) for some \( a_1, a_2, c_1, c_2 \in \mathbb{R} \). We have \( A_1 + A_2 = \begin{bmatrix} a_1 + a_2 & 0 \\ c_1 + c_2 & 0 \end{bmatrix} \in I \), and \(-A_1 = \begin{bmatrix} -a_1 & 0 \\ -c_1 & 0 \end{bmatrix} \in I \). This shows that \((I, +)\) is an additive subgroup of \((M_2(\mathbb{R}), +)\).

To see that \( I \) is a left ideal, let \( X = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \) be any element of the ring \( M_2(\mathbb{R}) \). Then for \( A_1 \in I \) as above, \( XA_1 = \begin{bmatrix} a_1 x + c_1 y & 0 \\ a_1 z + c_1 w & 0 \end{bmatrix} \in I \), showing \( I \) is a left ideal.

To see that \( I \) is not a right ideal, it suffices to find just one instance of an \( A \in I \) and an \( X \) in \( M_2(\mathbb{R}) \) such that \( AX \notin I \). We can take \( A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( X = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \). Then \( AX = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \notin I \). Hence \( I \) is not a right ideal.

8. Let \( \varphi : R_1 \rightarrow R_2 \) be a ring homomorphism.

Comment: There is a “slick” proof that works if you accept that we already know lots of group theory. Omitting the multiplication, \( \varphi : (R_1, +) \rightarrow (R_2, +) \) is a group homomorphism of abelian groups, and \( \ker(\varphi) \) as a ring homomorphism is the same as \( \ker(\varphi) \) as just a group homomorphism. We already know the result for groups, so we are done. However, I’ll give a direct proof too.

\((\Rightarrow)\) Assume that \( \varphi \) is 1-1. We know that, since \( \varphi \) is a homomorphism, \( \varphi(0_{R_1}) = 0_{R_2} \). Suppose that \( a \in \ker \varphi \). Then \( \varphi(a) = 0_{R_2} = \varphi(0_{R_1}) \). Since \( \varphi \) is 1-1, \( a = 0_{R_1} \). Thus \( \ker \varphi = \{0\} \).

\((\Leftarrow)\) Assume that \( \ker \varphi = \{0\} \). Let \( a, b \in R_1 \) such that \( \varphi(a) = \varphi(b) \). Then \( \varphi(a - b) = \varphi(a + (-b)) = \varphi(a) + \varphi(-b) = \varphi(a) - \varphi(b) = 0_{R_2} \). Hence \( a - b \in \ker(\varphi) \), so \( a - b = 0_{R_1} \), or \( a = b + 0 = b \). Thus \( \varphi \) is 1-1.

9. Since \( F \) is a field, the only ideals of \( F \) are \( F \) and \( \{0\} \). We know that \( \ker \varphi \) is an ideal of \( F \), and since \( \varphi(1) = 1 \neq 0 \), \( \ker \varphi \neq F \). Thus \( \ker \varphi = \{0\} \), and hence by problem 8, \( \varphi \) is 1-1.