Theorem 1. Let $G$ be a group and $\varphi : G \rightarrow G$ the function $\varphi(a) = a^{-1}$. Then $\varphi$ is a homomorphism if and only if $G$ is abelian. In the case that $G$ is abelian, $\varphi$ is an automorphism.

Proof. Suppose first that $G$ is not abelian. Then there exist $a, b \in G$ such that $ab \neq ba$. Taking inverses shows that $b^{-1}a^{-1} \neq a^{-1}b^{-1}$. For these $a, b \in G$ we get $\varphi(ab) = (ab)^{-1} = b^{-1}a^{-1} \neq a^{-1}b^{-1} = \varphi(a)\varphi(b)$, hence $\varphi$ is not a homomorphism when $G$ is non-abelian.

Assume now that $G$ is abelian. Then $\varphi(ab) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = \varphi(a)\varphi(b)$, where we are using that all elements in $G$ commute. If $a \in \ker(\varphi)$ then $\varphi(a) = a^{-1} = e$, hence $a = (a^{-1})^{-1} = e$. Thus $\ker \varphi = \{e\}$. [It is not enough to show that $e \in \ker(\varphi)$ to be able to conclude that it is the only thing in $\ker(\varphi)$.] By class Theorem 23, $\varphi$ is 1-1. Also, $\varphi$ is onto since for any $a \in G$, $a = \varphi(a^{-1})$. [You are not using here that every element has an inverse, you are using that every element is an inverse of something.]

□

Theorem 2. For any group $G$, the center $Z(G)$ is normal in $G$.

Proof. Since the elements of $Z(G)$ commute with every element in $G$ we have that if $z \in Z(G)$ and $g \in G$ then $g^{-1}zg = zg^{-1} = ze = z \in Z(G)$, i.e., for all $g \in G$, $g^{-1}Z(G)g \subseteq Z(G)$. Hence $Z(G) \triangleleft G$.

□

Theorem 3. Let $\varphi : G \rightarrow G'$ be a surjective homomorphism. If $G$ is abelian then $G'$ is abelian.

Proof. Let $a', b' \in G'$. Since $\varphi$ is onto, there exist $a, b \in G$ such that $a' = \varphi(a)$ and $b' = \varphi(b)$. Then $a'b' = \varphi(a)\varphi(b) = \varphi(ab) = \varphi(ba) = \varphi(b)\varphi(a) = b'a'$. Thus $G'$ is abelian.

□

Theorem 4. Let $G$ be a group and $H \subseteq G$ a fixed subgroup of $G$. Set $N = \cap_{a \in G} a^{-1}Ha$. Then $N \triangleleft G$.

Remark: Many people did not understand what $N$ is. $N$ is not, in general, equal to $a^{-1}Ha$ for any $a$. $N$ is contained in each $a^{-1}Ha$. In particular $N \subseteq e^{-1}He = H$, so $N$ is a subgroup of $H$. In fact, $N$ is the largest subgroup contained in $H$ and normal in $G$. Note that $N = H$ if and only if $H \triangleleft G$ [can you show this?].
Proof. Let $n \in N$ and $g \in G$. We must show that $g^{-1}ng \in N$, i.e., for any $a \in G$ we must show that $g^{-1}ng \in a^{-1}Ha$ (showing that $g^{-1}ng \in \cap_{a \in G}a^{-1}Ha = N$). Consider $c = ag^{-1}$. Since $n \in c^{-1}Hc$ we have $n = c^{-1}hc = ga^{-1}hag^{-1}$ for some $h \in H$. Then $g^{-1}ng = g^{-1}(ga^{-1}hag^{-1})g = a^{-1}ha$. Hence $g^{-1}ng \in a^{-1}Ha$. Thus $N \triangleleft G$. \hfill \Box

Theorem 5. Let $S$ be any set having more than two elements. For a fixed $s \in S$, define $H(s) = \{ f \in A(S) | f(s) = s \}$. Then $H(s)$ is not a normal subgroup of $A(S)$.

Proof. To show that $H(s)$ is not normal in $A(S)$ we need only produce an $f \in H(s)$ and a $g \in A(S)$ such that $g^{-1}fg \notin H(s)$, i.e., $g^{-1}fg(s) \neq s$.

Let $s, t, u \in S$ be three distinct elements (available by the hypothesis). Define $f$ by $f(t) = u$, $f(u) = t$, and $f(x) = x$ for $x \in S - \{ t, u \}$. Clearly $f \in H(s)$. Define $g$ by $g(s) = t$, $g(t) = s$ and $g(x) = x$ for $x \in S - \{ t, s \}$. Then $g^{-1}fg(s) = g^{-1}f(t) = g^{-1}(u) = u \neq s$. \hfill \Box

Theorem 6. Let $G$ be a cyclic group and $N$ a subgroup. Then $G/N$ is a cyclic group.

Proof. Let $G = \langle a \rangle$. Since $G$ is cyclic, it is abelian, and hence every subgroup is normal. Thus $G/N$ is a group. For any coset $Nb$ we have $Nb = Na^i$ for some $i \in \mathbb{Z}$, since $b = a^i$ for some $i \in \mathbb{Z}$. Thus $G/N = \langle Na \rangle$ is cyclic. \hfill \Box

Theorem 7. Suppose that $G$ is a group and $G/Z(G)$ is cyclic. Then $G$ is abelian.

Proof. We already know that $Z(G) \triangleleft G$, so $G/Z(G)$ is a group. By hypothesis, $G/Z(G)$ is cyclic, so there is an $a \in G$ such that $G/Z(G) = \langle Z(G)a \rangle$, i.e., every element of $G/Z(G)$ is of the form $Z(G)a^i$ for some $i \in \mathbb{Z}$. Let $c, d \in G$. Each element is in some coset of $Z(G)$. Say that $c \in Z(G)a^i$ and $d \in Z(G)a^j$ for some $i, j \in \mathbb{Z}$. Thus $c = z_1a^i$ and $d = z_2a^j$ for some $z_1, z_2 \in Z(G)$.

Thus $cd = (z_1a^i)(z_2a^j) = z_1z_2a^{i+j} = z_2z_1a^ia^j = z_2a^jz_1a^i = dc$, where we have used that $z_1$ and $z_2$ commute with every element of $G$ and we have used properties of exponents. Since $c$ and $d$ were arbitrary elements of $G$, we have shown that $G$ is abelian. \hfill \Box