Solutions to Math 331 — Quiz 1

Sept. 5, 2003

1a). (8 points) Put the matrix
\[
\begin{bmatrix}
1 & 3 & 1 & 5 \\
0 & 2 & -2 & 4 \\
3 & 1 & 11 & -1
\end{bmatrix}
\]
into reduced row-echelon form.

\[
\begin{bmatrix}
1 & 3 & 1 & 5 \\
0 & 2 & -2 & 4 \\
3 & 1 & 11 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 1 & 5 \\
0 & 2 & -2 & 4 \\
0 & -8 & 8 & -16
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 1 & 5 \\
0 & 1 & -1 & 2 \\
0 & -8 & 8 & -16
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 4 & -1 \\
0 & 1 & -1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

1b). (4 points) Use your work from (1a) to give all the solutions to the system of equations
\[
\begin{align*}
x_1 + 3x_2 + x_3 &= 5 \\
2x_2 - 2x_3 &= 4 \\
3x_1 + x_2 + 11x_3 &= -1
\end{align*}
\]
The matrix in part (1a) is the augmented matrix for this system of equations, so the work done above shows that \(x_3\) is the free variable, and \(x_1 = -4x_3 - 1\), and \(x_2 = x_3 + 2\). We should really write the solutions as column vectors. Here are two ways to write the solution set:

\[
\left\{ \begin{bmatrix} -4\alpha - 1 \\ \alpha + 2 \\ \alpha \end{bmatrix} \mid \alpha \in \mathbb{R} \right\} \quad \text{or} \quad \left\{ \alpha \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \mid \alpha \in \mathbb{R} \right\}
\]

2) (4 points) Find \(2 \times 2\) matrices \(A\) and \(B\) that both are not the zero matrix for which \(AB = 0\).

There are infinitely many correct answers to this problem. The main point is that unlike with numbers, two non-zero matrices can multiply to the zero matrix. One solution is \(A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\) and \(B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\).

3) (4 points) Let’s prove that the inverse of a square matrix \(A\) is unique (if \(A\) is invertible). Suppose that both \(B\) and \(C\) are inverses of \(A\). This means that \(AB = I = BA\) and \(AC = I = CA\). Compute the product \(BAC\) by parenthesizing in two different ways to show that \(B = C\).

\[(BA)C = IC = C \quad \text{and} \quad B(AC) = BI = B.\] Since both products are \(BAC\) we see that \(C = B\). It is not ok at any point to switch the order of the products, since matrix multiplication does not commute, in general.