1) (15 points) $L_1$ is the line given parametrically by $x = 3t + 1$, $y = -2t + 4$, and $z = t - 3$; $L_2$ is the line given by the symmetric equations $\frac{x+4}{2} = \frac{y-10}{4} = \frac{z+1}{8}$.

a) Give parametric equations for $L_2$.

b) These two lines intersect (you do not need to show this). Find an equation for the plane defined by the two lines.

a) (5 pts) Set each part equal to $t$ to get $x = 2t - 4$, $y = 4t + 10$, and $z = 8t - 1$.

b) (10 pts) $L_1$ has direction $\mathbf{v}_1 = (3, -2, 1)$, and $L_2$ has direction $\mathbf{v}_2 = (2, 4, 8)$. A normal vector for the plane is perpendicular to both, so we may take $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (-20, -22, 16)$. Any point on either line is a point on the plane, so, from $L_1$, $(1, 4 - 3)$ is on the plane. One equation for the plane is $-20(x - 1) - 22(y - 4) + 16(z + 3) = 0$.

2) (15 points) A curve $C$ is given parametrically by the vector function

$$\mathbf{r}(t) = \ln(t) \mathbf{i} + (t^2 + 5) \mathbf{j} + \frac{1}{t} \mathbf{k}.$$  

a) Find parametric equations for the tangent line to $C$ when $t = 2$.

b) What are the velocity, speed, and acceleration when $t = 1$ (label your answers clearly, so I know (that you know) which is which).

a) (6 pts) The velocity function is $\mathbf{v}(t) = \frac{1}{t} \mathbf{i} + 2t \mathbf{j} - \frac{1}{t^2} \mathbf{k}$. At $t = 2$, $\mathbf{v}(2) = \langle 1/2, 4, -1/4 \rangle$, which is the direction for the tangent line. Since $\mathbf{r}(2) = \langle \ln(2), 9, 1/2 \rangle$, equations for the tangent line are $x = t/2 + \ln(2)$, $y = 4t + 9$, and $z = -t/4 + 1/2$.

b) (9 pts) $\mathbf{a}(t) = -\frac{1}{t^2} \mathbf{i} + 2 \mathbf{j} + \frac{2}{t^3} \mathbf{k}$. The velocity (using $\mathbf{v}(t)$ from (a)) is $\mathbf{v}(1) = \langle 1, 2, -1 \rangle$. The speed is $|\mathbf{v}(1)| = |\langle 1, 2, -1 \rangle| = \sqrt{6}$. The acceleration is $\mathbf{a}(1) = \langle -1, 2, 2 \rangle$.

3) (15 points) Suppose that for the vector function $\mathbf{r}(t)$, you have calculated that $|\mathbf{v}(t)| = t^2 e^{t^3}$.

a) Find the length of the portion of the curve travelled between $t = 0$ and $t = 2$.

b) Find the tangential component of acceleration $a_T$ as a function of $t$.

a) (9 pts) The arc length is $L = \int_0^2 t^2 e^{t^3} \, dt = \frac{1}{3} e^{t^3} \bigg|_0^2 = \frac{1}{3} (e^8 - e^0) = \frac{1}{3} (e^8 - 1)$.

b) (6 pts) $a_T(t) = \frac{d}{dt} |\mathbf{v}(t)| = \frac{d}{dt} \left( t^2 e^{t^3} \right) = 2te^{t^3} + t^2 \cdot 3t^2 e^{t^3} = t(2 + 3t^3)e^{t^3}$.

4) (25 points) A particle follows a curve $C$ given by a vector function $\mathbf{r}(t)$. You observe that at time $t = 10$, the velocity and acceleration are $\mathbf{v}(10) = \langle 2, 2, -1 \rangle$ and $\mathbf{a}(10) = \langle -5, 1, -2 \rangle$.

a) Find $a_T \mathbf{T}$ when $t = 10$ by using projection (the answer does involve “modest” fractions).

b) What is $a_T$ at $t = 10$?

c) Find $a_N \mathbf{N}$, $a_N$, and $\kappa$ when $t = 10$.

d) Verify that at time $t = 10$, $a_T \mathbf{T} \perp a_N \mathbf{N}$.

e) At time $t = 10$, is the particle speeding up, slowing down, or neither? Justify briefly.

All the answers are computed for $t = 10$, so I’ll just write, for example, $a_T$ rather then $a_T(10)$.

a) (6 pts) $a_T \mathbf{T} = \text{Proj}_\mathbf{v} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = \frac{-6}{9} \langle 2, 2, -1 \rangle = -\frac{2}{3} \langle 2, 2, -1 \rangle$.

b) (3 pts) $a_T = \text{Comp}_\mathbf{v} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{v}|} = -\frac{6}{3} = -2$. 


c) \( (9 \text{ pts}) \) \( a_N \mathbf{N} = \mathbf{a} - a_T \mathbf{T} = (-5, 1, -2) - (-\frac{2}{3}) (2, 2, -1) = \frac{1}{3} (-11, 7, -8) \). Then \( a_N = \sqrt{|\mathbf{a}|^2 - (a_T)^2} = \sqrt{30 - (-\frac{2}{3})^2} = \sqrt{26} \), and \( \kappa = a_N/|v|^2 = \sqrt{26}/9 \).

d) (4 pts) We need to check that \( a_T \mathbf{T} \cdot a_N \mathbf{N} = 0: \)
\[
-\frac{2}{3} (2, 2, -1) \cdot \frac{1}{3} (-11, 7, -8) = \left(-\frac{2}{3}\right) \left(\frac{1}{3}\right) (2, 2, -1) \cdot (-11, 7, -8) = (-2/9)(-22 + 14 + 8) = 0.
\]
e) (3 pts) The change in the speed is given by \( a_T \). At time \( t = 2 \), we’ve computed that \( a_T = -2 \) is negative. So the particle is slowing down.

5) (15 points) For the function \( f(x, y, z) = yx^3 + \ln(x) + e^{yz} + \pi \):

a) Find \( f_x, f_y, \) and \( f_z \).

b) Find \( f_{xx} \) and \( f_{yz} \).

a) (9 pts) \( f_x = 3yx^2 + \frac{1}{x} + 0 + 0 = 3yx^2 + \frac{1}{x} \). \( f_y = x^3 + zye^{yz} + 0 = x^3 + zye^{yz} \). \( f_z = 0 + 0 + ye^{yz} = ye^{yz} \).

b) (6 pts) \( f_{xx} = \frac{\partial}{\partial x} \left(3yx^2 + \frac{1}{x}\right) = 6yx - \frac{1}{x^2} \). \( f_{yz} = \frac{\partial}{\partial z} (x^3 + zye^{yz}) = 0 + 1 \cdot e^{yz} + zye^{yz} = (1 + zy)e^{yz} \).

6) (15 points) For the function \( z = y^2 \sin(x) \):

a) Find an equation for the tangent plane to the surface at \((a, b) = (\pi/6, 3)\).

b) Find an approximate value for \((3.05)^2 \sin(3\pi/20)\) (hint: \(3\pi/20 = (9/10)(\pi/6)\)) (simplify where it is most reasonable to do so, but you may leave your answer as a sum).

a) (10 pts) \( f_x = y^2 \cos(x) \) and \( f_y = 2y \sin(x) \), so we get:
\( f(\pi/6, 3) = 9 \sin(\pi/6) = 9/2 \), \( f_x(\pi/6, 3) = 9\sqrt{3}/2 \), and \( f_y(\pi/6, 3) = 6(1/2) = 3 \). Thus the equation for the tangent plane is \( z = f_x(\pi/6, 3)(x - \pi/6) + f_y(\pi/6, 3)(y - 3) + f(\pi/6, 3) \),
or \( z = \frac{9\sqrt{3}}{2}(x - \pi/6) + 3(y - 3) + 9/2 \).

b) (5 pts) \((3.05)^2 \sin(3\pi/20) = f(3\pi/20, 3.05) \approx \frac{9\sqrt{3}}{2}(9/10)(\pi/6) - \pi/6) + 3(3.05 - 3) + 9/2 \)
\[
= \frac{9\sqrt{3}}{2} \left(-\frac{1}{10}\right) \frac{\pi}{6} + 3(0.05) + 9/2 = -\frac{\sqrt{3}\pi}{40} + \frac{3}{20} + \frac{9}{2}.
\]

**Extra Credit** (10 points) Let \( \mathbf{v} \) and \( \mathbf{w} \) be two vectors, and assume that \( \mathbf{w} \neq \mathbf{0} \). Show that it is always true that \((\mathbf{v} - \text{Proj}_\mathbf{w} \mathbf{v}) \perp \mathbf{w} \). (A picture is not sufficient here. Your proof should work in the case that the vectors are either 2D or 3D (or any other D.).)

Remember that if \( c \) is a scalar (i.e., a number), and \( \mathbf{a} \) and \( \mathbf{b} \) are vectors, then \((c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) \). We will use this in the case that \( c \) is the scalar \( \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \).

To check for perpendicularity, we compute the dot product:
\[(\mathbf{v} - \text{Proj}_\mathbf{w} \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \cdot \mathbf{w}) \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} - \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} (\mathbf{w} \cdot \mathbf{w}) = \mathbf{v} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w} = 0.
\]
Notice that I didn’t need to specify whether these were 2D or 3D vectors, and I never needed to list components of the vectors!