Analytical Solution of Melting in a Subcooled Semi–infinite Solid with Boundary Conditions of the Second Kind

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ABSTRACT

The melting problem in a semi–infinite region with constant heat flux boundary condition was solved by semi–exact method and integral approximate method. Effect of subcooling on the transient solid–liquid interface location and surface temperature were also discussed in this paper.

INTRODUCTION

Solid–liquid phase change phenomena exist widely in nature and industrial processes such as freezing of water and melting of ice, thermal energy storage, casting and metallurgical process, blood preservation under very low temperature etc. The difficulties involved in solving this kind of problem are the location of the solid–liquid interface (where latent heat released or absorbed) varies with time, and the interface location can be known only when the problem has been solved, that is the so–called "moving boundary problem".

Many investigations have been done concerning the one–dimensional melting or solidification in a semi–infinite body. For the boundary conditions of first kind, either one phase or two phase problems have been solved with various methods [1]. For the boundary conditions of second kind, the solutions of one phase problem have been published [2–5], but the solutions of two–phase problem have received very limited attention. Boley [6] obtained series solutions which are valid for short times by using "embedding technique".

The melting problem in a subcooled semi–infinite body with the boundary conditions of second kind is a two phase problem, its distinctive feature from the melting with boundary conditions of first kind is that, owing to the existence of subcooling, melting does not occur simultaneously with the beginning of heating, Only after a definite period of heating, the wall temperature reaches the melting point, then melting starts. So, there will be an initial temperature non–uniformity before melting begins. As a matter of fact, melting problem with subcooling and with boundary conditions of second kind is equivalent to a melting problem with the same boundary condiions and an initial temperature distribution but without

subcooling. Integration method was used to obtain the duration of preheating and the temperature profile of the solid while melting began. Then both semi–exact and approximate solutions of the melting processes were obtained and compared. The effect of subcooling on the interface location was discussed and the numerical examples of the melting of n–heptadecane and n–eicosane were calculated.

DURATION OF PREHEATING

At the beginning of heating, no melting occurs, the problem is actually a pure conduction problem with boundary conditions of second kind. Its mathematical description is as follows:

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{a_2}\frac{\partial T_2}{\partial t}, \quad 0 < x < \infty, \quad 0 < t < t_m \tag{1}$$

$$\frac{\partial T_2}{\partial x} = -\frac{1}{k_2} q, \quad x = 0, \quad 0 < t < t_m \tag{2}$$

$$T_2(x,t) \to T_i, \quad x \to \infty, \quad 0 < t < t_m \tag{3}$$

$$T_2(x,t) = T_i, \quad t = 0, \quad 0 < x < \infty \tag{4}$$

where t_m is the duration of preheating. This problem could be solved by integration method [1], assume the temperature profile as a second–degree polynomial and one can obtain the temperature profile as

$$T_2(x,t) = T_i + \frac{q\delta}{2k_2}(1 - \frac{x}{\delta})^2 \tag{5}$$

where δ is the pentration depth, that is the depth to which the thermal effects are felt. It can be obtained with the integration method

$$\delta = \sqrt{6a_2 t} \tag{6}$$

of course, the highest temperature exists at the surface (x = 0) of the semi–infinite body and can be expressed as

$$T_s(t) = T_i + \frac{q\delta}{2k_2} \tag{7}$$

Melting occurs when the surface temperature reaches its melting point T_m, and the corresponding penetration depth is

$$\delta_m = \frac{2k_2(T_m - T_i)}{q} \tag{8}$$

Then, the duration of pre–heating can be calculated from eq.(6)

$$t_m = \frac{2k_2^2(T_m - T_i)^2}{3a_2 q^2} \tag{9}$$

and the temperature profile will be

$$T_2(x,t_m) = T_f + (T_m - T_f)(1 - \frac{x}{\delta_m})^2 \tag{10}$$

SEMI-EXACT SOLUTION OF THE MELTING PROBLEM

Mathematically the melting process can be expressed as follows:
liquid region:

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{a_1}\frac{\partial T_1}{\partial t}, \quad 0 < x < s(t), \quad t > t_m \tag{11}$$

$$\frac{\partial T_1}{\partial x} = -\frac{1}{k_1} q, \quad x = 0, \quad t > t_m \tag{12}$$

solid region:

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{a_2}\frac{\partial T_2}{\partial t}, \quad s(t) < x < \infty, \quad t > t_m \tag{13}$$

$$T_2(x,t) \to T_f, \quad x \to \infty, \quad t > t_m \tag{14}$$

$$T_2(x,t) = T_f + (T_m - T_f)(1 - \frac{x}{\delta_m})^2, \quad t = t_m \tag{15}$$

solid–liquid interface:

$$T_1(x,t) = T_2(x,t) = T_m, \quad x = s(t), \quad t > t_m \tag{16}$$

$$k_2\frac{\partial T_2}{\partial x} - k_1\frac{\partial T_1}{\partial x} = \rho_2 L\frac{ds}{dt}, \quad x = s(t), \quad t > t_m \tag{17}$$

provided that the physical properties of solid and liquid remain unchanged when temperature varies and the natural convection in the liquid phase is negligibly weak. Defining the following non-dimensional variables:

$$\left.\begin{aligned} X &= \frac{q}{a_1\rho_1 L}x, \qquad \tau = \frac{q^2}{a_1\rho_1^2 L^2}t \\ S &= \frac{q}{a_1\rho_1 L}s, \qquad \triangle m = \frac{q}{a_1\rho_1 L}\delta_m \\ \theta_1 &= \frac{C_{p_1}}{L}[T_1(x,t) - T_m], \qquad K_a = \frac{a_1}{a_2} \\ \theta_2 &= \frac{C_{p_2}}{L}[T_2(x,t) - T_m], \qquad K_\rho = \frac{\rho_1}{\rho_2} \end{aligned}\right\} \tag{18}$$

Then, eqs. (11)–(17) can be made dimensionless

$$\frac{\partial^2 \theta_1}{\partial X^2} = \frac{\partial \theta_1}{\partial \tau} \quad , \quad 0 < X < S(\tau), \quad \tau > \tau_m \tag{19}$$

$$\frac{\partial \theta_1}{\partial X} = -1, \quad X = 0, \quad \tau > \tau_m \tag{20}$$

$$\frac{\partial^2 \theta_2}{\partial X^2} = K_a \frac{\partial \theta_2}{\partial \tau} \quad , \quad S(\tau) < X < \infty, \quad \tau > \tau_m \tag{21}$$

$$\theta_2(X,\tau) \to -S_c, \quad X \to \infty, \quad \tau > \tau_m \tag{22}$$

$$\theta_2(X,\tau) = S_c[(1 - \frac{X}{\Delta_m})^2 - 1], \quad \tau = \tau_m \tag{23}$$

$$\theta_1(X,\tau) = \theta_2(X,\tau) = 0, \quad X = S(\tau), \quad \tau > \tau_m \tag{24}$$

$$\frac{\partial \theta_2}{\partial X} - K_\rho K_a \frac{\partial \theta_1}{\partial X} = K_a \frac{dS}{d\tau}, \quad X = S(\tau), \quad \tau > \tau_m \tag{25}$$

where S_c is a subcooling parameter and defined as

$$S_c = \frac{C_{p_2}(T_m - T_i)}{L} \tag{26}$$

obviously, S_c has significant effect on the process.

One could assume a temperature profile in the liquid–phase region in the similar fashion as the exact solution of conduction in a semi–infinite body with boundary conditions of second kind [4]:

$$\theta_1(X,\tau) = A + 2\sqrt{\tau}\, ierfc(\frac{X}{2\sqrt{\tau}}) \tag{27}$$

where A is an unknown quantity which can be then computed from the boundary condition at the solid–liquid interface:

$$A = -2\sqrt{\tau}\, ierfc(\frac{S}{2\sqrt{\tau}}) \tag{28}$$

then, the temperature profile in the liquid–phase has the form

$$\theta_1(X,\tau) = 2\sqrt{\tau}\,[ierfc(\frac{X}{2\sqrt{\tau}})] - ierfc(\frac{S}{2\sqrt{\tau}})] \tag{29}$$

ierfc is differential complementary error function whose definition is

$$ierfc(X) = \frac{1}{\sqrt{\pi}} e^{-x^2} - X erfc(X) \tag{30}$$

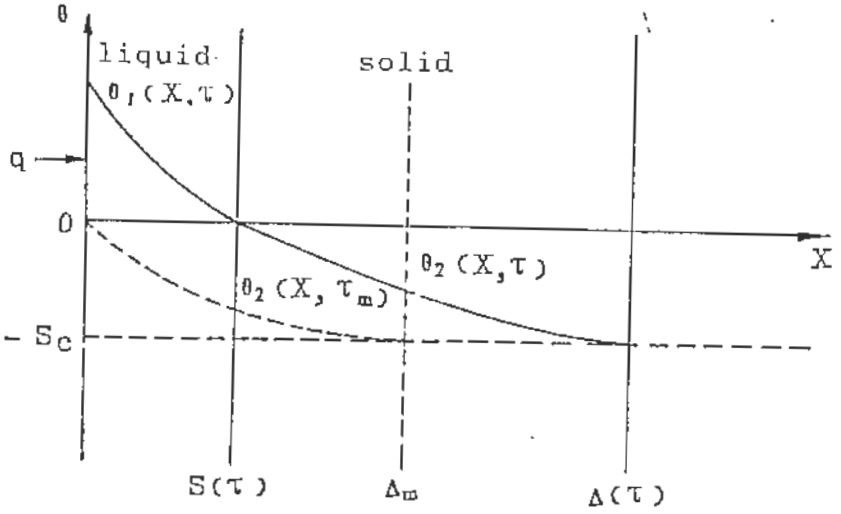

Fig.1 Physical model of melting

Temperature profile in the solid–phase region can be calculated again with the integration method, introducing the dimensionless variables in eq. (18), we get the non–dimensional preheating duration and the corresponding penetration depth as:

$$\tau_m = \frac{2}{3}\frac{1}{K_\beta^2 K_a} S_c^2 \tag{31}$$

$$\Delta_m = \frac{2}{K_\beta K_a} S_c \tag{32}$$

interface location and penetration depth at time τ are shown in Fig.1. Integrate eq. (21) in the interval (S, Δ), and from the definition of penetration depth and eq. (24), We obtain the integrated equation of the problem as

$$-\frac{\partial \theta_2}{\partial X}\Big|_{x=s} = K_a \frac{d}{d\tau}[\Theta_2 + S_c\Delta] \tag{33}$$

Where

$$\Theta_2 = \int_s^\Delta \theta_2 dX \tag{34}$$

We could assume the temperature profile in the solid–phase region of the form

$$\theta_2(X,\tau) = B_0 + B_1(X-S) + B_2(X-S)^2 \tag{35}$$

constants in eq. (35) could be computed from the boundary conditions eq. (24) and the difinition of penetration depth, finally the temperature profile could be obtained

$$\theta_2(X,\tau) = S_c[(\frac{\Delta - X}{\Delta - S})^2 - 1] \tag{36}$$

substitute the above equation into eq. (33), then we get

$$\frac{6}{K_a(\triangle - S)} = 2\frac{dS}{d\tau} + \frac{d\triangle}{d\tau} \tag{37}$$

which describes the relationship between the interface location and penetration depth.

Combining the temperature profiles in liquid phase eq. (29) and in solid phase eq. (36) with interface eq. (25), the location of interface can be expressed as

$$\frac{dS}{d\tau} = K_p erfc(\frac{S}{2\sqrt{\tau}}) - \frac{1}{K_a} \cdot \frac{2S_c}{\triangle - S} \tag{38}$$

substitute eq. (38) into eq. (37)

$$\frac{d\triangle}{d\tau} = \frac{1}{K_a} \cdot \frac{6}{\triangle - S}(1 + \frac{2}{3}S_c) - 2K_p erfc(\frac{S}{2\sqrt{\tau}}) \tag{39}$$

the initial conditions of eqs. (38), (39) are

$$S(\tau_m) = 0 \tag{40}$$

$$\triangle(\tau_m) = \triangle_m \tag{41}$$

with the use of these initial conditions and the Rung–Kutta method, the dependence of the interface location with dimensionless time τ can be predicted.

APPROXIMATE SOLUTION OF MELTING

In order to compare the exact solution with approximate solution of the liquid region temperature profile, integration method was also used to obain the temperature profile. We could assume a temperature profile of the quadric form, from eqs. (20), (24) and (25) then the temperature profile in liquid phase will be

$$\theta_1(X,\tau) = \frac{S}{2}(\frac{X - S}{S})^2 - \frac{1}{2}p\frac{X^2 - S^2}{S^2} \tag{42}$$

where

$$p = (\frac{S_c}{K_p K_a}\frac{S}{\triangle - S} - \frac{1}{2K_p}) + \sqrt{(\frac{S_c}{K_p K_a}\frac{S}{\triangle - S} - \frac{1}{2K_p})^2 + \frac{S}{K_p}} \tag{43}$$

Integrating the differential equation of the liquid phase eq. (19) in (0,S), and using boundary

conditions eqs. (20) and (24),then the integrated equation of liquid phase region will be

$$\frac{\partial \theta_1}{\partial X}\Big|_{x=s} + 1 = \frac{d\Theta_1}{d\tau} \tag{44}$$

where

$$\Theta_1 = \int_0^s \theta_1 dX \tag{45}$$

substitute the integrated equations (44) and (33) into the interface equation (25)

$$\frac{dS}{d\tau} + \frac{d}{d\tau}[\Theta_2 + S_c\Delta] + K_\rho \frac{d\Theta_1}{d\tau} = K_\rho \tag{46}$$

integrate both sides of eq. (46) with respect to τ within (τ_m, τ), then we obtain the interface equation as

$$\frac{1}{2}K_\rho S^2 + (K_\rho p + 3 + 2S_c)S + S_c(\Delta - \Delta_m) = 3K_\rho(\tau - \tau_m) \tag{47}$$

Solving simultaneously eqs. (37) and (47) with numerical method, then the variation relationship of interface location and penetration depth with dimensionless time could be calculated easily. Eqs. (37) and (47) are applicable to $S_c \neq 0$, if $S_c = 0$, that is no subcooling, interface location can be calculated by the following expression

$$S(K_\rho S + 5 + \sqrt{1 + 4K_\rho S}) = 6K_\rho \tau \tag{48}$$

if the difference of densities between solid and liquid phases is neglected, or $K_\rho = 1$, eq. (61) becomes

$$S(S + 5 + \sqrt{1 + 4S}) = 6\tau \tag{49}$$

which is in good agreement with Goodman's results [3].

RESULTS AND DISCUSSION

We calculated the interface location and the surface temperature profile of a semi-infinite body with different subcooling by both the semi-exact and approximate solution. Comparison is shown in Fig. 2 while $K_\rho = K_a = 1$. It is obvious that they agree well when ($\tau - \tau_m$) is small, but the deviation increases with increasing ($\tau - \tau_m$). Within the calculated range of Fig. 2, interface locations calculated with the approximate method are always slightly ahead the semi-exact solution, the maximum difference is 7.5%. The maximum deviation of surface temperature is 12%. It is not surprising that melting begins only when $\tau > \tau_m$, or preheating is

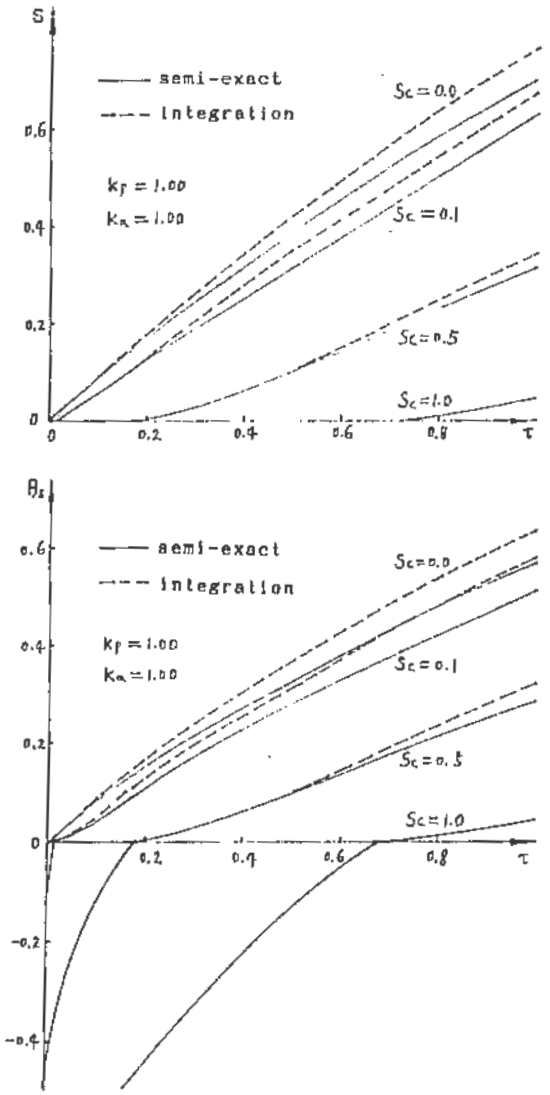

Fig.2 Comparison of different methods

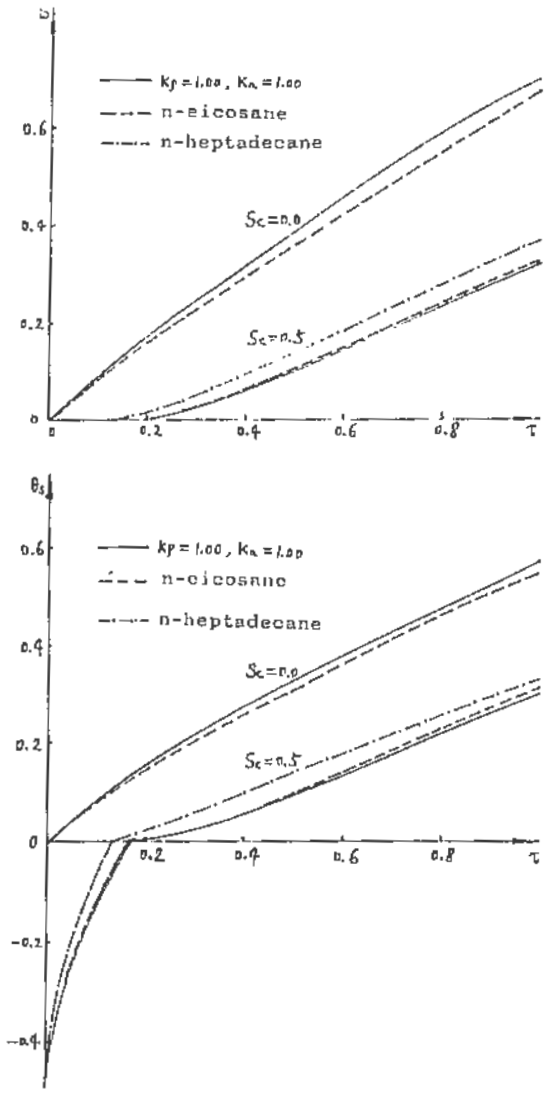

Fig.3 Comparison of different material

necessary for the melting process with subcooling, this is the unique feature of melting process with boundary conditions of second kind.

From previous explanations, the differential complementary error function must be calculated when using the semi-exact method, but the temperature profile in the liquid phase exactly satisfied the governing equations. The solving process of the integral approximate method is simpler than the semi-exact method, but the temperature profile in the liquid must be assumed and had slight influences on the obtained results. Both the semi-exact method and integral approximate method could serve as powerful tools for solving one-dimensional phase change problems.

In addition, we calculated the melting processes of n–heptadecane ($C_{17}H_{36}$), $K_\rho = 0.99$, $K_a = 1.34$ and n–eicosane ($C_{20}H_{42}$), $K_\rho = 0.91$, $K_a = 1.22$ so as to reveal the effect of physical properties on melting process as shown in Fig. 3. While no subcooling exists, ($S_c = 0$) the differences between various substances are not so evident as with subcooling (e.g. $S_c = 0.5$ shown in Fig.3), the results of n–heptadecane approximately coincide with those of constant property substance on the scale of Fig.3.

NOMENCLATURE

a	thermal diffusivity	x	coordinate
C_p	specific heat	δ	thickness of thermal layer
k	thermal conductivity	θ	dimensionless temperature
K_a	thermal diffusivity ratio	Θ	see eqs. (34) and (45)
K_ρ	density ratio	ρ	density
L	latent heat of melting	Subcripts	
q	surface heat flux	1	liquid phase
s	phase change position	2	solid phase
S_c	subcooling parameter	i	initial
t	time	m	melting
T	temperature		

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