A new radial integration boundary element method (RIBEM) for solving transient heat conduction problems with heat sources and variable thermal conductivity is presented in this article. The Green’s function for the Laplace equation is served as the fundamental solution to derive the boundary-domain integral equation. The transient terms are first discretized before applying the weighted residual technique that is different from the previous RIBEM for solving a transient heat conduction problem. Due to the strategy for dealing with the transient terms, temperature, rather than transient terms, is approximated by the radial basis function; this leads to similar mathematical formulations as those in RIBEM for steady heat conduction problems. Therefore, the present method is very easy to code and be implemented, and the strategy enables the assembling process of system equations to be very simple. Another advantage of the new RIBEM is that only 1D boundary line integrals are involved in both 2D and 3D problems. To the best of the authors’ knowledge, it is the first time to completely transform domain integrals to boundary line integrals for a 3D problem. Several 2D and 3D numerical examples are provided to show the effectiveness, accuracy, and potential of the present RIBEM.

1. Introduction

Transient heat conduction problems can be encountered in many engineering applications [1]; numerical methods, such as the finite-difference method (FDM) [2, 3], the finite-element method (FEM) [4–6], the meshless method [7–9], and the boundary element method (BEM) [10–14] are frequently employed to solve these problems. As a numerical and semi-analytical method, BEM has the advantage of only boundary discretization, and it could reduce the dimensionality of the problem [15–19]. In addition, it is very suitable for inverse problems [6, 20–23] that only boundary physical quantities could be measured. However, there would be domain integrals besides boundary integrals, if heat source, variable thermal conductivity, and transient terms are involved in BEM; this weakens its advantage of only boundary discretization.

To circumvent this deficiency, two types of methodologies are used to eliminate the domain integrals [18]. The first one is to find the fundamental solutions [24], but these methods have certain restrictions. For example, they may depend on the specific form of thermal conductivity, so it is difficult to develop general codes [18]. The second type is to convert the domain integrals into equivalent boundary integrals. The frequently used transformation technique is the dual reciprocity
method [25], triple reciprocity method [17], and multiple reciprocity method [26], which have been applied in heat conduction problems [27, 28, 19]. Besides, Gao [29] proposed the radial integration method, which can transform any complicated domain integrals into the boundary integrals. Based on this method, a radial integration boundary element method (RIBEM) has been developed and applied to solve heat transfer and mechanical problems [18, 30–35], and satisfactory results have been obtained.

In this article, a new RIBEM is presented to solve transient heat conduction problems with heat sources and spatially varying thermal conductivities. The transient terms are first discretized before applying the weighted residual technique that is different from the previous RIBEM for solving a transient heat conduction problem. Due to the strategy for dealing with the transient terms, temperature, rather than transient terms, is approximated by the radial basis function (RBF), which leads to similar mathematical formulations as those in RIBEM for steady-state heat conduction problems. Therefore, the present method is very simple to code and be implemented. Moreover, the strategy enables the assembling process of system equations to be very simple. Another advantage of the new RIBEM is that only 1D boundary line integrals are involved in both 2D and 3D problems. To the best of the authors’ knowledge, it is the first time to completely transform domain integrals into boundary line integrals for a 3D problem. It is expected that this methodology would facilitate different shapes of boundary meshes in further study. First, the domain integrals are transformed to equivalent boundary integrals by using radial integration method. For a 3D problem, radial integration method is employed once again to convert equivalent boundary integrals into boundary line integrals. Several 2D and 3D numerical examples are provided to show the effectiveness, the accuracy, and the potential of the present BEM.

2. Transient heat conduction problem

The transient heat conduction problem in the Cartesian coordinate can be expressed as follows:

$$\rho c_p \frac{\partial T(x, t)}{\partial t} = \frac{\partial}{\partial x_i} [k(x) \frac{\partial T(x, t)}{\partial x_i}] + Q(x) \quad (x \in \Omega)$$

(1)

with the following initial condition:

$$T(x, t)|_{t=0} = T_0(x)$$

(2)
the boundary conditions are

\[ T(x, t) = T(x, t), \quad x \in \Gamma_B \]  

(3)

\[ q(x, t) = -k(x) \frac{\partial T(x, t)}{\partial n} = \tilde{q}(x, t), \quad x \in \Gamma_\mathcal{Q} \]  

(4a)

\[ q(x, t) = -k(x) \frac{\partial T(x, t)}{\partial n} = h [T(x, t) - T_i], \quad x \in \Gamma_q \]  

(4b)

In Eqs. (1)–(4), \( T \) is the temperature, °C. \( t \) is time, s. \( \rho \) is the density, kg/m³. \( c_p \) is the mass specific heat, J/(kg°C). \( k \) is the thermal conductivity, W/(m°C). \( i \) changes from 1 to 3. \( q(x, t) \) is the normal heat flux on the boundary \( \Gamma \) of the computational domain \( \Omega \), W/m²; \( n \) is the unit normal to \( \Gamma \). \( h \) is heat convective coefficient. Subscript \( f \) represents ambient surrounding. In Eqs. (3) and (4), \( \tilde{T}(x, t) \) and \( \tilde{q}(x, t) \) are the given temperature and heat flux on the boundary. The thermal conductivity is assumed to be only varying with coordinates and is independent from temperature. The focused problems and solutions have the potential in multilayer nonhomogeneous media.

3. The new RIBEM to transform domain integral into boundary line integral

The Green’s function for the Laplace equation is served as the weight function to derive the boundary integral equation:

\[ G(x, y) = \begin{cases} \frac{1}{2\pi} \ln \left( \frac{d}{r} \right) & \text{for 2D problem} \\ \frac{1}{4\pi} \ln \left( \frac{r}{d/a} \right) & \text{for 3D problem} \end{cases} \]  

(5)

In the previous RIBEM, both sides of Eq. (1) were first directly multiplied by the Green’s function, and then the weighted residual technique [18, 34] was applied. It is different in the new RIBEM that the left side of Eq. (1), i.e., the transient item, is first discretized before applying the weighted residual technique. The forward FDM is used to discretize Eq. (1), and the result is

\[ \rho c_p \frac{T(x, t) - T(x, t - \Delta t)}{\Delta t} = \frac{\partial}{\partial x_i} \left[ k(x) \frac{\partial (x, t)}{\partial x_i} \right] + Q(x) \quad (t \geq \Delta t, \ x \in \Omega) \]  

(6)

In Eq. (6), \( t - \Delta t \) means the previous time step, and \( T(x, t - \Delta t) \) is known for each time step. Applying the weighted residual technique to Eq. (6) and using the Gauss’ divergence theorem, the boundary-domain integral equation for solving transient heat conduction problems can be established [18]:

\[ c k(y) T(y, t) = - \int_\Gamma G(x, y) q(x, t) d\Gamma(x) - \int_\Gamma \frac{\partial G(x, y)}{\partial n} k(x) T(x, t) d\Gamma(x) \]

\[ + \int_\Omega G(x, y) Q(x) d\Omega(x) + \int_\Omega V(x, y) k(x) T(x, t) d\Omega(x) \]

\[ - \int_\Omega \rho c_p G(x, y) \frac{T(x, t) - T(x, t - \Delta t)}{\Delta t} d\Omega(x) \]  

(7)

where \( c = 1 \) for internal points and 1/2 for smooth boundary points. \( \partial G(x, y)/\partial n \) and \( V(x, y) \) are as follows.

\[ \frac{\partial G(x, y)}{\partial n} = \frac{\partial G(x, y)}{\partial x_i} n_i(x) = \frac{1}{2\pi \rho r^2} \frac{\partial r}{\partial x_i} n_i \]  

(8)
In Eqs. (5)–(9), \( r \) is the distance between source point \( y \) and field point \( x \); \( \partial r / \partial x_i \) is the partial derivative of \( r \) with respect to coordinate \( x_i \); \( n_i \) is the \( i \)-th component of \( n \). Repeated subscripts mean summation. \( \alpha \) is 1 for 2D problems and 2 for 3D problems.

The last term on the right side of Eq. (7) can be rewritten as

\[
\int_{\Omega} \rho C_p G(x, y) \frac{T(x, t) - T(x, t - \Delta t)}{\Delta t} \, d\Omega(x) = \int_{\Omega} \frac{\rho C_p}{\Delta t} G(x, y) [T(x, t) - T(x, t - \Delta t)] \, d\Omega(x) \tag{10}
\]

Substituting Eq. (10) into Eq. (7), one can obtain

\[
ck(y)T(y, t) = -\int_{\Gamma} G(x, y) Q(x) \, d\Gamma(x) - \int_{\Gamma} k(x) \frac{\partial G(x, y)}{\partial n} T(x, t) \, d\Gamma(x)
+ \int_{\Omega} G(x, y) Q(x) \, d\Omega(x) + \int_{\Omega} \frac{\rho C_p}{\Delta t} G(x, y) T(x, t - \Delta t) \, d\Omega(x)
+ \int_{\Omega} [V(x, y) - \frac{\rho C_p}{\Delta t} \frac{1}{k(x)} G(x, y)] k(x) T(x, t) \, d\Omega(x)
\]

which is the general boundary-domain integral equation for transient heat conduction problems in nonhomogeneous media.

Radial integration method is applied to directly convert the 3rd and 4th term on the right-hand side of Eq. (11) into boundary integration:

\[
\int_{\Omega} G(x, y) Q(x) \, d\Omega(x) + \int_{\Omega} \frac{\rho C_p}{\Delta t} G(x, y) T(x, t - \Delta t) \, d\Omega(x) = \int_{\Gamma} \frac{1}{r^2(z, y)} \frac{\partial r}{\partial n} F(z, y) \, d\Gamma(z)
\]

where

\[
F(z, y) = \int_{0}^{r(z, y)} G(x, y) [Q(x) + \frac{\rho C_p}{\Delta t} T(x, t - \Delta t)] r^2 \, dr
\]

For the last term on the right side of Eq. (11), radial integration method could not be directly applied because the domain integral contains unknowns \( T(x, t) \). Therefore, normalized temperature \( \bar{T}(x, t) \) is introduced, which is approximated by a series of RBFs:

\[
\bar{T}(x, t) = k(x) T(x, t) = \sum_{A=1}^{N_A} \alpha^A \phi^A(R) + \sum_{k=1}^{A} a^k x_k + a^0
\]

\[
\sum_{A=1}^{N_A} \alpha^A = \sum_{A=1}^{N_A} \alpha^A x_k^A = 0
\]

where \( N_A \) is the number of application points consisting of all the boundary and internal nodes; \( \alpha^A \), \( a^k \), and \( a^0 \) are the coefficients to be determined, and \( x_k^A \) represents the Cartesian coordinates at the application point \( A \).

In Eq. (14), \( R = ||x - x^A|| \) is the distance from application point \( A \) to the field point \( x \), \( \phi^A(R) \) is the RBF, and compactly supported fourth-order spline RBF is commonly used, which could give stable results [18]. The coefficients \( \alpha^A \), \( a^k \), and \( a^0 \) can be determined by collocating the application point \( A \) at all nodes. A set of algebraic equations can be written in the matrix form as

\[
\{ \bar{T} \} = [\phi] \{ \alpha \}
\]

where \( \{ \alpha \} \) is a vector consisting of the coefficients \( \alpha^A \) for all application nodes and the polynomial coefficients \( a^k \), \( a^0 \). If no two nodes share the same coordinates, the matrix \([\phi]\) is invertible to obtain
Representing \( V(x, y) - \frac{\partial}{\partial t} \frac{1}{\kappa \Omega(x)} G(x, y) \) by \( \tilde{V}(x, y) \) and substituting Eq. (14) into the last item on the right side of Eq. (11) yield

\[
\int_{\Omega} \tilde{V}(x, y) \tilde{T}(x, t) d\Omega(x) = x^A \int_{\Gamma} \frac{1}{r^2(z, y)} \frac{\partial r}{\partial n} F^A(z, y) d\Gamma(z) + x^k \int_{\Gamma} \frac{1}{r^2(z, y)} \frac{\partial r}{\partial n} F^l(z, y) d\Gamma(z) + a^0 \int_{\Gamma} \frac{1}{r^2(z, y)} \frac{\partial r}{\partial n} F^0(z, y) d\Gamma(z)
\]

(18)

where

\[
F^A(z, y) = \int_0^{r(z, y)} \tilde{V}(x, y) \phi A r^2 dr
\]

(19)

\[
F^l(z, y) = \int_0^{r(z, y)} \tilde{V}(x, y) r^2 x_k dr
\]

(20)

\[
F^0(z, y) = \int_0^{r(z, y)} \tilde{V}(x, y) r^2 dr
\]

(21)

In Eqs. (19)–(21), \( x \) for calculating the radial integration is determined by Eq. (22), and it is varying with radius \( r \).

\[
x_i = y_i + r \xi_i,
\]

(22)

It can be seen that the expressions are different from those in Ref. [18] or Ref. [35] for a transient heat conduction problem. The expressions are similar to those in RIBEM for solving steady heat conduction problems in Ref. [30]. Therefore, the present method is very simple to code and easy to be implemented. The radial integrals shown in Eqs. (13), (19)–(21) are not singular, and they could be numerically evaluated by the Gauss formula.

In the present work, the equivalent 2D boundary integrals for a 3D problem are further converted to boundary line integrals using the radial integration method. Then, only 1D boundary line integrals are involved for any 2D or 3D problems. To the best of the authors’ knowledge, it is the first time to completely transform domain integrals to boundary line integrals for a 3D problem. It is expected that this methodology could facilitate arbitrary shapes of boundary meshes in further work. Taking Eq. (18) as an example, the boundary line integrals for a 3D problem (\( \alpha = 2 \)) can be obtained as

\[
\int_{\Omega} \tilde{V}(x, y) \tilde{T}(x, t) d\Omega(x) = x^A \int_{L} \frac{1}{R(z, z')} \frac{\partial R}{\partial n} F^A(z, z') dL(z') + x^k \int_{L} \frac{1}{R(z, z')} \frac{\partial R}{\partial n} F^l(z, z') dL(z') + a^0 \int_{L} \frac{1}{R(z, z')} \frac{\partial R}{\partial n} F^0(z, z') dL(z')
\]

(23)
where

\[
F^0(z, z') = \int_0^{R(z, z')} \frac{1}{r^3(z, y)} \frac{\partial r}{\partial n} F^0(z, y) RdR
\]

\[
F^1(z, z') = \int_0^{R(z, z')} \frac{1}{r^3(z, y)} \frac{\partial r}{\partial n} F^1(z, y) RdR
\]

\[
F^Q(z, z') = \int_0^{R(z, z')} \frac{1}{r^3(z, y)} \frac{\partial r}{\partial n} F^Q(z, y) RdR
\]

Similarly, all other equivalent boundary integrals can also be converted to the boundary line integrals for a 3D problem. The points \(y, z, z_0\), coupled with the distances \(R\) and \(r\) in Eqs. (23)-(26), are illustrated in Figure 1. \(L\) is the boundary line of the 2D boundary \(\Gamma\), and \(z\) coupled with \(z'\) is located on \(\Gamma\). \(y\) is the source point, and it is three dimensional.

For \(T(x, t - \Delta t)\) in Eq. (13), it can be evaluated by

\[
T(x, t - \Delta t) = [\varphi_1x \varphi_2x \varphi_3x \ldots \varphi_{N_x}x] x_1 x_2 x_3 1 \{\alpha\}
\]

where the vector \(\{\alpha\}\) is as follows:

\[
\{\alpha\} = \{x_1 x_2 x_3 \ldots x_{N_x} a_1 a_2 a_3 a_0\}^T
\]

Figure 1. Points \(y, z, z'\), coupled with the distances \(R\) and \(r\).

4. Formation of system equations

The boundary is discretized into a series of boundary elements. By collocating the source point \(y\) through all boundary and internal nodes, system equations for Eq. (11) can be formed. The boundary element model has \(N_b\) boundary nodes and \(N_i\) internal nodes, so the total number of nodes is \(N_A = N_b + N_i\). For any arbitrary time step, the system equations are

\[
[A_b] \{x_b\} = \{y_b\} - \{y_{b0}\} + [\tilde{V}_b] \{\tilde{T}\}
\]

\[
[\tilde{T}_i] \{x_b\} = [A_i] \{x_b\} + \{y_i\} - \{y_{i0}\} + [\tilde{V}_i] \{\tilde{T}\}
\]
Eq. (30) is for the boundary nodes, and Eq. (31) is for the internal nodes. \( \{ y_b \} \) and \( \{ y_i \} \) are generated by the integration of the heat source. The integration results for the previous time step are assembled into \( \{ y_bQ \} \) and \( \{ y_iQ \} \) for boundary and internal nodes, respectively. The integration results based on Eq. (23) are put in \( \tilde{V} \).

Then, we can obtain the following assembled equations for all nodes:

\[
\begin{pmatrix}
[A_b] & [0] \\
-[A_i] & [I]
\end{pmatrix}
\begin{pmatrix}
\{ y_b \} \\
\{ y_i \}
\end{pmatrix}
= \begin{pmatrix}
\{ \tilde{V}_b \} \\
\{ \tilde{V}_i \}
\end{pmatrix}
- \begin{pmatrix}
\{ y_bQ \} \\
\{ y_iQ \}
\end{pmatrix}
\]

(32)

where \([I]\) is the identity matrix. It can be seen that the assembling process of system equations is very simple, similar to that for solving a steady heat conduction problem with heat source in nonhomogeneous medium. By solving the set of system equations, boundary unknowns and the normalized internal temperatures could be obtained. Finally, true temperatures could be calculated.

5. Results and discussions

FORTRAN codes are developed using the derived mathematical formulations in this article. The codes can deal with constant or nonuniform initial temperatures, which can also deal with boundary conditions and heat sources that are varying with both time and coordinates.

In this section, two 2D and two 3D examples are provided to validate the present BEM for transient heat conduction problems with heat sources and spatially varying thermal conductivities. Analytical solutions, including the FDM or the FEM, are employed to examine the effectiveness and the accuracy of the present method.

5.1. Example 1

Figure 2 shows a 2D conduction problem in a square geometry with dimensions of 1 x 1 m². The density is 1 kg/m³, and the mass specific heat is 1 J/(kg.°C). The varying thermal conductivity is \( \{ x + y \} \) W/(m.°C). Heat source varies with both time and coordinates as follows [34]:

\[
Q(x) = -6(x + y) + 10 \cos(10t)
\]

(33)

The initial condition is

\[
T(x, y, 0) = x^2 + y^2
\]

(34)
The boundary conditions are

\[
T(x, y, t)\big|_{y=1} = x^2 + 1 + \sin(10t)
\]

(35)

\[
T(x, y, t)\big|_{x=2} = y^2 + 4 + \sin(10t)
\]

(36)

\[
T(x, y, t)\big|_{y=2} = x^2 + 4 + \sin(10t)
\]

(37)

\[
T(x, y, t)\big|_{x=1} = 1 + y^2 + \sin(10t)
\]

(38)

The analytical solution of this problem is

\[
T(x, y, t) = x^2 + y^2 + \sin(10t)
\]

(39)

In the BEM, the geometry is divided into 20 two-noded boundary elements, and there are 20 boundary nodes and 16 internal nodes as shown in Figure 3.

Effects of the time step are investigated first with four different time steps: 0.001 s, 0.01 s, 0.1 s, and 1 s. Table 1 lists the relative errors of the temperatures compared with the analytical solutions when the time is 5 s. It can be seen that the accuracy would increase with the decreasing time step; however, the efficiency would decrease. Therefore, the time step is chosen as 0.01 s. In the following examples, the time step is chosen considering both accuracy and efficiency.

Figure 4 shows the transient temperatures using the BEM, and Figure 5 shows the relative errors of temperatures compared with the analytical solutions. It can be seen that temperatures using the present BEM are in excellent agreement with the analytical solutions, and the maximal relative error is below 0.45%; this implies the high accuracy of the present BEM for solving a transient heat conduction problem with heat source and spatially varying thermal conductivity in a regular 2D geometry.

5.2. Example 2

Figure 6 shows heat conduction in an irregular 2D geometry. The boundary elements and nodes are shown in Figure 7; there are 35 boundary elements, 35 two-noded boundary nodes, and 43 internal nodes. The density is 7,000 kg/m³, and the mass specific heat is 300 J/(kg·°C). The internal heat source

![Figure 3. Boundary elements and nodes for the square geometry.](image-url)
Table 1. Relative errors of temperatures by using the BEM compared with the analytical solutions in the square geometry, when the time is 5 s.

<table>
<thead>
<tr>
<th>ID</th>
<th>0.001 s</th>
<th>0.01 s</th>
<th>0.1 s</th>
<th>1 s</th>
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<td>21</td>
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<tr>
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</table>

BEM, boundary element method.

Figure 4. Transient temperatures in the square geometry using the BEM: (a) 2 s, (b) 3 s, (c) 4 s, and (d) 5 s. Note: BEM, boundary element method.
is 10,000 W/m³, and the thermal conductivity is \([30 + 80 \times (x + y)]\) W/(m·°C). The initial temperature is 200°C. The boundary conditions are

\[
q(x, y)|_{\text{upper}} = 0 \\
T(x, y)|_{\text{left}} = 300 \\
q(x, y)|_{\text{bottom}} = 10,000 \\
T(x, y)|_{\text{right}} = 400
\]

It means that the temperature on the left boundary is 300°C, and the temperature at the two sides of the right boundary is 400°C. The heat flux on the bottom boundary is kept to be \(+10000\) W/m², and the other boundaries are adiabatic. ‘+’ represents same with the outward normal. The total time is 50 s. Figure 8 shows the transient temperatures when the time step is 2 s. It can be seen that the overall temperature increases with the time, and heat is conducted from the right to the left. Meanwhile, the heat spreads from the upper to the bottom. For comparison, the FEM is adopted to solve this problem. The grid size in the FEM is 0.0025 × 0.0025 m², which has been validated by grid independence. Figure 9 shows the comparison of the between temperatures obtained from

\[\text{Figure 5. Relative errors of transient temperatures using the BEM compared with analytical solutions: (a) 2 s, (b) 3 s, (c) 4 s, and (d) 5 s. Note: BEM, boundary element method.}\]
the present BEM and the FEM, at nodes 54 (0.02, 0.02), 51 (0.05, 0.02), and 48 (0.08, 0.02). It can be seen that the results are in excellent agreement, and it can also be seen that the grid number is much less in the BEM than that in the FEM. For quantitatively comparing the temperatures, relative errors of temperatures using the present BEM compared with the results of the FEM are calculated and are shown in Figure 10. It can be seen that most of the relative errors are less than 2%, which demonstrates the effectiveness and the high accuracy of the present method for solving 2D transient heat conduction problem with heat source and spatially varying thermal conductivity in an irregular geometry.

5.3. Example 3

The third example is heat conduction in a 3D cubic geometry, and the dimensions are 0.1 × 0.1 × 0.1 m$^3$. The boundary elements and nodes are shown in Figure 11. There are 96 4-noded boundary elements, 98 boundary nodes, and 27 internal nodes. The density is 6,000 kg/m$^3$, and the mass specific heat is 400 J/(kg°C). Heat source is distributed along the $z$-axis, and it is equal to 10$z$ W/m$^3$. The thermal conductivity is $(30 + 20 \times z)$ W/(m°C), and the initial temperature is 800°C. The boundary conditions are

$$
\begin{align*}
q(x, y, z)\big|_{\text{bottom}} &= 80,000 - 1000 t \\
q(x, y, z)\big|_{\text{right,left}} &= 0 \\
q(x, y, z)\big|_{\text{upper}} &= h \left[ T(x, y, z) - T_f \right] \\
q(x, y, z)\big|_{\text{front,back}} &= 0
\end{align*}
$$

(41)

![Figure 6. Dimensions of the irregular 2D geometry.](image)

![Figure 7. Boundary elements and nodes for the irregular 2D geometry.](image)
The bottom heat flux is time dependent. The upper surface is imposed on convective boundary condition, with \( h = 100 \text{ W/(m}^2\text{.°C)} \) and \( T_f = 30\text{°C} \). All the other boundaries are adiabatic. The time is 50 s, and the time step is 2 s by consideration of both accuracy and efficiency. The transient temperatures at nodes 80 (0.1, 0.1, 0.025), 75 (0.1, 0.1, 0.05), and 70 (0.1, 0.1, 0.075) using the BEM and the FDM are shown in Figure 12. The temperature decreases along the \( z \)-direction, from the upper boundary to the bottom surface. This is because the intensity of heat source decreases from the upper boundary to the bottom surface, and the cubic is nearly symmetrically cooled on the upper and bottom boundaries. It can also be seen that the results of the BEM and FDM compared well with each other. For quantitatively evaluating the results of the BEM, relative errors of temperatures using the present BEM compared with the results of the FDM are calculated and are listed in Table 2. It can be seen that most of the relative errors are far below 0.5%, which demonstrates the high accuracy of the present method for solving a 3D transient heat conduction problem with heat source and spatially varying thermal conductivity in a regular 3D geometry.

![Figure 8](image1.png)

**Figure 8.** Transient temperatures in the irregular 2D geometry: (a) 20 s, (b) 30 s, (c) 40 s, and (d) 50 s.

![Figure 9](image2.png)

**Figure 9.** Transient temperatures in the irregular geometry using the BEM and the FEM. Note: BEM, boundary element method; FEM, finite-element method.
5.4. Example 4

Finally, a complex 3D geometry is considered, and the boundary elements as well as the nodes are shown in Figure 13. The heights of the two hexagonal prisms are both 0.04 m, and their side lengths are 0.02 and 0.04 m, respectively. There are 1,008 boundary elements, 1,010 4-noded boundary nodes, and 203 internal nodes. The density is 7,800 kg/m$^3$, and the mass specific heat is 465 J/(kg.°C). Heat source is a constant, and it is equal to 1,000 W/m$^3$. The varying thermal conductivity is $[40 + 100 \times (x + y + z)]$ W/(m.°C), and the initial temperature is 200°C. The bottom boundary is kept at 1,000°C, and all the other boundaries are assumed to be adiabatic. Figure 14 shows transient surface temperatures using the present method. It can be seen that temperatures increase with the time. In general, the heat is conducted from the bottom to the upper, and the heat spreads from the outer into...
the inner at the interface of the two hexagonal prisms. This is because temperatures on the bottom are higher than those on the upper surface, and the heat source is uniformly distributed. FEM is also employed to obtain the transient temperatures, and comparisons of temperatures at nodes 772 (0.03, 0.01732, 0.06667), 774 (0.03, 0.01732, 0.05333), 982 (0, 0, 0.02667), and 984 (0, 0, 0.01333) are shown in Figure 15. It can be seen that the temperatures obtained using the BEM and the FEM agree well with each other, although the BEM meshes are less. Table 3 lists the detailed relative

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BEM, boundary element method; FDM, finite-difference method.

Figure 12. Transient temperatures in the cubic geometry using the BEM and the FDM. Note: BEM, boundary element method; FDM, finite-difference method.
errors of the temperatures. It can be seen that the highest value is less than 0.6%, which demonstrates the high accuracy of the present method for solving a 3D transient heat conduction problem with heat source and spatially varying thermal conductivity in a complex 3D geometry.

Moreover, the two 3D examples demonstrate that the methodology of only 1D boundary line integrals are effective and accurate, which would provide a theoretical basis for setting up polygonal BEM to adapt different shapes of boundary meshes.

Figure 13. Boundary and finite elements for the 3D complex geometry: (a) boundary elements, (b) finite elements.

Figure 14. Transient temperatures in the complex 3D geometry: (a) 4 s, (b) 6 s, (c) 8 s, and (d) 10 s.
6. Conclusions

In this article, a new RIBEM is presented for solving transient heat conduction problems with heat sources and variable thermal conductivity. The transient terms are first discretized before applying the weighted residual technique, which is different from the previous RIBEM for solving a transient heat conduction problem. This strategy enables the present method to be very easy to code and be implemented, and it also enables the assembling process of system equations to be very simple. Another advantage of the new RIBEM is that only 1D boundary line integrals are involved in both 2D and 3D problems. To the best of the authors’ knowledge, it is the first time to completely transform domain integrals to boundary line integrals for a 3D problem.

Numerical examples show that the present BEM is simple, effective, and accurate, for solving 2D and 3D transient heat conduction problems with heat sources and spatially varying thermal conductivities in either simple or complex geometries. The initial temperatures could be constant or nonuniform. In addition, the boundary conditions and the heat sources could be varying with both time and coordinates.

The present work would provide a theoretical basis for setting up polygonal BEM, in which only 1D boundary line integrals are involved. Moreover, the present work has the potential in multilayer nonhomogeneous media.

Table 3. Relative errors of temperatures by using the BEM compared with the results of the FEM in the complex 3D geometry.

<table>
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<tr>
<th>Time, s</th>
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BEM, boundary element method; FEM, finite-element method.

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References


