Lattice Boltzmann Method Simulation of Natural Convection Heat Transfer in Horizontal Annulus

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The lattice Boltzmann method is employed to simulate the stability and transitions of natural convection in a horizontal annulus with the Prandtl number varying from 0.1 to 0.7 and the Rayleigh number ranging from $10^3$ to $5.0 \times 10^5$. Different flow patterns including steady upward flow, oscillatory upward flow, steady downward flow, and oscillatory downward flow are found for different values of the parameters. The flow pattern is very complex when the Rayleigh number is large. The critical Rayleigh number increases with the increase of the Prandtl number and with the decrease of the aspect ratio.

Nomenclature

- $A$ = aspect ratio
- $C_p$ = heat capacity
- $c$ = fluid particle speed
- $c_r$ = speed of sound in lattice scale
- $c_u$ = discrete lattice velocity
- $D_o$ = inner cylinder diameter
- $F_y$ = force term along the vertical direction
- $f_a$ = density distribution function
- $Gr$ = Grashof number
- $g$ = gravitational acceleration
- $g_a$ = temperature distribution function
- $j$ = Jacobian
- $k_{eq}$ = equivalent thermal conductivity
- $k_{eq,i}$ = mean equivalent thermal conductivity of the inner cylinder
- $k_{eq,o}$ = mean equivalent thermal conductivity of the outer cylinder
- $L$ = width gap
- $N$ = grid quantity
- $N_u$ = Nusselt number
- $Pr$ = Prandtl number
- $p$ = pressure, Pa
- $p'$ = lattice Boltzmann model macropressure
- $Q_w$ = constant heat flux on the inner surface
- $R$ = radius ratio, $R_o/R_i$
- $Ra$ = Rayleigh number
- $R_i$ = inner radius
- $R_o$ = outer radius
- $r$ = radial coordinate
- $T_i$ = dimensionless temperature of inner cylinder
- $T_o$ = dimensionless temperature of outer cylinder
- $t$ = temperature
- $t'$ = lattice Boltzmann model macrotemperature
- $u$ = dimensionless velocity
- $u_d$ = dimensionless horizontal components of velocity
- $v$ = dimensionless vertical components of velocity
- $w$ = weighting factors
- $\beta$ = thermal expansion coefficient
- $\Delta t$ = lattice time step
- $\Theta$ = dimensionless temperature
- $\theta$ = angle
- $\nu$ = kinematic viscosity
- $\rho$ = density
- $\rho'$ = lattice Boltzmann model macrodensity
- $\tau$ = lattice relaxation time

I. Introduction

The importance of natural convection between horizontal concentric cylinders has driven a great number of research in engineering and industrial technology, including nuclear reactor design, aircraft cabin insulation, and cooling systems in electronic components. The annular shape enclosure is one of the applicable geometries in engineering and industry; in particular, the horizontal circular annulus shape is commonly found in solar collector–receiver systems, underground electric transmission cables, vapor condensers for water distillation, heat exchangers, and food processing equipment. This topic has been widely investigated in the available literature [1,2], with studies focused on the large variety of flow structures that arises from different aspect ratios, Rayleigh numbers, and Prandtl numbers [3,4].

For the cases of low Rayleigh numbers, the basic flowfield consists of two two-dimensional crescent-shaped cells, which are symmetrical about the vertical plane containing the axes of the cylinders. In each of the two annular half-spaces, the fluid goes upward and downward along the hot inner and cold outer cylinders, respectively. Conduction is the major mode of heat transfer between the differentially heated boundaries of the annulus. As the Rayleigh number increases, the rotation of the main cells moves upward and a thermal plume starts to form at the upper part of the annulus with an impingement region on the outer cylinder. The distributions of the thermal fluxes along the inner and outer cylinders show that the largest part of the heat convection within the annulus is extracted from the lower part of the inner cylinder. The flow and temperature fields of the crescent-shaped convection have been extensively investigated experimentally and numerically [5,6].

The instability of the crescent-shaped convection with different aspect ratios and Rayleigh numbers at moderate Prandtl numbers has been investigated extensively. For example, Powe et al. [7] and Bishop et al. [8] depicted flow regimes and spatial patterns for air-filled annuli as a function of the aspect ratio and Rayleigh number. Their work showed that, for wide gap annuli with aspect ratios less than 2.8, transitions happened from two-dimensional steady to...
oscillatory flows. For moderate-gap annuli (2.8 < A < 8.5), a three-dimensional spiral flow was observed past the transition; whereas for narrow-gap annuli (A > 8.5), the basic single-cellular flow changed to multicellular flows. Later, the experimental and numerical results of Rao et al. [9] were reported in a qualitative agreement with Powe et al.'s experimental map for moderate- and narrow-gap annuli [7]. For narrow annuli (A > 10) in the case that three-dimensional flows, some discrepancies were observed. Therefore, the flow in the upper region of the narrow-gap annulus was either transverse rolls or a combination of longitudinal and transverse rolls with respect to the cylinder axis [10]. If the radius ratio was sufficiently small, an odd number of transverse rolls could occur so that a longitudinal flow was present at the midaxial plane [11].

With the in-depth research, among the two-dimensional studies, a large variety of research was focused on multicellular flow patterns and multiplicity of solutions for various sets of parameters [12–18]. These results revealed the existence of an imperfect bifurcation. The number of cells mainly depends on the aspect ratio and Rayleigh number. For a fixed supercritical value of Rayleigh number, the number of cells ascends as the aspect ratio increases. The aspect ratio of the three-dimensional flow approached a sufficiently large value, the number of cells tends toward infinity and the classical Rayleigh–Bénard problem characterized by a pitchfork bifurcation applies. The multicellular flows calculated in these studies were shown to undergo an unsteady secondary instability by increasing the Rayleigh number, provided the aspect ratio is large enough (for example, A = 14.3). The resulting periodic flow is composed of cellular patterns located in the basic flow in the lateral regions. This second type of instability is hydrodynamic in its origin, as in air-filled vertical slots.

There were also many studies about the stability in horizontal concentric annulus. Choi and Moon-Uhn [19] studied the linear stability of the crescent-shaped convection by solving the linear equation for three-dimensional disturbances with a time-marching method. It was shown that the transient behavior was valid for the cases of A ≥ 2.1. This implied that the resultant three-dimensional spiral convection was a steady-state flow, which was in contrast to the previous findings of Powe et al. [7] and Moukalled and Acharya [20].

On the other hand, for the cases where A ≤ 2.1, they could not find any critical Rayleigh number. The linear stability theory for parallel flows was applied to natural convection by Walton [21] and Dyko and Vafai [10] using an expansion in inverse power of the aspect ratio. Dyko and Vafai [10] evaluated critical Rayleigh numbers for three-dimensional instability by using a linear stability theory and energy method, and they concluded that the instability was subcritical. Bifurcation diagrams were obtained for various radius ratios, and the main thresholds were tracked as a function of the radius ratio by Petrone et al. [22]. The vertical symmetry model was also considered by Petrone et al. [23] to compute steady and oscillatory asymmetrical flows. A three-dimensional linear stability analysis using a spectral element method was conducted by Adachi and Imai [24] to evaluate a critical Rayleigh number with respect to three-dimensional disturbances over a wide range of aspect ratios. They proposed new transition lines of the critical Rayleigh number that consisted of three lines as a function of aspect ratio.

In addition, many studies have also considered the influence of Prandtl number [25–28]. Yoo [29] considered the natural convection problem in a narrow horizontal annulus with A = 12 and investigated the effects of the Prandtl number on the stability of the conduction regime. It was observed that the stability of the conduction regime of natural convection could be classified into two regimes. In other research, Yoo [30] studied the natural convection in a narrow horizontal concentric annulus for a fluid of Pr = 0.4. He found that the combined effects of thermal and hydrodynamic instability resulted in complex multicellular flow patterns. The influence of the Prandtl number to bifurcation and solutions in the horizontal annulus were studied within the range of 0.3 ≤ Pr ≤ 1 [31].

Many lattice Boltzmann model (LBMs) are designed to simulate natural convection considering heat transfer. Chatterjee and Chakraborty [32,33] and Jourabian et al. [34] used the lattice Boltzmann method instead of Navier–Stokes equations. Ren et al. [35] reported a LBM boundary condition–enforced immersed boundary method to study the natural convection in a concentric annulus [31–32]. Liao and Lin adopted an LBM to simulate the natural convection flows with heat transfer [36]. Fattahi et al. [37] simulated natural convection heat transfer in an eccentric annulus using the lattice Boltzmann model based on a double-population approach. They investigated mixed convection heat transfer in an eccentric annulus based on a multidistribution function, which is a double-population approach [38]. Osman and Sidik [39] investigated natural convection from a concentrically and eccentrically inner-heated cylinder placed inside a cold outer cylinder.

The flow patterns in a horizontal concentric annulus have been studied by different methods, and their linear or nonlinear stability has been investigated with different approaches. However, the stability and transitions of flowfields still require a deeper degree of investigation in some ranges of Rayleigh and Prandtl numbers. In addition, none of the available research works simultaneously considered all the main parameters influencing the flow behavior in this system. Therefore, the present study is motivated by the need of revealing the transitions and the flow structure while taking into account all the main parameters by using the lattice Boltzmann method as a new solution technique. The flow and stability are discussed in detail, and the effects of the aspect ratio and of the Prandtl number on the flow patterns, as well as the critical Rayleigh number, are systematically investigated.

II. Problem Statement and Mathematical Model

A schematic diagram of the physical model of the investigated system is shown in Fig. 1. The natural convection in a horizontal cylindrical annulus with an inner radius \( R_i \) and an outer radius \( R_o \) is simulated (The inner and outer diameters are denoted by 2Di and 2Do, respectively). The angle \( \theta \) is measured clockwise from the upper vertical plane. The radial aspect ratio is denoted as \( A = D_o / (R_o - R_i) \).

In this paper, \( k_{eq} \) is the local equivalent thermal conductivity and \( Q_0 \) is the constant heat flux on the inner surface. The velocity \( \hat{u} \) and temperature \( \hat{T} \) are nondimensionalized as follows:

\[
\hat{u} = \frac{\hat{u} - \hat{u}_m}{\hat{u}_m}
\]

\[
\hat{T} = \frac{\hat{T} - \hat{T}_m}{\hat{T}_0 - \hat{T}_m}
\]

The flow is assumed to be incompressible, laminar, Newtonian, and two-dimensional. The Boussinesq-approximated Navier–Stokes equations and the energy equation with negligible viscous dissipation are used to determine the buoyancy-induced flowfield.
The governing equations written in dimensionless form can be written as follows:

\[ \nabla \cdot \mathbf{u} = 0 \quad (3) \]

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + Pr \cdot \Delta \mathbf{u} + Ra \cdot Pr \cdot \mathbf{\Theta} \quad (4) \]

\[ \frac{\partial \mathbf{\Theta}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{\Theta} = \Delta \mathbf{\Theta} \quad (5) \]

These equations show that this problem depends on one geometrical parameter (the aspect ratio) and two dimensionless parameters (Prandtl number and Rayleigh number). They are defined as

\[ R_a = \frac{c_p \beta g (R_0 - R_1)^2 Q_{in}}{k_{eq} v_k}, \quad Pr = \frac{C_p v_k}{k_{eq}} \quad (6) \]

where \( \rho, C_p, \) and \( v_k \) are the density, specific heat capacity, and kinetic viscosity of the fluid, respectively. The Prandtl number is defined as the ratio of momentum diffusion (kinematic viscosity) to thermal diffusivity and the Rayleigh number is defined as the product of the Grashof number, which describes the relationship between buoyancy and viscosity within a fluid, and the Prandtl number, which describes the relationship between momentum diffusivity and thermal diffusivity. Hence, the Rayleigh number itself can also be viewed as the ratio of buoyancy to viscosity forces times the ratio of momentum to thermal diffusivities. The boundary conditions at the inner and outer cylinders can be written as follows:

\[ r = R_i \quad \text{and} \quad 0 \leq \theta \leq 2\pi: \quad u = v = 0 \quad \text{and} \quad \mathbf{\Theta} = 1 \]

\[ r = R_o \quad \text{and} \quad 0 \leq \theta \leq 2\pi: \quad u = v = 0 \quad \text{and} \quad \mathbf{\Theta} = 0 \]

The local equivalent thermal conductivity is given by

\[ k_{eq} = -\frac{R}{r} \int_0^{2\pi} \left( \frac{\partial \mathbf{\Theta}}{\partial r} \right)_{r=R_i} \, d\theta \quad (7) \]

The mean equivalent thermal conductivities on the inner and outer cylinders are calculated by integrating the local equivalent thermal conductivity on the surface area of either cylinder:

\[ \bar{k}_{eq} = -\frac{R}{2\pi} \int_0^{2\pi} \left( \frac{\partial \mathbf{\Theta}}{\partial r} \right)_{r=R_i} \, d\theta \]

\[ \bar{k}_{eq} = -\frac{R}{2\pi} \int_0^{2\pi} \left( \frac{\partial \mathbf{\Theta}}{\partial r} \right)_{r=R_o} \, d\theta \quad (8) \]

### III. Lattice Boltzmann Method and Interpolation Method

#### A. Lattice Boltzmann Method

The lattice Boltzmann method is a numerical scheme for the solution of Boltzmann’s transport equation (BTE) for particle distribution function. The mass, momentum, and energy conservation equations of fluid dynamics can be derived from BTE through a Chapman–Enskog expansion [40–42]. The same scheme can be extended to show that the Lattice Boltzmann Equations (LBE) method leads to the mass and momentum conservation equations of fluid dynamics [43–52].

The discrete velocity set of two-dimensional nine velocity (D2Q9) lattice Boltzmann equation method in the Bhatnagar Gross Krook (BGK) approximation is used in this paper. The general form of the LBE in the presence of an external force can be written as follows:

\[ f_{\alpha}(x + c_{\alpha} \Delta t, t + \Delta t) - f_{\alpha}(x, t) = \frac{\Delta t^2}{\tau} \frac{f_{\alpha}^{eq}(x, t) - f_{\alpha}(x, t)}{\tau} + \Delta t \cdot F \quad (9) \]

where \( \Delta t \) is the lattice time step; \( c_\alpha \) (\( \alpha = 0, 1 \ldots 8 \)) is the lattice velocity along the direction \( \alpha \) in the discrete velocity set of two-dimensional nine velocity; \( F \) is the external force; \( f_{\alpha} \) is the particle distribution function along the direction \( \alpha \); and \( \tau \) denotes the lattice relaxation time (the average time interval between particles collision), which is related to the kinematic viscosity by the following relation:

\[ \tau = \left( \frac{1}{\Delta \ell^2} \right) \nu_0 + 0.5 \quad (10) \]

where \( \nu_0 \) is the kinematic viscosity, \( c_s \) is the speed of sound, and \( f_{\alpha}^{eq} \) is the equilibrium distribution function that depends on the type of problem. The equilibrium distribution functions for the fluid field are calculated according to the following equation:

\[ f_{\alpha}^{eq} = w_{\alpha} \rho \left[ 1 + \frac{(c_s \cdot u)}{c_s^2} + \frac{(c_s \cdot u)^2}{2c_s^2} - u^2 \right] \quad (11) \]

where \( w_{\alpha} \) is the set of weighting coefficients along each direction. The values of \( w_0 = 4/9 \) for \( |\alpha| = 0 \), \( w_{1-4} = 1/9 \) for \( |\alpha| = 1 \), and \( w_{5-8} = 1/36 \) for \( |\alpha| = 2 \) are assigned in this model. Also, \( \rho \) and \( u \) are the macroscopic fluid density and velocity, and they are calculated as follows:

\[ \rho = \sum_{\alpha} f_{\alpha} \quad (12) \]

\[ \rho u = \sum_{\alpha} f_{\alpha} u_{\alpha} \quad (13) \]

The thermal Lattice Boltzmann equation can be written as follows:

\[ g_{\alpha}(x + c_{\alpha} \Delta t, t + \Delta t) - g_{\alpha}(x, t) = \frac{\Delta t^2}{\tau_c} \frac{g_{\alpha}^{eq}(x, t) - g_{\alpha}(x, t)}{\tau_c} \quad (14) \]

where the components of the thermal equilibrium distribution function are given by the following:

\[ g_{\alpha}^{eq} = w_{\alpha} T \left[ 1 + 2 \left( \frac{c_{\alpha} \cdot u}{c_s^2} \right) \right] \quad (15) \]

where \( T \) is the fluid temperature, which can be calculated as follows:

\[ T = \sum_{\alpha} g_{eq}^{\alpha} \quad (16) \]

The temperature relaxation time is related to diffusivity by the following equation:

\[ \tau_c = \left( \frac{2}{\Delta \ell^2} \right) \alpha + 0.5 \quad (17) \]

To incorporate the buoyancy force in the model, the Boussinesq approximation was applied; therefore, the force term in Eq. (2) needs to be calculated as follows, in the vertical direction \( \gamma \):

\[ F_{\gamma} = 3 w_\gamma g \beta \Delta T \quad (18) \]
Fig. 2 Schematic diagram of the grid model.

where \( g_y \) is the acceleration of gravity acting in the \( y \) direction of the lattice links, \( \beta \) is the thermal expansion coefficient, and \( \Delta T \) is the temperature difference.

### B. Interpolation Method Boundary Conditions

In the present study, the computational domain is set to be \(-L < x < L\), where the cylinder’s center is taken to be the coordinate origin and is covered by a square lattice. As shown in Fig. 2, the evolution process at the lattice grids of the square is preformed using the D2Q9 model. To obtain the variables in the horizontal cylindrical annulus, it is divided into a series of concentric circles as virtual grids. LBM grids variables are obtained by the neighboring virtual grid layer in order to use the nonequilibrium part, the physical parameters can be calculated by a second-order Taylor expansion:

\[
[S] = \begin{bmatrix}
1 & \Delta x_0 & \Delta y_0 & \frac{(\Delta x_0)^2}{2} & \frac{(\Delta y_0)^2}{2} & \Delta x_0 \Delta y_0 \\
1 & \Delta x_1 & \Delta y_1 & \frac{(\Delta x_1)^2}{2} & \frac{(\Delta y_1)^2}{2} & \Delta x_1 \Delta y_1 \\
- & - & - & - & - & - \\
- & - & - & - & - & - \\
1 & \Delta x_M & \Delta y_M & \frac{(\Delta x_M)^2}{2} & \frac{(\Delta y_M)^2}{2} & \Delta x_M \Delta y_M
\end{bmatrix}_{(M+1)\times 6}
\]  

where \( M \) is the number of neighboring points around \( P \). The \( \Delta x \) and \( \Delta y \) values in the matrix \([S]\) are given by

\[
\Delta x_0 = c_{x_0} \Delta t, \quad \Delta y_0 = c_{y_0} \Delta t
\]  

\[
\Delta x_i = x_i + c_{x_0} \Delta t - x_0, \quad \Delta y_i = y_i + c_{y_0} \Delta t - y_0,
\] for \( i = 1, 2, \cdots, M \)

When the coordinates of all mesh points are given and the particle velocity and time-step size are specified, the matrix \([S]\) is determined. To minimize the error used in the least-squares method, we obtain

\[
\{V\} = ([S]^T[S])^{-1}[S]^T \{h\} = [B] \{h\}
\]  

where

\[
\{h\} = \{h_0, h_1, \cdots, h_M\}^T,
\]

\[
h_i = f_a(x_i, y_i, t) + [f^{eq}_a(x_i, y_i, t) - f_a(x_i, y_i, t)]/\tau
\]  

Note that \([B]\) in Eq. (22) is a \( 6 \times (M + 1) \)-dimensional matrix. From Eq. (22), we find

\[
f_a(x_0, y_0, t + \Delta t) = V_1 = \sum_{n=1}^{M+1} b_{i,n}^1 h_{i,n}^{eq}
\]  

where \( b_{i,n}^1 \) are the elements of the first row of matrix \([B]\), which are precomputed before the LBM is applied.

For the processing of boundary conditions, we choose the nonequilibrium extrapolation boundary, which has a wide application range, and for which the convergence and numerical accuracy can meet the requirement. The equilibrium distribution functions can be decomposed into two parts, corresponding to equilibrium and nonequilibrium:

\[
f_a = f^{eq}_a + f^{inc}_a
\]  

For the equilibrium part, all the physical parameters can be calculated by the adjacent internal nodes; whereas for the nonequilibrium part, the physical parameters can be calculated by a second-order Taylor expansion:

\[
f^{inc}_a(\text{neighbor}) = f^{inc}_a + O(\delta_t^2)
\]  

The equilibrium distribution functions can be written as follows:

\[
f_a = f^{eq}_a + (f_a - f^{eq}_a)(\text{neighbor})
\]
C. Numerical Solution Procedure

1) Initialize Macroparameters $\rho$, $p$, $t$, and $u$.
2) Initialize equilibrium distribution functions $f_\alpha$ and $T_\alpha$.
3) Initialize the equilibrium distribution functions in the boundary.
4) Calculate equilibrium distribution functions after the collision and stream $f_0^\alpha$ and $T_0^\alpha$.
5) Calculate macroparameters $\rho_0$, $p_0$, $t_0$, and $u_0$.
6) Perform the convergence test.
7) Calculate macrocharacteristic parameters

$$\max|\Theta_{i,j}^{n+1} - \Theta_{i,j}^n| < 10^{-8}$$

$$\max|u_{i,j}^{n+1} - u_{i,j}^n| < 10^{-8}$$

If convergence is not achieved, the process resumes from step 3; otherwise, proceed to the last step.

IV. Grid Testing and Code Validation

To test and assess the grid independence of the solution method, numerical simulations are performed as shown in Table 1. The grid quantity is used as

$$N = \sum_{n=1}^{N} 2\pi R_i + n \frac{R_o - R_i}{n_y} d_{xy}$$

(28)

where $n_y$ indicates the number of grid layers in the radial direction, and $d_{xy}$ denotes the distance between nodes on the same layer. To make the lattice Boltzmann method operate properly, the value of $d_{xy} = 1.5$ is adopted.

To evaluate the performance of the present method, the averaged equivalent thermal conductivity $k_{eq}$ is derived for natural convection in a concentric horizontal annulus and the results are compared with those of Kuehn and Goldstein [5] in Fig. 3 and Table 1, finding good agreement between the isotherms and streamlines obtained from the present work and those taken from the literature. The relative error in

![Fig. 3 Averaged equivalent thermal conductivity as a function of the angle.](image)

![Fig. 4 Transient development of flow patterns for $Pr = 0.3$ and $Ra 10000$.](image)

![Fig. 5 Isotherms at $A = 2$ and $Pr = 0.3$: a) $Ra = 2.0 \times 10^4$; b) $Ra = 3.0 \times 10^4$; and c) $Ra = 1.0 \times 10^5$.](image)
terms of equivalent thermal conduction is less than 2% for $Ra = 10^3$. As can be seen from Fig. 3, the mean equivalent thermal conductivities of both cylinders, at different Rayleigh numbers, closely match the values from [5].

![Graph](image)

**Fig. 6** Variation of dimensionless values (horizontal and vertical velocity, temperature) at $Pr = 0.3$: a) $Ra = 2.0 \times 10^4$; b) $Ra = 3.0 \times 10^4$; and c) $Ra = 1.0 \times 10^5$.

V. Results and Discussions

In the present study, the effects of the Rayleigh number, the Prandtl number, and the aspect ratio on natural convection in a horizontal concentric annulus are investigated using the LBE method as discussed previously. Calculations are performed in the parameter ranges of $10^3 \leq Ra \leq 10^6$, $0.1 \leq Pr \leq 0.7$, and $0.4 \leq A \leq 10$; and the initial condition is assumed to be $u = v = \Theta = 0$ with the inner cylinder suddenly heated to $\Theta = 1$.

A. Effects of Rayleigh Number

We first investigate the effects of the Rayleigh number on the stability and the transitions of natural convection in a horizontal concentric annulus. In the simulations discussed in this section, the aspect ratio and the Prandtl number are fixed at 2 and 0.3, respectively.

The flow patterns are here labeled according to the classification by Yoo [30] based on the flow direction atop the annulus. The flow pattern in which the fluid ascends along the center plane is called “upward” flow; the other, which descends, is called “downward” flow. The variation of flow patterns at $Ra = 10^3$ is displayed in Fig. 4, where the fluid is seen to form a crescent-shaped eddy at the early stage ($t = 0.0949$). In particular, the fluid rises near the inner hot cylinder and sinks near the outer cold cylinder, forming a kind of upward flow in which the fluid ascends in the top of the annulus ($\Theta = 0$). The flow patterns gradually change with time (Figs. 4b and 4c). At $t = 0.269$, the downward flow pattern is established with the fluid descending in the top of the annulus ($\Theta = 0$) by forming two counter-rotating eddies (see Fig. 4d). This kind of transition is in agreement with Yoo’s description. However, at a later stage, the flow pattern transforms to upward again, as illustrated in the figure for $t = 12.65$ (Fig. 4g) and eventually maintains this steady upward flow (see Fig. 4h). The detailed transition process from downward flow to upward flow is shown in Figs. 4e and 4f.

To systematically study the effects of the Rayleigh number, a series of Rayleigh numbers (in the range $10^3 \leq Ra \leq 10^6$) are selected to study the transition phenomenon, and the final flowfields are presented in Fig. 5. When the Rayleigh number is rather small (i.e., $Ra \leq 6.0 \times 10^3$), the upward flow prevails at the beginning until the steady upward flow pattern is established. At higher values of the Rayleigh number (i.e., $Ra = 10^5$), the downward flow occurs at the beginning and then transforms to steady upward flow. At $Ra = 2.0 \times 10^5$, a right side-skewed downward flow is observed after an initial phase of symmetric downward flow. The trends of dimensionless velocity and temperature over time, corresponding to the flows of Fig. 5, are displayed in Fig. 6, which also reports the transition of the flow pattern and the oscillation phenomenon. From

![Graph](image)

**Fig. 7** Equivalent thermal conductivity of the inner cylinder at $Pr = 0.3$ when the inner cylinder is suddenly heated as an initial condition.
the figure, we can see that, at $Ra = 2.0 \times 10^4$, the dimensionless values increase sharply to a certain level. The transition process is indicated by a gradually appearing steady decrease, and the steady flow pattern is achieved with a similar right downward-skewed shape. At $Ra = 3.0 \times 10^4$, the flow becomes oscillatory with a regular periodicity. At $Ra = 1.0 \times 10^5$, the steady upward flow pattern is achieved again, after an initial stage of downward flow.

To understand the effects of the Rayleigh number on flow pattern, we performed simulations for different Rayleigh numbers; the results are shown in Fig. 7. When the inner cylinder is suddenly heated, the steady upward flow can be established in the ranges $Ra \leq 1.5 \times 10^4$, $2.45 \times 10^4 \leq Ra \leq 2.6 \times 10^4$, and $9.9 \times 10^4 \leq Ra \leq 1.1 \times 10^5$. The steady right-skewed downward flow is maintained at $1.55 \times 10^4 \leq Ra \leq 2.4 \times 10^4$, whereas the oscillatory flow appears at $2.65 \times 10^4 \leq Ra \leq 9.7 \times 10^4$. The presence of these patterns can be explained by noting that, for all the simulation conditions, momentum diffusion is weak (i.e., $Pr = 0.3$) and the flow itself is not very stable. When the Rayleigh number is low, thermal diffusion is a predominant factor. The flow pattern evolves into an upward flow as the Rayleigh number increases gradually. The flow of buoyancy lift gradually increases, and momentum diffusion becomes stronger, although it cannot balance the flow due to heat transfer. The flow pattern in these conditions is a periodical downward flow. If the Rayleigh number is further increased, momentum diffusion becomes the predominant factor and the flow pattern evolves into an oscillation upward flow.

The influence of initial conditions is also studied, and the results are shown in Fig. 8. The initial upward flow patterns are the same as discussed before, whereas the range of Rayleigh numbers leading to the steady upward flow can be extended to $Ra = 3.89 \times 10^4$; although, the flow symmetry at $3.5 \times 10^4 \leq Ra \leq 3.89 \times 10^4$ is not complete, as can be seen in Fig. 5c. The range of values leading to the steady right-skewed flow can be extended to $Ra \leq 2.85 \times 10^4$, anticipating the formation of the right-skewed flow while maintaining the zone of oscillatory flow in the same range. Because the periodic oscillation can be regarded as a kind of dynamic bifurcation, it is in fact a combination of many solutions. We can conclude from Fig. 8 that, because of different initial conditions, with an increasing Rayleigh number, dual solutions exist for $1.55 \times 10^4 \leq Ra \leq 2.65 \times 10^4$ and multiple solutions are indeed valid for $2.65 \times 10^4 \leq Ra \leq 9.5 \times 10^4$. The transitions between flow pattern regimes follow a clear route (i.e., the flow pattern evolves from steady to oscillating, then back to steady, and then to oscillating again) when the Rayleigh number is sufficiently high.

Fig. 8 Equivalent thermal conductivity of the inner cylinder at $Pr = 0.3$ when the inner cylinder is suddenly heated as an initial condition.

Fig. 9 Isotherms at $Pr = 0.7$ and $Ra = 5.0 \times 10^4$ with different aspect ratios: a) $A = 2$, b) $A = 3$, and c) $A = 4$.

Fig. 10 Time variation of dimensionless temperature at $A = 2$, $A = 3$, and $A = 4$. 
As mentioned previously, the most important parameter governing fluid patterns transition is the Rayleigh number, for which the increase can change the state of flow and heat transfer from steady to oscillatory.

B. Effects of Aspect Ratio

We now consider the effects of the aspect ratio on natural convection in a horizontal concentric annular system. In the following simulations, the Prandtl number is kept constant at $Pr = 0.7$.

The isotherms corresponding to different aspect ratios ($A = 2$, $3$, and $4$) at $Ra = 5.0 \times 10^3$ are illustrated in Fig. 9, which clearly shows that all the obtained isotherms are stable, although they are not characterized by the same flow conditions. For the case of an aspect ratio of $A = 2$, the flow is initially upward and evolves into steady, with an according temperature stabilization (see Fig. 10). For the cases of aspect ratios of $A = 3$ and $A = 4$, the flow transforms from downward to steady upward.

We also consider the case of larger aspect ratios, obtaining the results shown in Fig. 11. When $A = 6$, the flow forms a stable crescent-shaped eddy, whereas a pair of cells appear at the top part of the annulus for $A = 8$ and $A = 10$. This is mainly caused by the wall effect, which occurs in a limited space by interaction between the flow and thermal boundary layers. Moreover, the formation of flow patterns at $A = 6$ undergoes a transition from a less symmetric downward flow to a steady upward flow, whereas the upward flow is maintained at all times when $A = 8$. The oscillatory upward flow appears at $A = 10$ (see Fig. 11). We also observe a temperature increase until a steady level is reached (see Fig. 12).

From the aforementioned results, we can conclude that the transition phenomenon does not occur when the aspect ratio is small.

We now discuss the relationship between flow patterns and Rayleigh numbers. A classification of the flow patterns at different aspect ratios on a wide range of Rayleigh numbers is reported in Fig. 13, where TIUF represents time-independent upward flow, TF represents transition flow, POF represents periodic oscillatory flow, and NPOF represents nonperiodic oscillatory flow. As can be seen in the figure, for $A = 3$, the transition occurs for Rayleigh numbers in the range $3.4 \times 10^3 \leq Ra \leq 6.1 \times 10^4$, whereas for $6.3 \times 10^3 \leq Ra \leq 4.1 \times 10^7$, the upward flow is maintained until the steady state is achieved. For $Ra \geq 4.3 \times 10^7$, a periodic oscillatory upward flow occurs. At $A = 4$, interestingly, the oscillation phenomenon occurs for $2.5 \times 10^7 \leq Ra \leq 3.2 \times 10^8$ because the hydrodynamic effect is not strong and the flow in this limited space is strongly affected by the heat transfer process. After that, the transition occurs at $3.4 \times 10^4 \leq Ra \leq 4.7 \times 10^4$ and the steady upward flow is sustained in the range $4.9 \times 10^4 \leq Ra \leq 3.5 \times 10^6$. Another transition is found when $3.8 \times 10^4 \leq Ra \leq 2.6 \times 10^5$, leading to a periodic oscillatory upward flow at $Ra \geq 2.8 \times 10^5$. When $A = 6$, the upward flow is present at $Ra \leq 2.3 \times 10^3$ and transition flow occurs at $2.4 \times 10^3 \leq Ra \leq 6.1 \times 10^3$. Periodic oscillatory flow happens at $6.5 \times 10^3 \leq Ra \leq 2.2 \times 10^4$. Time-independent upward flow is formed again at $Ra = 2.5 \times 10^4$, and another transition in the flow pattern occurs at $2.8 \times 10^4 \leq Ra \leq 5.5 \times 10^4$. Then, periodic oscillations are established at $Ra = 6.0 \times 10^4$ and nonperiodic oscillations occur at $6.8 \times 10^4 \leq Ra \leq 8.5 \times 10^4$. Another transition in the flow pattern occurs at $9.8 \times 10^4 \leq Ra \leq 2.5 \times 10^5$, and nonperiodic oscillatory flow is eventually established at $Ra \geq 3.5 \times 10^5$. When $A = 8$, time-independent upward flow is found at $Ra \leq 2.8 \times 10^5$ and the first transition in the flow pattern occurs at $2.9 \times 10^5 \leq Ra \leq 3.3 \times 10^5$. Periodic oscillatory flow is then established at $3.5 \times 10^5 \leq Ra \leq 8.0 \times 10^7$, and nonperiodic oscillatory flow occurs at $8.5 \times 10^7 \leq Ra \leq 4.7 \times 10^8$. If the Rayleigh number is further increased, a transition in the flow behavior occurs at $4.9 \times 10^8 \leq Ra \leq 2.5 \times 10^9$ and nonperiodic

![Fig. 11 Streamlines at Pr = 0.7 and Ra = 3.0 x 10^3 for a) A = 6, b) A = 8, and c) A = 10.](image1)

![Fig. 12 Time variation of dimensionless temperature at A = 6, A = 8, and A = 10.](image2)

![Fig. 13 Categorization of flow patterns in a horizontal concentric cylindrical annulus; above the solid line, the flow becomes oscillatory.](image3)
oscillatory flow is finally established at $Ra \geq 3.5 \times 10^5$. When $A = 10$, as the distance between the two cylinders is very small, the transition phenomenon seldom occurs and the flow transforms directly from steady upward flow to periodic (at first) and to nonperiodic oscillations with the increase of Rayleigh number. This analysis allows us to state that, in general, nonperiodic oscillations are only found for $A \geq 6$. The solid line in Fig. 13 divides the region of oscillatory flow patterns from the steady flow zone.

Figure 14 shows the averaged equivalent thermal conductivity of the inner cylinder for different aspect ratios on a wide range of Rayleigh numbers. As can be seen from the figure, no oscillations are found for $A \leq 3$; the oscillation zones for $A = 4$ and $A = 6$ are also shown, significantly pointing out the effects of variations in the aspect ratio.

To summarize the results of this analysis, the increase of parameter $A$ corresponds to a narrower gap width, which means that the thermal boundary layer and the flow boundary layer influence each other, making the flow behavior more chaotic. The critical Rayleigh number for the transition of the flow and heat transfer from a steady state to an oscillatory state becomes lower.

C. Effects of Prandtl Number

To investigate the effects of the Prandtl number on the characteristics of natural convection in a horizontal concentric annulus, the flow behavior and properties are studied as a function of the Prandtl number ($0.1 \leq Pr \leq 0.7$), whereas the aspect ratio $A$ is fixed at two.

Yoo investigated [31] dual steady solutions for the specific case of $Pr = 0.7$, obtaining the downward flow by introducing artificial numerical disturbances. The upper part of an annulus with a heated inner cylinder is thermally unstable; accordingly, the instability of the crescent-shaped upward flow that yields downward flow can persist, if the unstable stratification in the top region is sufficiently strong.

Our results in terms of the equivalent thermal conductivity of the inner cylinder are presented with $Pr = 0.2$ for a wide range of Rayleigh numbers in Fig. 15, in which SU represents steady upward flow, OU represents oscillatory upward flow, SLSD represents steady left-skewed downward flow, SRSD represents steady right-skewed downward flow, OBLRSD represents oscillatory between left- and right-skewed downward flow, ORSD represents oscillatory right-skewed downward flow, OLSD represents oscillatory left-skewed downward flow, OBUD represents oscillatory between upward and downward flow, and SSD represents steady symmetric downward flow. The figure shows that nine different types of flow patterns appear for different Rayleigh numbers. Specifically, at $Ra \leq 4.3 \times 10^3$, time-independent steady upward flow is found, whereas at $4.7 \times 10^3 \leq Ra \leq 6.5 \times 10^3$, oscillatory upward flow is established. With further increase of the Rayleigh number, steady left- and right-skewed downward flows appear at $7.0 \times 10^3 \leq Ra \leq 4.2 \times 10^4$ and $4.3 \times 10^4 \leq Ra \leq 7.5 \times 10^4$, respectively. Afterward, the oscillatory downward flow swings from left to right as the Rayleigh number ranges from $7.7 \times 10^4$ to $1.44 \times 10^5$. We remark that this kind of oscillation is different from the oscillation described in the previous subsection with $Pr = 0.3$. In fact, this oscillation has a fairly small amplitude and the flow mostly fluctuates from left- to right-skewed with a rather small deviation angle from the center symmetric line, as opposed to the large deviation angle found at $Pr = 0.4$. At higher Rayleigh numbers, oscillatory left- and right-skewed downward flows appear at $1.45 \times 10^5 \leq Ra \leq 1.65 \times 10^5$ and $1.7 \times 10^5 \leq Ra \leq 1.93 \times 10^5$, respectively. The oscillatory flow between upward and downward occurs when the Rayleigh number is approximately $1.95 \times 10^5$. Then, a steady symmetrical downward flow is established at $1.98 \times 10^5 \leq Ra \leq 2.15 \times 10^5$. Beyond this range, the oscillatory right-skewed downward flow appears again. We may thus conclude that, for $Pr = 0.2$, the transitions in the flow pattern do not follow the same order as we indicated for the case of $Pr = 0.3$; but they follow a different, more complex route, which can
be schematized as a “steady, oscillation, steady, oscillation, steady, oscillation” sequence.

For \( Pr = 0.4 \), the same route situation can also be observed on the whole range of Rayleigh numbers, as shown in Fig. 16. For \( Ra \leq 2.4 \times 10^4 \) or \( Ra \geq 4.7 \times 10^4 \), a steady upward flow pattern is observed, whereas the oscillatory flow appears in the intermediate range \( 2.5 \times 10^4 \leq Ra \leq 4.5 \times 10^4 \). The simpler transition route “steady, oscillation, steady, oscillation” is found in this case as well, which is also consistent with existing studies.

Figure 17 shows the averaged equivalent thermal conductivity for different Prandtl numbers, with a fixed aspect ratio of \( A = 2 \) and a Rayleigh number of \( Ra = 10^4 \). As can be seen from the figure, in this region, \( k_{eq} \) exhibits a sharp decrease when the Prandtl number increases from 0.2 to 0.3 because the flow is downward at \( Pr = 0.2 \). However, further increasing the Prandtl number results in a gradual increase of \( k_{eq} \) due to the upward flowfield in this region. These results allow us to state that the transition does not appear at \( Pr \geq 0.4 \). The critical line beyond which the periodic oscillatory upward flow occurs is displayed in Fig. 18. The critical Rayleigh number increases with the increase of Prandtl number for the case of \( A = 2 \).

As a final conclusion to this analysis, we may state that increasing the Prandtl number reduces the amount of possible flow pattern regimes.

![Fig. 16 Equivalent thermal conductivity of the inner cylinder for different Rayleigh numbers at \( Pr = 0.4 \).](image1)

![Fig. 17 Equivalent thermal conductivity of the inner and outer cylinders at \( A = 2 \) and \( Ra = 10^4 \) at different Prandtl numbers.](image2)

![Fig. 18 Critical Rayleigh numbers for different Prandtl numbers at \( A = 2 \).](image3)

**VI. Conclusions**

The lattice Boltzmann method is employed to simulate natural convective flow in a horizontal concentric annulus, focusing on its stability and on the transitions between different flow patterns. The effects of the Rayleigh number, the annulus aspect ratio, and the Prandtl number on flow patterns and on the temperature distribution are systematically investigated. Different scenarios of transitions are studied, revealing two important features of flows:

1) At low Rayleigh numbers \( (Ra < 10^4) \) and with an aspect ratio of \( A = 2 \), the presence of crescent-shaped or kidney-shaped eddies is revealed, regardless of the values of the Prandtl number and aspect ratio. When \( Pr = 0.2 \), the evolution of flow patterns as the Rayleigh number is increased can be schematized as a threefold switching between steady and oscillation. The transition scheme is simpler (i.e., steady, oscillation, steady, oscillation) at \( Pr = 0.3 \) and \( Pr = 0.4 \). At \( Pr \geq 0.5 \), the transitional flow does not occur, leading to a single “steady, oscillation” switching sequence.

2) For aspect ratios of \( A \leq 2 \), the flow keeps upward until stabilizing to steady upward flow and no transition occurs over the whole investigated range of Rayleigh numbers of \( 10^3 \leq Ra \leq 5.0 \times 10^5 \). For \( A \geq 3 \), the transition in flow pattern occurs at certain Rayleigh numbers. When \( A = 3 \), the flow patterns undergo a transitional route that can be schematized with the sequence “sustained stable flow, transitional flow, sustained stable flow, periodic oscillatory flow.” When \( A = 4 \), a more complex transition route is observed with the increase of the Rayleigh number, namely, “sustained stable flow, periodic oscillatory flow, transitional flow, sustained stable flow, transitional flow, periodic oscillatory flow.” Finally, when \( A \geq 6 \), the nonperiodic oscillatory flow occurs after the appearance of periodic oscillatory flow.

3) The Rayleigh number is the most important parameter governing flow pattern transitions. As the Rayleigh number increases, the state of flow and heat transfer evolves from steady to oscillatory.

4) When the aspect ratio is increased, the critical Rayleigh number for the transition of the flow and heat transfer from steady state to oscillatory state becomes lower.

5) When the Prandtl is increased, fewer different flow patterns are found for the fluid in the investigated horizontal annulus.

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**References**


