A HYBRID LATTICE BOLTZMANN AND MONTE CARLO METHOD FOR NATURAL CONVECTION SIMULATION

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A hybrid lattice Boltzmann and Monte Carlo method is proposed to solve the natural convection problem. The lattice Boltzmann method solves the velocity field while the temperature field is obtained by the Monte Carlo method. The two-dimensional nine discrete velocities (D2Q9) model is used in the lattice Boltzmann method, and random walker is employed in the Monte Carlo method. Natural convection is via a differentially heated rectangular enclosure under different Rayleigh numbers which are solved for validation and the results agree well with the benchmark. It indicates the proposed hybrid method is valid for the natural convection simulation.

KEY WORDS: lattice Boltzmann method, Monte Carlo method, natural convection

1. INTRODUCTION

The lattice Boltzmann method (LBM) has been developed into a promising numerical method for fluid flow and heat transfer problems in the last two decades. Chen et al. (1991) and Qian et al. (1992) proposed the lattice Bhatnagar-Gross-Krook model (LBGK); the Navier-Stokes equation could be obtained from this model and it also reduced the computing time significantly. Since then, the LBGK has become a widely used approach in LBM. Chen and Doolen (1998) reviewed the LBM theory and its isothermal applications that velocity, density, and pressure can be obtained from density distribution in LBM. It has been used to solve incompressible flow (Guo and Zhao, 2002), flow in porous media (Tang et al., 2005), and multiphase flow (Luo, 2000).

On the other hand, the thermal LBM solves the fluid flow problem with heat transfer. The thermal LBM falls into two categories: the multispeed (MS) and the multidistribution function (MDF) approaches (Orazio and Succi, 2004). The MS approach obtains the temperature field by adding more discrete velocities to the density distribution (Chen et al., 1994), while MDF employs an additional independent distribution to solve the energy equation (Li et al., 2015). However, both methods have some limitations: the MS approach has limits on numerical instability and a narrow range of temperature variation, while MDF must assume that the Mach number of the flow is small and the density varies slowly to obtain the correct macroscopic equations (Guo et al., 2002).

The main advantage of LBM lies in the velocity field simulation process (Succi, 2001). Therefore, one can use LBM to solve the velocity field only and apply another numerical method to solve the temperature field for the fluid flow and heat transfer problems. There are several reported results about combining LBM with other numerical methods, such as the finite difference method (FDM) and the finite volume method (FVM). Mezrhab et al. (2004) used hybrid LBM and FVM to solve convective flow (Mezrhab et al., 2004). Mishra and Roy solved transient conduction and radiation heat transfer problems using the LBM and FVM (Mishra and Roy, 2007). Li et al. (2014a) applied hybrid LBM and FVM to solve a melting problem with natural convection in a rectangular enclosure.
2. ALGORITHM

The Monte Carlo method (MCM) is a widely used numerical method for heat transfer problems (Howell, 1998). It can solve conduction (Haji-Sheikh and Sparrow, 1961), convection (Chandler et al., 1968), and radiation (Perlmutter and Howell, 1964) heat transfer problems. In addition, both MCM and LBM belong to the mesoscopic scale method. It is possible to combine these two methods to solve convection heat transfer problems. However, there is no report on combining these two methods for a convection heat transfer problem. The objective of this paper is to verify the hybrid LBM-MCM method by solving a natural convection in a differentially heated enclosure. The Boussinesq assumption is applied to the simulation process in order to treat the fluid as incompressible. The results are compared with the benchmark results in Davis (1983).

2.1 Lattice Boltzmann Method for Fluid Flow

The lattice Boltzmann equation can describe the statistical behavior of a fluid flow:

\[ f(r + e \Delta t, t + \Delta t) - f(r, t) = \Omega + F, \]  

where \( f \) is the density distribution, \( \Omega \) is the collision term, and \( F \) is the body force. Since the problem under consideration is two-dimensional, the D2Q9 model is preferred. In this model, the discrete velocities are shown in Fig. 1 and they are defined as follows:

\[ e_i = \begin{cases} 
(0, 0) & i = 1 \\
 c \left( - \cos \frac{i\pi}{2}, - \sin \frac{i\pi}{2} \right) & i = 2, 3, 4, 5 \\
 \sqrt{2c} \left( - \cos \frac{(2i + 1)\pi}{4}, - \sin \frac{(2i + 1)\pi}{4} \right) & i = 6, 7, 8, 9 
\end{cases} \]
where \( c \) is the lattice speed that relates to the speed of sound, \( c_s \), as
\[
c^2 = 3c_s^2. \tag{3}
\]
Then the macroscopic properties (velocity \( V \), pressure \( P \), and density \( \rho \)) can be obtained from the density distribution:
\[
\rho = \sum_{i=1}^{9} f_i, \tag{4}
\]
\[
\rho V = \sum_{i=1}^{9} e_i f_i, \tag{5}
\]
\[
p = \rho c_s^2. \tag{6}
\]
In the LBGK model, the collision term in Eq. (1) is assumed as
\[
\Omega_i = \frac{f_i^{eq}(r,t) - f_i(r,t)}{\tau_v} \quad i = 1, 2, \ldots 9, \tag{7}
\]
where \( \tau_v \) is the relaxation time and \( f_i^{eq} \) is the equilibrium distribution function:
\[
f_i^{eq} = \rho \omega_i \left[ 1 + \frac{e_i \cdot V}{c_s^2} + \frac{(e_i \cdot V)^2}{2c_s^4} - \frac{V \cdot V}{2c_s^2} \right], \tag{8}
\]
where
\[
\omega_i = \begin{cases} 
4/9 & i = 1 \\
1/9 & i = 2, 3, 4, 5 \\
1/36 & i = 6, 7, 8, 9.
\end{cases} \tag{9}
\]
Equation (1) is then simplified to
\[
f_i (r + e_i \Delta t, t + \Delta t) - f_i (r, t) = \frac{1}{\tau_v} \left[ f_i^{eq}(r,t) - f_i(r,t) \right] + F_i, \quad i = 1, 2, \ldots 9, \tag{10}
\]
where the force term can be obtained as
\[
F_i = \Delta t G \cdot \frac{(e_i - V)}{p} f_i^{eq}, \tag{11}
\]
where $G$ is the effective gravitational force,

$$G = -\beta (\theta - \theta_i) g.$$  \hfill (12)

The density distribution can lead to the Navier-Stokes equation using the Chapman-Enskog expansion (Peng et al., 2003) when the relaxation time is defined as

$$\nu = \frac{c_s^2}{\tau_v} \left( \frac{1}{2} \right) \Delta t,$$  \hfill (13)

where $\nu$ is the kinematic viscosity.

### 2.2 Monte Carlo Method (MCM) for Heat Transfer

A rectangular grid system of mesh size $\Delta x \times \Delta y$ is selected for the two-dimensional computational domain shown in Fig. 2. The inner computing node $(i, j)$ relates to its surrounding nodes by

$$T_{i,j} = P_{x+} T_{i+1,j} + P_{x-} T_{i-1,j} + P_{y+} T_{i,j+1} + P_{y-} T_{i,j-1},$$  \hfill (14)

where the possibilities $P_{x+}$, $P_{x-}$, $P_{y+}$ and $P_{y-}$ are positive; their sum has to be 1.

For the conduction heat transfer problem, these possibilities are

$$\begin{cases}
P_{x+} = P_{x-} = \frac{\Delta y/\Delta x}{2(\Delta y/\Delta x + \Delta x/\Delta y)} \\
P_{y+} = P_{y-} = \frac{\Delta x/\Delta y}{2(\Delta y/\Delta x + \Delta x/\Delta y)}
\end{cases}$$  \hfill (15)

For the convection heat transfer problem, the horizontal velocity $u$ and vertical velocity $v$ have effects on $P_{x+}$, $P_{x-}$, $P_{y+}$, and $P_{y-}$. The possibilities should be modified to the following equation:

![FIG. 2: Computing grid](image)

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\[
\begin{align*}
\mathcal{P}^+ &= \frac{\alpha \Delta y/\Delta x - u \Delta y}{D} \quad \mathcal{P}^- = \frac{\alpha \Delta y/\Delta x}{D} \\
\mathcal{P}^y &= \frac{\alpha \Delta x/\Delta y - v \Delta x}{D} \quad \mathcal{P}^y = \frac{\alpha \Delta x/\Delta y}{D},
\end{align*}
\]

where
\[
D = 2 (\Delta y/\Delta x + \Delta x/\Delta y) - u \Delta y - v \Delta x.
\]

The statistical procedure MCM uses random walkers to solve the heat transfer problem (Minkowycz et al., 2006). A random walker locates at node \((i, j)\) at the beginning and a random number \(RN\) is chosen in the uniform distribution set from 0 to 1. This random walker will change its position by the following rules:

\[
\begin{align*}
\text{if } 0 < RN < \mathcal{P}^+ & \text{ from } (i, j) \text{ to } (i + 1, j) \\
\text{if } \mathcal{P}^+ < RN < \mathcal{P}^+ + \mathcal{P}^y & \text{ from } (i, j) \text{ to } (i, j + 1) \\
\text{if } \mathcal{P}^+ + \mathcal{P}^y < RN < \mathcal{P}^+ + \mathcal{P}^y + \mathcal{P}^- & \text{ from } (i, j) \text{ to } (i - 1, j) \\
\text{if } 1 - \mathcal{P}^- < RN < 1 & \text{ from } (i, j) \text{ to } (i, j - 1).
\end{align*}
\]

Once the random walker has completed its first step, the procedure continues for the second step. This process goes on until that random walker reaches the boundary. Then the boundary condition \(T_w(1)\) is recorded for the inner node \((i, j)\). This process is repeated \(N\) times and the recorded boundary conditions are \(T_w(2)\) to \(T_w(N)\). With these \(N\) results, the MCM estimation for \(T(i, j)\) can be expressed as

\[
T(i, j) = \frac{1}{N} \sum_{n=1}^{N} T_w(n).
\]

Then all the inner computing nodes can be obtained by this method. The treatments for different kinds of boundaries can be found in Minkowycz et al. (2006).

### 2.3 Hybrid LBM-MCM Method

A hybrid LBM-MCM method is designed to solve a fluid flow and heat transfer problem with LBM and MCM simultaneously. The same grid system is used in both LBM and MCM. The velocity field is obtained by LBM while MCM solves the temperature field. Figure 3 shows the flowchart for the hybrid method. For the natural convection problem under consideration, temperature and velocity have effects on each other. Therefore this method needs to run LBM and MCM in sequence for each step. Prevost (2014) and Sun et al. (2013) pointed out that multiphysical sequential solvers without iteration may lead to wrong solutions in the strong coupling cases. To avoid this, MCM is iterated until the convergence criterion is satisfied in each step.

After setting the initial condition, LBM solves the velocity field with the initial temperature field. Then the temperature field is obtained by MCM with the LBM velocity result. This temperature can be applied to LBM in the next step. This process is repeated until converged results are reached.

### 3. PROBLEM STATEMENT

Incompressible fluid natural convection in a square enclosure is shown in Fig. 4. For the velocity field, a non-slip boundary condition is applied to all boundaries. The left boundary is kept at a constant temperature \(T_h\) while the right boundary has a lower constant temperature of \(T_c\). The top and bottom boundaries are adiabatic. Applying the Boussinesq assumption, the problem can be described by the following governing equations:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.
\]
Equations (1)–(4) are subject to the following boundary conditions:

\[
x = 0, \quad u = 0, \quad v = 0, \quad T = T_h, \tag{24}
\]

\[
x = H, \quad u = 0, \quad v = 0, \quad T = T_c, \tag{25}
\]
\[ y = 0, \ u = 0, \ v = 0, \ \partial T / \partial y = 0, \] (26)
\[ y = H, \ u = 0, \ v = 0, \ \partial T / \partial y = 0. \] (27)

The velocity field will be solved by LBM; therefore the nondimensional process is based on LBM settings. Applying the following nondimensional variables,
\[
\begin{align*}
X &= \frac{x}{H}, \quad Y = \frac{y}{H}, \quad u_c = \sqrt{g \beta (T_h - T_m) H}, \quad Ma = \frac{u_c}{c_s}, \\
\tau &= \frac{t \sqrt{3c_s}}{H}, \quad \theta = \frac{T - T_c}{T_h - T_c}, \quad P = \frac{p}{3c_s^2}, \quad Pr = \frac{v}{\alpha}, \quad Ra = \frac{g \beta (T_h - T_c) H^2 Pr}{\sqrt{2}},
\end{align*}
\] (28)

to Eqs. (20)–(27), one can reach
\[
\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0,
\] (29)
\[
\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = - \frac{\partial P}{\partial X} + Ma \sqrt{Pr} \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right),
\] (30)
\[
\frac{\partial V}{\partial \tau} + U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} = - \frac{\partial P}{\partial Y} + Ma \sqrt{Pr} \left( \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) + \frac{Ma^2 \theta}{3},
\] (31)
\[
\frac{\partial \theta}{\partial \tau} + U \frac{\partial \theta}{\partial X} + V \frac{\partial \theta}{\partial Y} = Ma \sqrt{\frac{1}{3 Ra \cdot Pr}} \left( \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2} \right),
\] (32)
\[
X = 0, \quad U = 0, \quad V = 0, \quad \theta = 1,
\] (33)
\[
X = 1, \quad U = 0, \quad V = 0, \quad \theta = 0,
\] (34)
\[
Y = 0, \quad U = 0, \quad V = 0, \quad \partial \theta / \partial Y = 0,
\] (35)
\[
Y = 1, \quad U = 0, \quad U = 0, \quad \partial \theta / \partial Y = 0.
\] (36)

Nusselt number Nu is the ratio of convection to conduction heat transfer across the boundary. It shows the convection effect to the heat transfer process,
\[
Nu = \frac{h}{k/H} \bigg|_{x=0} = - \frac{\partial \theta}{\partial X} \bigg|_{x=0}.
\] (37)

4. RESULTS AND DISCUSSIONS

4.1 Validation of LBM

A pure incompressible fluid flow problem is solved to verify LBM. As shown in Fig. 5, a cavity is filled with incompressible fluid and a constant horizontal velocity \(u_c\) at the top of the cavity which drives the inside fluid flow. All other boundaries are no-slip. The governing equations and the nondimensionalization process based on LBM variables can be found in Li et al. (2014b). The Reynolds number is defined as
\[
Re = \frac{u_c H}{\nu},
\] (38)

where \(H\) is the cavity height and \(\nu\) is the kinematic viscosity.

Three cases are solved for Re equaling 100, 400, and 1000, respectively. Figure 6 shows their streamlines. For the case of Re = 100, one primary vortex exists in the cavity and its center is (0.62, 0.74). Another two vortexes locate at the left and right bottom corners of the cavity, respectively. When Re increases to 400, there are still three vortexes in the cavity at similar locations. In this case, the primary vortex center is (0.56, 0.61). The steam is much stronger with Re increasing to 1000 and the primary vortex center changes to (0.53, 0.57). The benchmark solutions for these three cases can be found in Ghia et al. (1982). They also reported three vortexes in the cavity for these three cases.
the primary vortex locations of these three cases are (0.62, 0.73), (0.55, 0.61), and (0.53, 0.56), respectively. The differences between LBM results and that in Ghia et al. (1982) are insignificant for all three cases. Meanwhile, these three LBM streamline tendencies are also similar to the benchmark solutions. Therefore, the LBM in this paper can give good predictions to this incompressible fluid flow.

4.2 Validation of MCM

As discussed in Section 3.2, MCM has the same approach in solving conduction and convection heat transfer problems. A pure conduction problem is solved to test the MCM. In Fig. 7, the left, right, and bottom of the two-dimensional domain are kept at $T_c$ and the top boundary has a higher temperature of $T_h$. It is assumed that the thermal conductivity is independent from the temperature. The energy equation for this problem is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad 0 < x < H, \quad 0 < y < H,$$

(39)

with the following boundary conditions:

$$T = T_c, \quad x = 0 \quad \text{or} \quad H, \quad 0 < y < H,$$

(40)
FIG. 7: Two-dimensional steady-state heat conduction

\[ T = T_c, \quad y = 0, \quad 0 < x < H, \quad (41) \]

\[ T = T_h, \quad y = H, \quad 0 < x < H. \quad (42) \]

Defining the following nondimensional variables,

\[
\begin{align*}
X &= \frac{x}{H}, \quad Y = \frac{y}{H}, \\
\theta &= \frac{T - T_c}{T_h - T_c},
\end{align*}
\quad (43)
\]

Eqs. (39)–(42) become

\[ \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2} = 0, \quad 0 < X < 1, \quad 0 < Y < 1, \quad (44) \]

with the following boundary conditions:

\[ \theta = 0, \quad X = 0 \quad \text{or} \quad 1, \quad 0 < Y < 1, \quad (45) \]

\[ \theta = 0, \quad Y = 0, \quad 0 < X < 1, \quad (46) \]

\[ \theta = 1, \quad Y = 1, \quad 0 < X < 1. \quad (47) \]

This problem can be solved analytically by the separation of variables method (Faghri et al., 2010):

\[ \theta = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \frac{\sinh(n\pi Y)}{\sinh(n\pi)} \sin(n\pi X). \quad (48) \]

Figure 8 shows the comparison of the analytical and MCM temperature fields. As discussed in Section 2.2, MCM is a statistical method to simulate the heat transfer process. Its results are based on a large number of random walkers’ results. The MCM isothermal lines are not that smooth due to the nature of the procedure. These two methods’ isothermal lines agree well with each other. Therefore, MCM in this paper is valid for the heat transfer problem.
FIG. 8: Conduction temperature field comparison

4.3 Natural Convection in Rectangular Enclosure

Figure 9 shows the streamlines for Case 1 that Ra and Pr are $10^4$ and 0.71, respectively. There is a vortex in the cavity due to the convection effect. Figure 10 is the temperature field obtained from the hybrid method. Both streamlines and temperature field agree very well with the benchmark solutions except for the unsmoothness of some isothermal lines. As discussed in the pure conduction problem, the unsmoothness is led by the nature of MCM and its effect on the streamlines is insignificant. Meanwhile, Table 1 summarizes maximum Nusselt number $\text{Nu}_{\text{max}}$, maximum Nusselt number location $Y_{\text{Nu}_{\text{max}}}$, and average Nusselt number $\text{Nu}_{\text{ave}}$ obtained from the LBM-MCM and Davis (1983). It can be seen that the hybrid LBM-MCM method has a good agreement with benchmark standard solutions for these three parameters for Case 1.

Case 2 is also studied when Ra grows to $10^5$ and Pr is kept at 0.71. Convection plays a more important role due to the increased Ra. Two vortexes located in the cavity are shown in Fig. 11, and the temperature field in Fig. 12 also indicates a pronounced convection effect. Table 1 also showed the comparison of $\text{Nu}_{\text{max}}$, $Y_{\text{Nu}_{\text{max}}}$, and $\text{Nu}_{\text{ave}}$. 

FIG. 9: Streamlines for Case 1

FIG. 10: Temperature field for Case 1
TABLE 1: Comparison of Nusselt number

<table>
<thead>
<tr>
<th>Results</th>
<th>Nu_{max}</th>
<th>Y_{Nu_{max}}</th>
<th>Nu_{ave}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ra = 10^4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LBM-MCM</td>
<td>3.50</td>
<td>0.15</td>
<td>2.21</td>
</tr>
<tr>
<td>Davis (1983)</td>
<td>3.53</td>
<td>0.15</td>
<td>2.24</td>
</tr>
<tr>
<td>Error</td>
<td>1.0%</td>
<td>0</td>
<td>1.4%</td>
</tr>
<tr>
<td>Ra = 10^5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LBM-MCM</td>
<td>7.38</td>
<td>0.08</td>
<td>4.38</td>
</tr>
<tr>
<td>Davis (1983)</td>
<td>7.72</td>
<td>0.08</td>
<td>4.52</td>
</tr>
<tr>
<td>Error</td>
<td>4.4%</td>
<td>0</td>
<td>3.1%</td>
</tr>
</tbody>
</table>

FIG. 11: Streamlines for Case 2

FIG. 12: Temperature field for Case 2

obtained from the present LBM-MCM with the benchmark solutions for this case. The streamlines, temperature field, and Nusselt number results in hybrid LBM-MCM method all agree well with the benchmark solutions. Similar to Case 1, the unsmooth isothermal lines effect is insignificant. Therefore the hybrid LBM-MCM method can give good predictions to these two natural convection cases. Two hybrid LBM-FVM models (Li et al., 2014c,d) were employed to solve these two natural convection cases. They can also reach the same accuracy with similar CPU time.

5. CONCLUSIONS

A pure incompressible fluid flow and a pure conduction problem are solved by LBM and MCM, respectively. The results indicate LBM and MCM are valid for those problems. Then, a hybrid LBM-MCM approach is proposed for the fluid flow and heat transfer problem. The LBM is applied to solve the velocity field and the temperature field is obtained by the MCM. This hybrid method is employed to solve two cases of natural convection in a cavity. The streamlines’ temperature field and Nusselt number obtained from the present LBM-MCM approach agree well with that of the benchmark solutions. Thus, the hybrid LBM-MCM is reliable for the natural convection simulation.

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