Inverse Estimation of Surface Heating Condition in a Finite Slab With Temperature-Dependent Thermophysical Properties

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Temperature and heat flux at the heated surface can be estimated by solving an inverse heat conduction problem (IHCP) based on measured temperature and/or heat flux at the accessible locations (e.g., back surface). Most of the previous studies used temperature measurement data in the objective function, and little work has been done for the inverse numerical algorithm based on heat flux measurement data. In this study, a one-dimensional IHCP in a finite slab is solved by using the conjugate gradient method. The heat flux measurement data are, for the first time, incorporated into the objective function for a nonlinear heat conduction problem with temperature-dependent thermophysical properties. The results clearly show that the inverse approach of using heat flux measurement data in the objective function can provide much better predictions than the traditional approaches in which the temperature measurements are employed in the objective function. Parametric studies are performed to demonstrate the robustness of the formulated IHCP algorithm by testing it for two different materials under different frequencies of the imposed heat flux along with random errors of the measured heat flux at the back surface.

INTRODUCTION

High-energy laser (HEL) weapons can remotely deliver high-power laser at the speed of light onto a military target. It is critical to know the transient of temperature in the target in order to accurately assess the resulting thermomechanical response. However, the heated surface is either inaccessible or too hot, so its temperature cannot be directly measured with thermal sensors. Similar problems can be found during reentry of a space vehicle into the atmosphere, as well as in some laser manufacturing processes [1]. Under these circumstances, the heated (front) surface temperature can be determined indirectly by solving an inverse heat conduction problem (IHCP) [2–4] based on the transient temperature and/or heat flux measured at the back surface.

To formulate an IHCP, either temperature or heat flux at some locations should be measured to provide some additional information for solving the ill-posed problem. Between them, temperature is often preferred because it can be measured with less uncertainty compared to the heat flux [5–8]. Recent studies, however, have shown that using the measured heat flux as additional information in an IHCP can increase the stability of the solution due to the less proneness to the inherent instability of the ill-posed IHCP [9, 10].

Although IHCPs have been extensively studied for different applications in the past decades, most of them used temperature measurement data in the objective function [11–24]. Little work has been done for the inverse numerical algorithm based on heat flux measurement data. Furthermore, in HEL weapon applications, the laser energy is delivered to the surface in a periodic way because of the target-spinning or atmosphere variations. Since the formulation of the IHCP is quite subjective, it is necessary to determine which formulation is more appropriate for applications with a periodic heat flux for which it may pose extra difficulties in the solution of the inverse problems.
Recently, the authors proposed a robust and error-insensitive IHCP formulation to reconstruct the front-surface heating condition by using the temperature measurement data as the boundary condition at the back surface while incorporating the heat flux measurement data in the objective function [25]. However, the thermophysical properties were assumed to be constant. As temperature significantly changes temporally during the high-power laser interaction, thermophysical properties could vary with temperature. For this case, new equations for the sensitivity problems and adjoint problems have to be derived to incorporate heat flux measurement data in the objective function. To the best knowledge of the authors, such derivations have never been reported in existing literatures.

The objective of this article is to develop an effective algorithm that can accurately recover the front surface temperature based on measured temperature and heat flux at the back surface for a target with temperature-dependent thermophysical properties. The performance of the developed algorithm is investigated for different frequency of the imposed periodic heat flux and different random error of the measured heat flux at the back surface.

MODEL DESCRIPTION

For the case when a laser beam size is much larger than the thickness of a heated target, the IHCP can be treated as a one-dimensional problem. To illustrate the methodology of the inverse heat transfer algorithms employed in this study, a finite slab with a thickness of $L^{*}$, as shown in Figure 1, is considered. Initially, the slab is uniformly at temperature $T_{0}^{*}$ and is subjected to a high intensity laser heating from $t^{*} = 0^{*}$ at its front surface. The purpose of this study is to demonstrate the effectiveness and accuracy of the proposed IHCP formulation in reconstructing the heat flux $q^{*}_{L}(t)$ and temperature $T^{*}_{L}(t)$ at the front surface of a target with temperature-dependent thermophysical properties, based on the measured temperature and heat flux at the back surface. Due to the fact that temperature measurement contains much less errors compared to the heat flux measurement [5–8], the back surface temperature $T^{*}_{L}(t)$ is used as the boundary condition and the back surface heat flux $q^{*}_{L}(t)$ is adopted in the objective function. Both the specific heat and the thermal conductivity of the slab considered are temperature dependent.

![Figure 1 Physical model.](image)

The Direct Problem

The direct problem can be expressed as follows:

$$C(T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left[ k(T) \frac{\partial T}{\partial x} \right]$$

(2)

$$T = T_{0} \quad \text{for} \quad 0 \leq x \leq L, \quad t = 0$$

(3)

$$-k(T) \frac{\partial T}{\partial x} = q_{l}(t) \quad \text{for} \quad x = 0, \ t > 0$$

(4)

$$T(L, t) = Y_{TL}(t) \quad \text{for} \quad x = L, \ t > 0$$

(5)

where $C(T)$ is temperature-dependent volume specific heat, and $k(T)$ is temperature-dependent thermal conductivity.

In the direct problem described here, the front-surface heat flux $q_{l}(t)$ and the back-surface temperature $Y_{TL}(t)$ are considered to be known. The objective of the direct problem here is to determine the transient temperature and heat flux distribution in the target.

The Inverse Problem

For the inverse problem, the heat flux at $x = 0$ is unknown and needs to be recovered, but everything else in the direct problem is known. The additional information needed in the estimation of the front surface heat flux is available from the readings of a heat flux sensor installed at the back surface. The inverse problem can thus be stated as: The heat flux measurements at the back surface, $Y_{q_L}(t)$, is utilized to recover the front surface...
heat flux. The front surface temperature is computed according to the temperature–heat flux relation defined by the classical Fourier’s law.

It must be pointed out that in the inverse approach presented in this study, the temperatures at the back surface are also measured. But they are used as a known boundary condition at the back surface.

If the measurement data are dense in time, they can be approximated as continuous. For this case, the inverse solution can be obtained by minimizing the following ordinary least-squares norm:

$$ S[q(t)] = \int_0^{t_f} \left( Y_{qL}(t) - q[L; t; q(t)] \right)^2 dt $$

where $Y_{qL}(t)$ and $q[L; t; q(t)]$ are the measured and computed heat fluxes at the back surface, respectively.

### Conjugate Gradient Method for Minimization

The iterative process based on the conjugate gradient method (CGM) [3, 4] is now derived for the estimation of unknown heat flux $q_1(t)$ by minimizing the objective function $S$ given by Eq. (6). The front surface heat flux $q_1(t)$ at iteration $k + 1$ is advanced by

$$ q_1^{k+1}(t) = q_1^k(t) - \beta^k d^k(t) $$

where $\beta^k$ is the search step size from iteration $k$ to $k + 1$, which is addressed in the next section, and $d^k(t)$ is the direction of descent (i.e., search direction), given by:

$$ d^k(t) = \nabla S[q_1^k(t)] + \gamma^k d^k-1(t) $$

which is a conjugation of the gradient direction $\nabla S[q_1^k(t)]$ at iteration $k$ and the direction of descent $d^k-1(t)$ at iteration $k-1$. The conjugate coefficient $\gamma^k$ is determined by:

$$ \gamma^k = \frac{\int_0^{t_f} \nabla S[q_1^k(t)] \cdot \nabla S[q_1^{k-1}(t)] dt}{\int_0^{t_f} \left( \nabla S[q_1^{k-1}(t)] \right)^2 dt} $$

with $\gamma^0 = 0$. To perform the iterations according to Eq. (7), the step size $\beta^k$ and the gradient of the functional $\nabla S[q_1^k(t)]$ need to be determined. To do so, a sensitivity problem and an adjoint problem are constructed in the following.

### Sensitivity Problem and Search Step Size

The sensitivity and adjoint problems can be obtained by the limiting approach described in reference [4]. It is assumed that the unknown heat flux $q_1(t)$ is perturbed by an amount $\xi \Delta q_1(t)$ with $\xi$ being a real number. Thus, the temperature $T(x, t)$ undergoes a variation $\xi \Delta T(t)$, that is:

$$ T_\xi(x, t) = T(x, t) + \xi \Delta T(x, t) $$

where the subscript $\xi$ refers to a perturbed variable.

The perturbation of temperature causes variations on the thermophysical properties. The resulting perturbed quantities are linearized as:

$$ C_\xi(T_\xi) = C(T) + \frac{dC}{dT} \cdot \xi \cdot \Delta T $$

$$ k_\xi(T_\xi) = k(T) + \frac{dk}{dT} \cdot \xi \cdot \Delta T $$

For convenience in the subsequent analysis, the differential equation (2) is rewritten as:

$$ D(T) = C(T) \frac{dT}{dt} - \frac{\partial}{\partial x} \left[ k(T) \frac{dT}{dx} \right] $$

The perturbed form of the above equation becomes:

$$ D_\xi(T_\xi) = C_\xi(T_\xi) \frac{dT_\xi}{dt} - \frac{\partial}{\partial x} \left[ k_\xi(T_\xi) \frac{dT_\xi}{dx} \right] $$

Figure 3 Thermal properties of stainless steel 304 and aluminum alloy 2024-T6 [24].
To formulate the sensitivity problem, we apply the limiting process [4] to Eqs. (13) and (14):

\[ \lim_{\xi \to 0} \frac{D_x(T) - D(T)}{\xi} = 0 \]  

(15)

A similar limiting process is employed to the boundary and initial conditions of the direct problem. After some manipulations, the sensitivity problem is described by:

\[ \frac{\partial(C\Delta T)}{\partial t} = \frac{\partial^2(k\Delta T)}{\partial x^2} \]  

(16)

\[ \Delta T(x, 0) = 0 \quad \text{for} \quad 0 \leq x \leq L, \quad t = 0 \]  

(17)

\[ -\frac{\partial(k\Delta T)}{\partial x} = \Delta q_1(t) \quad \text{for} \quad x = 0, \quad t > 0 \]  

(18)

\[ \Delta T(x, t) = 0 \quad \text{for} \quad x = L, \quad t > 0 \]  

(19)

The preceding equations are to determine the temperature variation \( \Delta T(x, t) \) that is caused by the perturbation \( \Delta q_1(t) \).

The objective function at iteration \( k + 1 \) can be obtained by replacing \( q_1 \) in Eq. (6) with \( q_1^{k+1} \) given by Eq. (7):

\[ S[q_1^{k+1}(t)] = \int_0^{t_f} [Y_q(t) - q[L, t; q_1^k - \beta^k dq^k)]^2 dt \]  

(20)

Linearizing \( q[L, t; q_1^k - \beta^k dq^k] \) by the Taylor-series expansion, Eq. (20) takes the following form:

\[ S[q_1^{k+1}(t)] = \int_0^{t_f} [Y_q(t) - q[L, t; q_1^k] - \beta^k \Delta q(dq^k)]^2 dt \]  

(21)

In Eq. (21), the heat flux \( q[L, t; q_1^k] \) is solved from the direct problem [Eqs. (2)–(5)] with the estimated \( q_1^k \) as the boundary condition at \( x = 0 \). The sensitivity function \( \Delta q(dq^k) \) is the heat flux variation at \( x = L \) and time \( t \), which is calculated using Fourier’s law based on the temperature variation \( \Delta T \) that is obtained from Eqs. (16)–(19) by letting \( \Delta q_1 = dq^k \). The calculation of the heat flux variation \( \Delta q(dq^k) \) will be addressed in detail in the next section. The search step size \( \beta^k \) can be determined by minimizing the objective function, given by Eq. (21), with respect to \( \beta^k \):

\[ \beta^k = \int_0^{t_f} [q[L, t; q_1(t)] - Y_q(t)] \cdot \Delta q[L, t; dq^k(t)] \cdot dt \]  

(22)

\[ \int_0^{t_f} \{\Delta q[L, t; dq^k(t)]\}^2 \cdot dt \]
Adjoint Problem and Gradient Equation

Multiplying the right-hand side of Eq. (13) by a Lagrange multiplier $\lambda(x, t)$, integrating over the time and space domains, and then adding the result to Eq. (6) leads to:

$$S[q_1(t)] = \int_0^{t_f} \left[ Y_{qL}(t) - q[L, t; q_1(t)] \right]^2 dt$$

$$+ \int_{x=0}^L \int_{t=0}^{t_f} \left[ C(T) \frac{\partial T}{\partial t} - \frac{\partial}{\partial x} \left[ k(T) \frac{\partial T}{\partial x} \right] \right] \cdot \lambda dx dt$$

(23)

The preceding extended functional $S[q_1(t)]$ will undergo a variation $\Delta S[q_1(t)]$ when the unknown quantity $q_1(t)$, temperature $T$, heat flux $q$, volume specific heat $C$, and thermal conductivity $k$ undergo variations $\xi \Delta q_1(t)$, $\xi \Delta T(t)$, $\xi \Delta q(t)$, $\frac{dC}{dT} \cdot \xi \cdot \Delta T$, $\frac{dk}{dT} \cdot \xi \cdot \Delta T$. The variation $\Delta S[q_1(t)]$ can be obtained by applying the following limiting process:

$$\Delta S[q_1(t)] = \lim_{\xi \to 0} \frac{S[q_1(T_0)] - S[q_1(T)]}{\xi}$$

(24)

where the term $S_0[q_1(T_0)]$ is obtained by reexpressing Eq. (23) with the perturbed quantities defined in Eqs. (10)–(12).

After performing integration, applying the boundary and initial conditions of the sensitivity problems, and letting the terms containing $\Delta T(x, t)$ be zero, we get the following adjoint problem for $\lambda(x, t)$:

$$C \frac{\partial \lambda}{\partial t} + k \frac{\partial^2 \lambda}{\partial x^2} + 2[q[x, t; q_1(t)] - Y_{qL}(t)] \cdot \frac{\Delta q[x, t; d^k(t)]}{\Delta T[x, t; d^k(t)]}$$

$$\times \delta(x - L) = 0 \quad \text{in} \quad 0 < x < L, \quad \text{for} \quad t > 0$$

(25)

$$\lambda(x, t_f) = 0 \quad \text{in} \quad 0 < x < L, \quad \text{for} \quad t = t_f$$

(26)

$$\frac{\partial \lambda(0, t)}{\partial x} = 0 \quad \text{at} \quad x = 0, \quad \text{for} \quad t > 0$$

(27)

$$\lambda(L, t) = 0 \quad \text{at} \quad x = L, \quad \text{for} \quad t > 0$$

(28)

where $\delta(\cdot)$ is the Dirac delta function.

The adjoint problem, Eqs. (25)–(28), is different from the standard initial value problems, Eqs. (2)–(5), for which the final time condition at time $t = t_f$ is specified instead of the customary initial condition ($t = 0$). However, this problem can be converted to an initial value problem by transformation of the time variables as $\tau = t_f - t$.

The heat flux variation $\Delta q[x, t; d^k(t)]$ in Eq. (25) can be calculated by Fourier’s law based on the temperature variation $\Delta T(x, t; d^k(t))$. However, considering the fact that the thermal conductivity is temperature dependent, special care should be given to the derivation of $\Delta q[x, t; d^k(t)]$, which is next described with the aid of Figure 2.

Figure 2 shows the one-dimensional finite difference mesh used in this study. All the quantities in Figure 2 are non-dimensional. In total, there are $M$ grid points, which are located at the centers of control volumes. The dash lines represent the faces of control volumes. We focus on the control volume of grid point $M - 1$ adjacent to the boundary $x = L$ (shadowed area in
Figure 6 Inverse results for stainless steel 304 with measurement error of $\phi^* = 5\% \cdot \left[ Y_q^{\text{exact}}(t) \right]_{\text{max}}$ and frequency of $f = 1 \text{ Hz}$: (a) back-surface heat flux measurements; (b) estimated front-surface heat flux by CGM inverse method; (c) front-surface temperature as a by-product.

Figure 7 Inverse results for stainless steel 304 with measurement error of $\phi^* = 1\% \cdot \left[ Y_q^{\text{exact}}(t) \right]_{\text{max}}$ and frequency of $f = 5 \text{ Hz}$: (a) estimated front-surface heat flux by CGM inverse method; (b) front-surface temperature as a by-product.

The heat flux at $x = L$ can be approximately estimated by:

$$q[L, t] = k_m \frac{T_{M-1} - T_M}{(\delta x)_e}$$  \hspace{1cm} (29)

where $k_m$ stands for the mean thermal conductivity of $k_{M-1}$ and $k_M$. In the present study, $k_m$ is calculated as the harmonic mean \[26\] of $k_{M-1}$ and $k_M$:

$$k_m = \frac{2k_{M-1}k_M}{k_{M-1} + k_M}$$  \hspace{1cm} (30)

Substituting Eq. (30) into Eq. (29), one gets:

$$q[L, t] = \frac{2k_{M-1}k_M}{k_{M-1} + k_M} \frac{T_{M-1} - T_M}{(\delta x)_e}$$  \hspace{1cm} (31)

If the thermal conductivities are constant, the calculation of the heat flux variation $\Delta q[L, t]$ is straightforward:

$$\Delta q[L, t] = \frac{2k_{M-1}k_M}{k_{M-1} + k_M} \frac{\Delta T_{M-1} - \Delta T_M}{(\delta x)_e}$$  \hspace{1cm} (32)
It can be seen by comparing Eqs. (33) and (32) that when the front-surface heat flux \( q(t) \) is subject to a variation \( \Delta q(t) \), the variation of back-surface heat flux \( \Delta q[L, t] \) calculated with temperature-dependent thermal properties will be larger than that calculated with constant thermal properties.

After letting the terms containing \( \Delta T(x, t) \) be zero, the following integral term is left:

\[
\Delta S[q(t)] = \int_0^{t_f} \lambda(0, t) \cdot \Delta q(t) \, dt \tag{34}
\]

By assuming that the unknown function \( q(t) \) belongs to the Hilbert space of square-integrable functions in the time domain \( 0 < t < t_f \), we can write [4, 27]:

\[
\Delta S[q(t)] = \int_0^{t_f} \nabla S[q(t)] \cdot \Delta q(t) \, dt \tag{35}
\]

A comparison of Eq. (34) with Eq. (35) leads to the following expression for the gradient of functional \( \partial S[q(t)] \):

\[
\nabla S[q(t)] = \lambda(0, t) \tag{36}
\]
**Stopping Criterion**

The discrepancy principle is used as the stopping criterion [3, 4]:

\[ S[q_1(t)] < \chi \]  

(37)

where \( \chi \) denotes the tolerance. Assume that the absolute value of the heat flux residuals can be approximated by:

\[ |Y_{qL}(t) - q[L, t; q_1(t)]| \approx \phi \]  

(38)

where \( \phi \) is the standard deviation of the measurements. Substituting Eq. (35) into Eq. (8), the tolerance \( \chi \) for the stopping criterion is obtained:

\[ \chi = \phi^2 I_f \]  

(39)

\[ \text{heat transfer engineering} \]

**RESULTS AND DISCUSSION**

**Generation of Measurement Data**

Instead of conducting an actual experiment, the measurement data of temperature and heat flux are generated numerically by solving the direct problem described by the governing Eq. (2) with initial condition given by Eq. (3) and the following boundary conditions:

\[ -k \frac{\partial T(L, t)}{\partial x} = q - h(T - T_\infty) - \varepsilon\sigma(T^4 - T_\infty^4) \text{ at } x = 0, \text{ for } t > 0 \]  

(40)

\[ -k \frac{\partial T(L, t)}{\partial x} = h(T - T_\infty) + \varepsilon\sigma(T^4 - T_\infty^4) \text{ at } x = L, \text{ for } t > 0 \]  

(41)

where \( q \) is the periodic heat flux imposed on the front surface. The results are obtained with the validated computer code [28]. In the inverse heat transfer analysis, the simulated measurement data of back surface temperature and heat fluxes are used as the boundary condition and are employed in the objective function, respectively, to estimate the heat flux and temperature at the front surface. These recovered heat flux and temperature will be compared with the front-surface temperature and heat flux calculated from the above direct problem [Eqs. (2), (3), (40)]

\[ \text{heat transfer engineering} \]
and (41)] to examine the accuracy of the present inverse heat conduction algorithm.

The simulated measurement data computed from Eqs. (2), (3), (40), and (41) provide the exact (errorless) measurement. To account for the measurement error in back surface heat flux, we add an error term to \( Y_{qLexact}(t) \) in the form:

\[
Y_{qL}(t) = Y_{qLexact}(t) + \omega \phi
\]

where \( Y_{qLexact}(t) \) are the data simulated from the direct problem described by Eqs. (2), (3), (40), and (41); \( \phi \) is the standard deviation of the measurements and is set as a percentage of the highest heat flux value at the back surface; and \( \omega \) is a random variable having a normal distribution with zero mean and unitary standard deviation. The measurement data obtained by Eq. (42) will contain random errors that have a normal distribution with standard deviation equal to \( \phi \).

The bias error in heat flux measurement is ignored. In addition, we assume that the back surface temperature contains no errors, since temperature can be measured with much less uncertainty compared to the heat flux [7, 8].

Results of IHCP

The following numerical analyses are performed for stainless steel 304 and aluminum alloy 2024-T6. Their temperature-dependent thermophysical properties [29] are plotted in Figures 3a and b, respectively. When the highest temperature exceeds the maximum temperature at which thermophysical properties are available, the thermophysical properties are obtained by linear extrapolation. Other parameters are: \( L = 2.5 \text{ mm}, T_0 = 300 \text{ K}, T_n = 300 \text{ K}, h = 5 \text{ W/(m}^2\cdot\text{K}), \varepsilon = 0.92 \). The front surface heat flux is assumed to be \( q^* = q_{const} + 0.1q_{const} \sin(2\pi ft^*) \) (W/m²). Unless specified otherwise, the following simulation parameters are used: \( q_{const} = 200 \text{ W/cm}^2, f = 1.0 \text{ Hz}, \phi^* = 1% \cdot [Y_{qLexact}(t)]_{max} \). Here, the simulation parameters are specified in real values, which can be converted to dimensionless quantities according to Eq. (1). The finite-difference method is used to solve the direct problem, sensitivity problem, and adjoint problem. Time discretization is obtained by applying a fully implicit scheme. Total grid number along the \( x \) direction is taken as 20 after a mesh refinement test. The final time is chosen as \( t_f^* = 3.6s \), and the time step \( \Delta t^* = 0.1s \). Therefore, a total
deviations between the recovered values and the exact solutions are also given in the figure. The RMS deviation between any two quantities $A$ and $B$ is defined as:

$$RMS = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (A_i - B_i)^2}$$

where $N$ is the total number of sample points.

As seen from Figure 5b and Figure 5c, both the recovered heat flux and the by-product temperature at the front surface are in excellent agreement with the exact results obtained from solving the direct problem. The RMS deviations between the recovered values and the exact solutions are so small that one cannot tell apart the two plotting curves in Figure 5b and Figure 5c. In the numerical simulation, no prior information on the functional form of the front-surface heating condition is required.

Figure 6 presents the effect of heat flux measurement error on the accuracy of the present IHCP formulation. The simulation parameters are almost the same as those in Figure 5 except for the random error on the measured heat flux. It can be seen from Figure 6b that when the heat flux measurement error is increased to 5%, the front surface heat flux can still be reconstructed with a good accuracy. The front surface temperature can also be recovered as a by-product with excellent accuracy (Figure 6c).

Figure 7 tests the effect of the frequency of the sinusoidal component on the accuracy of heat flux and temperature recovered from the proposed IHCP formulation. The simulation parameters are the same as those in Figure 5 except for the frequency of the sinusoidal component. It is found from Figure 7a that when the frequency $f$ is increased to 5 Hz, the phase of the estimated heat flux agrees well with that of the exact solutions, and there is a slight degradation in the accuracy of the amplitude of the recovered heat flux. Again, the front surface temperature can be recovered with high accuracy as a by-product of the inverse algorithm (Figure 7b).

Figure 8 shows the inverse solutions when both the measurement error and the frequency are increased (the measurement error is 5% and the frequency is 5 Hz). It appears that the accuracy of the recovered heat flux is still reasonably good, and the front surface temperature can be accurately estimated.

A focus of this paper is on the use of temperature-dependent thermal properties. To show the effect of using temperature-dependent thermal properties, Figure 9 shows the comparison between the inverse solutions obtained using temperature-dependent thermal properties and using constant thermal properties. The constant thermal properties used in Figure 9 are thermal conductivity $k^* = 14.9$ W/(m-K) and volume specific heat $C^* = 3768.3$ kJ/(m$^3$-K). Other simulation parameters are the same as those in Figure 8. The exact solutions in Figure 9 are obtained using temperature-dependent thermal properties. As is seen from the comparison between Figures 8 and 9, for the same object size and laser irradiation parameters, the constant property assumption will result in totally different inverse solutions. This indicates that constant property assumption may introduce considerable errors in IHCP analysis, which
underlines the necessity of using thermal properties in practical applications.

As mentioned earlier, one of the original contributions of the present study is using heat flux measurement data in the objective function. To show the superiority of using heat flux measurements in the objective function, Figure 10 presents the inverse solutions using temperature measurement data in the objective function while the heat flux measurement data are employed as the boundary condition. Other simulation parameters are the same as those in Figure 8. It is seen by comparing Figures 8 and 10 that there is not much difference between the recovered temperatures obtained using these two methods (comparing Figures 8b and 10b), but the accuracy in the recovered heat flux obtained by using temperature measurements is subject to severe degradation (comparing Figures 8a and 10a). The comparison between Figures 8 and 10 clearly indicates that the inverse approach using heat flux measurements in objective function has advantage over the traditional method using temperature measurements in objective function.

Figure 11 shows the calculated results when the measurement error and the frequency are further increased to $\phi^* = 20\% \cdot [Y_{qLexact}(t)]_{\text{max}}$ and $f = 6$ or 10 Hz. It can be seen from Figure 11a and b that for the case where $\phi^* = 10\% \cdot [Y_{qLexact}(t)]_{\text{max}}$ and $f = 6$ Hz, the estimation of the front-surface heat flux severely degrades (Figure 11a), but it is obvious that the recovery of the front-surface temperature is still excellent (Figure 11b). When the frequency $f$ is further increased to 10 Hz (Figure 11c and d), the temperature estimation is still very good. This indicates that the inverse formulations developed in this study are very robust in recovering the front-surface temperature. This is very important since the temperature, not the heat flux, will be used as the input information when subsequent thermomechanical analysis is performed to explore the damage mechanism.

The foregoing discussions were carried out for stainless steel 304. To demonstrate the applicability of the proposed IHCP formulation to different materials, the numerical model is tested for another commonly used industrial material, aluminum alloy 2024-T6, whose thermal properties are given in Figure 3b. The measurement error in back-surface heat flux is 5% and the frequency of the periodic heating flux is 5 Hz. Figure 12 shows the inverse results. As seen from Figure 12a, both the phase and the amplitude of the estimated heat flux agree excellently with those of the exact solutions.

Figure 13 shows the inverse results when the measurement error and the frequency are further increased to $\phi^* = 20\% \cdot [Y_{qLexact}(t)]_{\text{max}}$ and $f = 10$ Hz.
which are obtained in a similar way as described in Eq. (42). As can be seen from Figure 14, both the phase and amplitude of the estimated heat flux agree excellently with those of the exact solutions. This demonstrates that the numerical formulation proposed in present study is insensitive to the uncertainties in thermal properties.

As mentioned earlier, in this study, the front-surface heating condition is recovered based on the heat flux and temperature measurements at back surface. So far, we only consider the random errors in heat flux measurements and the temperature measurements are assumed to be errorless. It is necessary to know the effects of the back-surface temperature measurement data on the accuracy of the inverse solutions. Figure 15 tests the influence of the random errors in back-surface temperature measurements. The random error is introduced via Eq. (42), in which the standard deviation of the measurements is taken as $\phi^* = 0.1 \, \text{K}$. As can be seen in Figure 15a, the heat flux estimation is subject to pronounced degradation. But the temperature prediction still maintains a reasonable accuracy (Figure 15b). This confirms that the boundary condition choice and inverse

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**Figure 14** Influence of uncertainties in thermophysical properties (stainless steel 304, $\phi^* = 5\% \cdot [Y^{*}_{qLexac}]_{max}$, $f = 5 \, \text{Hz}$): (a) estimated front-surface heat flux by CGM inverse method; (b) front-surface temperature as a by-product.

**Figure 15** Influence of measurement error in back-surface temperature (stainless steel 304, $\phi^* = 0.1 \, \text{K}$, $f = 5 \, \text{Hz}$): (a) estimated front-surface heat flux by CGM inverse method; (b) front-surface temperature as a by-product.
algorithm formulated in this study are robust in retrieving the front-surface temperature.

CONCLUSIONS

A conjugate gradient method algorithm is presented to reconstruct heat flux and temperature at the front (heated) surface of a finite slab with temperature-dependent thermophysical properties and under high-intensity periodic heating based on the temperature and heat flux measurement data at the back surface. The inverse problem is formulated in such a way that the front-surface heat flux is chosen as the unknown function to be recovered, and the front-surface temperature is computed as a by-product of the IHCP algorithm. New equations for the sensitivity problems and adjoint problems are derived to incorporate heat flux measurement data in the objective function. It is shown that the constant thermal property assumption may introduce considerable errors in inverse solutions, which underscore the necessity of using thermal properties in practical applications. It is also demonstrated that the inverse approach using heat flux measurements in the objective function is superior to the traditional methods in which the temperature measurement data are used. The methodologies are tested for two commonly used industrial materials, stainless steel 304 and aluminum alloy 2024-T6. The effects of the uncertainties in back-surface temperature and thermal properties on inverse solutions are also tested. The excellent numerical results demonstrate that the proposed approach is a robust numerical algorithm for IHCP with temperature-dependent thermophysical properties.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>dimensionless volume-specific heat</td>
</tr>
<tr>
<td>$C^*$</td>
<td>volume specific heat, J/(m$^3$·K)</td>
</tr>
<tr>
<td>$d^k(t)$</td>
<td>dimensionless direction of descent at iteration $k$</td>
</tr>
<tr>
<td>$f$</td>
<td>frequency of periodic laser heat flux at front surface, Hz</td>
</tr>
<tr>
<td>$h$</td>
<td>dimensionless convection heat transfer coefficient</td>
</tr>
<tr>
<td>$h^*$</td>
<td>convection heat transfer coefficient, W/(m$^2$·K)</td>
</tr>
<tr>
<td>$k$</td>
<td>dimensionless thermal conductivity</td>
</tr>
<tr>
<td>$k^*$</td>
<td>thermal conductivity, W/(m·K)</td>
</tr>
<tr>
<td>$l_c$</td>
<td>characteristic length, m</td>
</tr>
<tr>
<td>$L$</td>
<td>dimensionless thickness of one-dimensional (1-D) slab</td>
</tr>
<tr>
<td>$L^*$</td>
<td>thickness of 1-D slab, m</td>
</tr>
<tr>
<td>$q$</td>
<td>dimensionless intensity of heating source at front surface</td>
</tr>
<tr>
<td>$q^*$</td>
<td>intensity of heating source at front surface, W/m$^2$</td>
</tr>
<tr>
<td>$q_c$</td>
<td>characteristic heat flux, W/m$^2$</td>
</tr>
<tr>
<td>$q_{const}$</td>
<td>constant component of the front-surface periodic heat flux, W/m$^2$</td>
</tr>
<tr>
<td>$q(t)$</td>
<td>dimensionless observed heat flux at front surface</td>
</tr>
<tr>
<td>$q^*_f(t)$</td>
<td>observed heat flux at front surface, W/m$^2$</td>
</tr>
<tr>
<td>$q[L, t; q(t)]$</td>
<td>dimensionless computed heat flux at the back surface</td>
</tr>
<tr>
<td>$\Delta q[L, t; d^k(t)]$</td>
<td>dimensionless heat flux variation, which is sometimes simplified as $\Delta q(d^k)$</td>
</tr>
<tr>
<td>$S$</td>
<td>dimensionless objective function</td>
</tr>
<tr>
<td>$\nabla S[q^*_f(t)]$</td>
<td>dimensionless gradient direction of objective functional at iteration $k$</td>
</tr>
<tr>
<td>$\Delta S[q^*_f(t)]$</td>
<td>dimensionless objective function variation</td>
</tr>
<tr>
<td>$t$</td>
<td>dimensionless time</td>
</tr>
<tr>
<td>$t^*$</td>
<td>time, s</td>
</tr>
<tr>
<td>$\Delta t^*$</td>
<td>time step, s</td>
</tr>
<tr>
<td>$t_c$</td>
<td>characteristic time, s</td>
</tr>
<tr>
<td>$t_f$</td>
<td>dimensionless final time</td>
</tr>
<tr>
<td>$t^*$</td>
<td>final time, s</td>
</tr>
<tr>
<td>$T$</td>
<td>dimensionless temperature</td>
</tr>
<tr>
<td>$T^*$</td>
<td>temperature, K</td>
</tr>
<tr>
<td>$T_c$</td>
<td>characteristic temperature, K</td>
</tr>
<tr>
<td>$T_0$</td>
<td>dimensionless initial temperature</td>
</tr>
<tr>
<td>$T_0^*$</td>
<td>initial temperature, K</td>
</tr>
<tr>
<td>$T_\infty$</td>
<td>dimensionless ambient temperature</td>
</tr>
<tr>
<td>$T_f(t)$</td>
<td>dimensionless front surface temperature</td>
</tr>
<tr>
<td>$T^*_f(t)$</td>
<td>front surface temperature, K</td>
</tr>
<tr>
<td>$\Delta T[x, t; q(t)]$</td>
<td>dimensionless temperature variation, which is sometimes simplified as $\Delta T$</td>
</tr>
<tr>
<td>$x$</td>
<td>dimensionless spatial coordinate variable</td>
</tr>
<tr>
<td>$x^*$</td>
<td>spatial coordinate variable, m</td>
</tr>
<tr>
<td>$Y(t)$</td>
<td>dimensionless measurement data (temperature or heat flux) with errors at back surface obtained by numerical simulations</td>
</tr>
<tr>
<td>$Y_{exact}(t)$</td>
<td>dimensionless measurement data (temperature or heat flux) without errors at back surface obtained by numerical simulations</td>
</tr>
<tr>
<td>$Y_{q_L}(t)$</td>
<td>dimensionless measurement heat flux at the back surface</td>
</tr>
<tr>
<td>$Y_{q_L'}(t)$</td>
<td>measurement heat flux at the back surface, W/m$^2$</td>
</tr>
<tr>
<td>$Y_{T_L}(t)$</td>
<td>dimensionless measurement temperature at the back surface</td>
</tr>
<tr>
<td>$Y_{T_L'}(t)$</td>
<td>measurement temperature at the back surface, K</td>
</tr>
</tbody>
</table>

Greek Symbols

- $\beta^k$ dimensionless search step size at iteration level $k$
- $\chi$ dimensionless tolerance used to stop the CGM iteration procedure
- $\delta$ Dirac delta function
- $\varepsilon$ surface emissivity
- $\phi$ dimensionless standard deviation of heat flux or temperature measurements
- $\psi^k$ dimensionless conjugate coefficient at iteration level $k$
\( \lambda(x, t) \) \hspace{1cm} \text{dimensionless Lagrange multiplier} \\
\( \sigma \) \hspace{1cm} \text{a dimensionless quantity related to Stefan–Boltzmann constant, defined by Eq. (1)} \\
\( \sigma^* \) \hspace{1cm} \text{Stefan–Boltzmann constant, } \sigma = 5.67 \times 10^{-8} \text{ W/(m}^2\text{-K}^4\text{)} \\
\( \omega \) \hspace{1cm} \text{a dimensionless random variable having a normal distribution with zero mean and unitary standard deviation} \\
\( \xi \) \hspace{1cm} \text{dimensionless perturbed variable} \\

**Subscripts** \\
\( 0 \) \hspace{1cm} \text{initial} \\
\( f \) \hspace{1cm} \text{final} \\
\( q \) \hspace{1cm} \text{heat flux} \\
\( T \) \hspace{1cm} \text{temperature} \\

**Superscripts** \\
\( * \) \hspace{1cm} \text{real physical quantities with dimensions} \\
\( k \) \hspace{1cm} \text{iteration level} \\

**REFERENCES** \\


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