Abstract—In this paper we introduce a novel strategy for generalizing existing puzzles and games by mathematically expressing the operations of the game; then deriving mathematical generalizations of those expressions; and finally implementing a new variant of the game using those generalized operations. The strategy is illustrated in a case study involving the adapting of a traditional game/puzzle to exploit the computational power of smart devices. The focus here is not so much on the end product as it is on the process and considerations underpinning its development by use of the proposed approach. Ancillary results of the venture include generalizations of the circular-shift operator and examination of its computational complexity.

Keywords: Aesthetic Image Transformations, Circular-Shift Operator, Computer Games, Engineering Education, Game Development, Sliding-Tile Puzzles, Solitaire, TriPeaks.

I. INTRODUCTION

The subject of “game design” is far too broad to admit any form of concise top-down treatment, but often it is possible to glean general concepts and principles from bottom-up consideration of concrete examples. In the case of particular board games, e.g., Monopoly, it is straightforward to identify specific features that people find to be engaging and entertaining[7], [10], and from these it’s possible to identify general principles that explain the appeal not only of similar board games but also a wider variety of other games as well. However, these general principles would not be sufficient to permit an alien scholar to predict, or derive as being inevitable, the actual existence of such games as a genre. That’s because the genre’s existence has less to do with objectively discernible attributes and more to do with historical happenstance: someone developed the first instance of the kind, it became popular, and subsequent similar games were developed to leverage not only the popular attributes of that first game but also the newly existing familiarity of those attributes among general consumers. In other words, once the initial instance became popular the genre could evolve incrementally with improvements that would be appreciated by consumers without the steep learning curve of a completely de novo game.

The availability of relatively low-cost computers in the 1970s created opportunities to not only design a new class of games tailored to leverage the unique capabilities of the new technology but also to adapt existing games for computer implementation. A good example is the popular card game Solitaire1. Variations of this game are found pre-installed on most home computers and smart devices. It is interesting to note which features of the game are simulated and which are not. In theory, it would seem that the game could be implemented using any set of distinct shapes and/or color designs instead of simulating the suits of a traditional 52-card deck of playing cards. But to do so would sacrifice the familiarity of the game to consumers. In other words, while the logic of the game would be identical it wouldn’t be Solitaire from the perspective of most consumers. In fact, current versions of Solitaire differ little from early computer implementations even though advances in computing power could permit a highly realistic simulation of human hands holding and manipulating cards to more closely emulate how the game is played with real cards. This suggests that current implementations include the salient features necessary to capture the experience and serve as an acceptable replacement for the traditional form of the game.

Traditional Solitaire (Klondike) emerged as one of the most popular and widely-played computer games, and it spawned a family of Solitaire-like games that also became popular despite the fact that they were not convenient to play with actual cards. One such example is TriPeaks[15], which requires cards to be arranged in three overlapping triangular configurations that would be cumbersome to maintain in a neat-and-orderly form using real cards on a real table. As such, it represents one of the first examples of a virtual card game designed specifically to be played on a computer. Now there are many other games that are similarly designed to exploit the familiarity of Solitaire in a form that leverages the

1This term is actually generic for a variety of single-player card games (also known as Patience) but is also commonly used to refer to the specific game Klondike[9], [11]. Because more people know that specific game under the name “Solitaire” instead of “Klondike,” we will use it in that sense here.
advantages of computer simulation. In this paper we will examine the process of adapting and redefining an existing game for computer simulation in a way that reveals a sequence of concrete steps in which mathematical and algorithmic considerations can be applied.

II. SLIDING-TILE PUZZLE GAMES

Before the prevalence of hand-held computer games there were many popular puzzle games requiring manual movement of objects to achieve a goal configuration. Some of these involved the movement of wooden pegs within an arrangement of holes in a wooden base while others involved the sliding of tiles within a rectangular frame [5]. Figure 1 shows an example of the latter,

![Scrambled 8-tile puzzle and its solution.](image)

where a chosen tile adjacent to the empty space can be moved into that space\(^2\) in such a way that a sequence of moves transforms the random initial configuration (left) to the ordered goal configuration (right). Traditionally the frame and tiles were made of wood, then later plastic, and the number of tiles could be larger, e.g., 15 in a 4 × 4 frame, and the numeric digits might be replaced with letters or other symbols, or even graphical elements forming an image when placed in the goal configuration.

Of course a direct simulation of the game is straightforward, and the above figure provides a reasonable prototype of a possible visual interface. If one starts with such a direct translation of the real-world interface, then the mechanics of the interaction might include touch sensitivity with velocity detection for the user to specify a particular tile and direction of motion using a finger swipe. On further consideration, however, it can be recognized that simple touch detection is sufficient because the direction of motion can be uniquely inferred based on the location of the space.

\(^2\)An alternative, though equivalent, interpretation of a move is that the space (or hole) swaps location with the tile [1].

An important question that arises is whether something critical to the “game experience” is lost if swiping is replaced with touch\(^3\). Fortunately in this case nothing is lost because touch detection will be triggered by a swipe to produce the same result, i.e., the user experience will be identical in most circumstances. At this point we can see how the traditional game can easily be implemented in a way that preserves an exact analogy between the real version and the simulated one. With this we can begin to consider ways in which the simulated version can go beyond what is possible with the real version due to the physical constraints of tiles on a fixed frame.

Simulation offers myriad opportunities for changing the shape of the frame and the shape of the tiles, e.g., to triangles or hexagons, because there is no need to support the physical sliding of pieces. Thinking even more broadly, it can be recognized that in the simulated version there is no need for an empty space/hole. More specifically, we could have nine tiles and define a new operation in which the swiping of a tile causes it to swap position with the adjacent tile in the direction of the swipe. Unfortunately, such an operation causes the game play to degenerate into simply executing a trivial sequence of swaps to move tile 1 directly to its correct final position, then doing the same for tile 2, and so on.

An alternative operation can also be defined that is in some ways more consistent with the spirit of the traditional game. That is to allow a swipe to circularly shift a given row or column. For example, if a right swipe is applied to a row with digits 4-3-7, the result would be for each tile to move one position to its right with the rightmost tile moving (wrapping around) to the first position, so 4-3-7 would circularly shift to become 7-4-3. This operation preserves many of the familiar aspects of the traditional game in that later moves become increasingly constrained as more tiles become fixed in their final positions. More specifically, the effectiveness of a simple greedy strategy is limited because it may move a particular piece closer to its destination while simultaneously moving other pieces away from their goal positions.

The circular-shift operation provides an interesting variation on the game, but can the idea be extended further? For example, instead of discrete integer-value shifts can the operation be generalized to allow a continuous range of motion, e.g., a circular shift of \(k = 1.83\)

\(^3\)Similarly, is it worthwhile to associate sound with the movement of tiles, e.g., as they slide and then “click” on impact? Sound can contribute immensely to user experience when playing a simulated physical-based game like this.
or \( k = -3.24 \), as determined by the duration of the player’s swipe? It isn’t immediately clear what such a generalization would look like, or whether it would contribute positively to the game-playing experience, but it’s certainly worth investigating because it’s a feature that can potentially be supported on a computer but not by any simple physical device. This motivates an examination of the circular-shift operator in a more abstract mathematical sense in order to assess whether it is possible to generalize it to, e.g., take on arbitrary real values.

### III. Generalized Circular-Shift (GCS) Operator

The *shift* operator is widely used in computer engineering and computer science to transform the state of a computer register, which can be treated abstractly as a vector of length \( n \). The shift operator takes an integer parameter \( k \) and moves each value at location \( i \) of an \( n \)-element vector to location \( i+k \), with each of the first \( k \) elements becoming zeros. The following is an example with \( k = 2 \):

\[
\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 & 2 & 3 \end{bmatrix}
\]

The *circular-shift* operator (or circular buffer [2]) is also familiar in engineering applications and is defined analogously except that values moved beyond the index range of the vector are moved (rotated) to locations modulo the length of the vector in the intuitively natural way we’ve already assumed:

\[
\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 5 & 1 & 2 & 3 \end{bmatrix}
\]

Unlike the shift operator, the circular shift is invertible, i.e., there always exists a circular shift that will return to the initial state. In fact, the operation can be expressed as an \( n \times n \) permutation matrix of the following form:

\[
P = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(n \times n)}
\]  
(1)

where \( I \) represents the identity matrix. In the cases of \( n = 3 \) and \( n = 4 \) this would give, respectively:

\[
P_{3 \times 3} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad P_{4 \times 4} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.
\]  
(2)

A simple circular shift of the vector \([1 2 3]\) can thus be obtained by applying \( P_{3 \times 3} \) as

\[
\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.
\]  
(3)

Raising \( P \) to an integral power \( k \), i.e., \( P^k \), has the effect of shifting a vector by \( k \) positions. In the case of \( n = 5 \), for example, a circular shift of \( k = 2 \) can be expressed as

\[
\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 4 \end{bmatrix}
\]  
(4)

This provides a clear generalization from an integer to a real-valued parameter \( k \) because the transformation \( P^k \) is well-defined for any real value for the exponent.

A natural question to ask is whether a simpler interpolation-based generalization, e.g., one that applies some sort of weighted average based on the fractional part of the parameter, might represent a superior alternative. The principal obstacle to defining such a generalization is ensuring that it is invertible, i.e., maintains all information. As an example, consider the vector \([1 3 1 3]\) with \( k = 1/2 \). In this case almost any simple interpolation scheme will produce a value for each location that is the mean of the value shifted halfway into the location and the value shifted halfway out of the location, thus yielding the result \([2 2 2 2]\). The vector of all equal values is a fixed point for almost any generalization of the circular-shift operator, and any non-invertible scheme will tend toward that fixed point at the expense of information about the original state\(^4\). In other words, performing a sequence of non-integer interpolation-type operations to rows and/or columns of the 8-tile puzzle would lose information in a way that prevents it from being solved, i.e., being able to reach the goal state.

Having identified a mathematically consistent generalization of the operator, a practical concern arises about the efficiency with which it can be evaluated. Specifically, for a plausible-size value of \( n \), say \( n = 5 \), can a given non-integral power of a \( 5 \times 5 \) matrix be evaluated efficiently enough to satisfy real-time constraints? The

\(^4\)It can be verified that the GCS vector transform defined using powers of the \( n \)-dimensional circular-shift matrix has the all-equal state as an eigenvector such that it is invariant with respect to any choice of \( k \).
answer of course depends on the computational resources available. The general algorithm for raising a matrix to an arbitrary real exponent is unlikely to take more than a fraction of a second on a typical smart phone, so at worst there might be a slight noticeable lag after completion of a swipe by the user because the display update cannot begin until after the transformation has been completed. Fortunately, the GCS operator does not involve an arbitrary matrix. A closer examination reveals that the circular-shift matrix is circulant, i.e., each row $i$ is equal to row $i – 1$ circularly shifted by one position. Circulant matrices are special ([4], [3]) in that they can be diagonalized in $O(n^2 \log(n))$ time, as opposed to $O(n^3)$ for a general matrix, and consequently can be raised to an arbitrary real power and multiplied by a given vector with the same complexity. The matrix $P$ is also special in that it is unitary [6], i.e., the Euclidean norm of $Pv$ will equal that of $v$, and it has equal row and column sums, which means that the sum of the elements of $Pv$ will equal that of $v$. Taken jointly, these two properties can be summarized as preserving the mean and variance of the elements of the transformed vector.

In the following section we present our main result: an $n$-parameter nonlinear matrix operator based on the GCS.

IV. GCS MATRIX TRANSFORM OPERATORS

We now generalize the vector GCS operator of the previous section to a matrix operator, denoted $\circ$. It must be emphasized that the operator under consideration is not of the form $P^k M$ or $M P^k$, where all rows or columns undergo the same linear transformation. Rather, the GCS matrix operator developed in this section performs separately-parameterized circular rotations of individual rows or columns. For example, in the case of a $3 \times 3$ matrix the rows can be circularly rotated with three independent parameters $\alpha, \beta, \gamma$ as:

$$
\begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix} \circ
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix} =
\begin{cases}
\text{GCS ([a b c], } \alpha) \\
\text{GCS ([d e f], } \beta) \\
\text{GCS ([g h i], } \gamma)
\end{cases}
\tag{5}
$$

where the operator $\circ$ indicates that the $i$th element of the parameter vector defines the circular shift to be applied to the $i$th row of the matrix.

A converse operator, denoted $\odot$, is analogously defined to operate on the columns of the matrix as

$$
\begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix} \odot
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix} =
\begin{cases}
\text{GCS ([a d g], } \alpha) \\
\text{GCS ([b e h], } \beta) \\
\text{GCS ([c f i], } \gamma)
\end{cases}
\tag{6}
$$

where the first column of the resulting matrix is GCS $([a d g], \alpha)$.

If $\circ$ and $\odot$ are taken as right-associative then a sequence of applications can be meaningfully interpreted, e.g.,

$$
\begin{bmatrix}
\xi \\
\psi \\
\omega
\end{bmatrix} \odot
\begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix} \circ
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix} =
\begin{bmatrix}
\xi \\
\psi \\
\omega
\end{bmatrix} \odot
\begin{bmatrix}
\alpha \circ \beta \circ \gamma \\
\alpha \circ \beta \circ \gamma \\
\alpha \circ \beta \circ \gamma
\end{bmatrix}
\tag{7}
$$

It can be verified that the inverse of either operator can be obtained by taking the negative of its parameters:

$$
\begin{bmatrix}
-\alpha \\
-\beta \\
-\gamma
\end{bmatrix} \circ
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix} =
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
\tag{8}
$$

and more generally that

$$
\begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix} \circ
\begin{bmatrix}
\xi \\
\psi \\
\omega
\end{bmatrix} =
\begin{bmatrix}
\alpha + \xi \\
\beta + \psi \\
\gamma + \omega
\end{bmatrix}
\odot
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
\tag{9}
$$

but that no similar composition is possible for a mixed sequence of $\circ$ and $\odot$ operations, e.g., they do not commute:

$$
\begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix} \odot
\begin{bmatrix}
\phi \\
\xi \\
\psi
\end{bmatrix} \circ
\begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
i & j & k & l
\end{bmatrix} \neq
\begin{bmatrix}
\phi \\
\xi \\
\psi
\end{bmatrix} \odot
\begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix} \circ
\begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
i & j & k & l
\end{bmatrix}
\tag{10}
$$

and, as shown above, do not have the same number of parameters if the matrix argument is not square.

The line (i.e., row or column) operations performed by the $\circ$ and $\odot$ operators are separately parameterized and thus do not generally represent a linear transformation. However, they preserve key structural properties such as the norm and sum of each line. In other words, while...
they can dramatically affect the determinant and other scalar functions of a given matrix, they do so without scaling any row or column. A simple though illustrative example is the following

\[
\begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix} \circledast \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\] (11)

where the result of the transformation is a reduction of a full-rank matrix to a matrix of rank 1. The following example shows that the same degree of rank reduction is also possible for a full-rank matrix even when all of its values are distinct and no two rows or columns have the same norm or sum:

\[
\begin{bmatrix}
1.2 \\
1.4 \\
1.8
\end{bmatrix} \circledast \begin{bmatrix}
3.1484 & 1.3213 & 1.5303 \\
9.2946 & 5.2798 & 3.4257 \\
4.9393 & 5.3574 & 1.7033
\end{bmatrix} = \begin{bmatrix}
1 & 2 & 3 \\
3 & 6 & 9 \\
2 & 4 & 6
\end{bmatrix} .
\] (12)

Of course this example was contrived to ensure a rank-1 result, but it nonetheless demonstrates that strong shift and scale relationships among the rows may be difficult to discern\(^6\). The principal motive for considering rank properties is that they can be exploited by an automated n-tile puzzle solver whenever the goal matrix is not of full rank, e.g., as would be true in cases such as:

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix} \text{ or } \begin{bmatrix}
0 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 0
\end{bmatrix} .
\] (13)

However, our main focus in this paper is on interactive game-play involving a human user, which is developed in the next section.

To summarize, the operators defined in this section are intended for use in representing circular shifts of rows and columns of a matrix as performed when solving a sliding-puzzle game. For example, suppose the final two moves of the game involve a circular shift of the third row by 2 positions, followed by a 1-position shift of the middle column, then the sequence can be expressed using the new operators as

\[
\begin{bmatrix}
0 \\
0 \\
2
\end{bmatrix} \circledast \begin{bmatrix}
1 & 5 & 3 \\
4 & 8 & 6 \\
9 & 7 & 2
\end{bmatrix} = \begin{bmatrix}
1 & 5 & 3 \\
4 & 8 & 6 \\
7 & 2 & 9
\end{bmatrix}
\] (14)

\(^6\)It can be hypothesized that spatial coherence of elements within matrices representing, e.g., natural images, may tend to create such relationships, at least locally. If so then GCS-based methods may provide a basis for defining new types of matrix decompositions/splittings [6], [8] and for approximate rank reduction that is distinct from conventional approaches. Again, this is a topic beyond the scope of the present paper.

followed by

\[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} \circledast \begin{bmatrix}
1 & 5 & 3 \\
4 & 8 & 6 \\
7 & 2 & 9
\end{bmatrix} = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
\] (15)

or the consecutive operations can be expressed jointly as

\[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} \circledast \begin{bmatrix}
1 & 5 & 3 \\
4 & 8 & 6 \\
9 & 7 & 2
\end{bmatrix} = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix} .
\] (16)

V. THE GCS 9-TILE PUZZLE

As suggested by the example of Eq. 12, the application of real-valued shifts will transform tiles with integer values to ones with non-integer values, e.g.,

\[
\begin{bmatrix}
1.08 \\
0.61 \\
2.64
\end{bmatrix} \circledast \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix} = \begin{bmatrix}
3.0823 & 1.1103 & 1.8074 \\
5.2637 & 3.8946 & 5.8417 \\
6.8758 & 8.7904 & 8.3337
\end{bmatrix} .
\] (17)

This of course should be expected, but now it is necessary to seriously consider whether such a generalization is conducive to engaging gameplay. The original version of the game had numbers on the tiles, but from a player's perspective they are just symbols to be moved. By contrast, the generalized form above displays an inherently mathematical aspect that most people are unlikely to find accessible, let alone enjoyable. Assuming the focus is not on entertaining our future robot overlords, there is a clear need to abstract away from raw numeric values and redefine in terms of something with more direct intuitive appeal. This can be accomplished by treating the numeric values as parameters for determining what is displayed on the tiles. For example, the numeric values could be mapped to grey-scale colors to give the alternative display depicted in Figure 2 for the original goal configuration.

\[\text{Figure 2 - Converting tiles from numeric to grey-scale colors.}\]

This is encouraging, but whereas the increasing sequence 1...9 represents a distinctive goal configuration in the case of tiles with numeric digits, the sequence of increasingly darker shades of grey is not so distinctive
because humans have difficulty distinguishing absolute, as opposed to relative, gradations of color intensity. In other words, a player may have difficulty assessing whether a given sequence of shades of grey exactly equals the desired sequence for any given row or column. What is needed is a completely unambiguous goal state. In this case the ideal goal state would be the uniform state in which all tiles have the same color.

Unfortunately, the GCS operation has no effect on a vector of identical values, so there is no way to initialize in the uniform state and “scramble” to a non-uniform state by applying a sequence of random circular shifts to the rows and columns. That’s not necessarily an issue, though, because the functional mapping of numeric values to grey-scale values can be changed to whatever we please to satisfy our needs. For example, we can define the numeric goal state to be

\[
\begin{bmatrix}
-1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & -1
\end{bmatrix}
\]  

and use their absolute values (unsigned magnitudes) to determine their grey-scale intensities. This will create a multiplicity of matrices that correspond to the goal state, e.g.,

\[
\begin{bmatrix}
1 & 1 & -1 \\
-1 & 1 & 1 \\
1 & -1 & 1
\end{bmatrix}
\]

but that can easily be accommodated by only checking absolute values when assessing whether the player has achieved a/the goal state. To summarize, the player will begin with a configuration of tiles having a random distribution of grey-scale intensities and will have to perform a sequence of swipes to circularly shift the rows and columns until the tiles have uniform color (Figure 3).

It should be noted that unlike the discrete values of the original game, the values in the generalized version are not suitable for the use of equality tests. This means that a threshold should be applied so that, e.g., if all values are within some $\epsilon$ of having unit magnitude then the user should be deemed to have achieved the goal state.

VI. FURTHER GENERALIZATIONS

The most obvious generalization would be to expand from a $3 \times 3$ matrix of tiles to a $4 \times 4$ matrix. However, an important fact that has not been mentioned about the generalized circular-shift operator is that it yields complex numbers in even dimensions, e.g.:

\[
\begin{bmatrix}
1.3 & 2.7 \\
3.2 & 2.4 \\
-2.2 & 1.2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 2 3 4 \\
5 6 7 8 \\
9 10 11 12
\end{bmatrix}
\]

\[
= \begin{bmatrix}
14.89+1.63i & 11.08-2.15i & 2.83+1.76i & 8.13-2.40i \\
9.67-2.65i & 14.17+3.32i & 7.93-2.56i & 14.77+0.29i \\
5.63-1.05i & 7.83+0.38i & 10.43-0.78i & 14.77+0.29i \\
3.19-2.36i & 6.40+3.13i & 8.87-2.45i & 0.78+2.86i
\end{bmatrix}
\]

Implementing a game in this case using a display of complex numeric values could pose a risk to humanity by potentially frustrating and provoking our future robot overlords, but we have seen that this may be avoided by transforming to a non-numeric display. In fact, the same absolute-value approach used in the previous section can be applied identically to complex numbers that arise in the $4 \times 4$ case. Alternatively, we could also exploit the angle and magnitude information given by complex numbers to generate a richer variety of colors or to display directional intensity gradients on the tiles.

Another possible generalization is to associate a matrix with each tile instead of a single numeric value. This is essentially what is done in the case of traditional puzzles in which the sliding tiles represent patches of an image. Figure 4 shows an example of a $3 \times 3$ puzzle in which 8 virtual tiles depict parts of an image and one space is empty to permit movement of the tiles. The image on the right is the goal state while the image on the left is the initial state obtained by jumbling the 8 tiles (and implicitly the hole) using a random sequence of ordinary tile shifts.
The generalization of the circular-shift matrix $P$ for the shifting of submatrices associated with tiles (e.g., to eliminate the need for an empty space/hole) can be achieved using the Kronecker matrix product

$$I \otimes P$$

where $I$ is the identity matrix of size equal to the size of the tiles to be shifted. Basically, this just replaces each $i,j$ element of $P$ with $P_{i,j}$ times the identity matrix of size equal to the block size. Figure 5 is the generalization of the previous example where the full image is represented on 9 tiles because generalized circular shift operations eliminate need for an empty space. Analogous to the previous example, the image on the left is the initial state obtained by jumbling the 9 tiles using a random sequence of generalized circular shifts:

As can be seen, the underlying matrix for the sub-image associated with each tile will generally be a superposition (mixture) of the tile matrices in its row and column, so the extent to which players can develop an intuitive feel for the effect that a swipe has on a given row or column will impact whether or not the puzzle is entertaining to solve. In the $3 \times 3$ case (e.g., like the above example) the puzzle is not difficult to solve and may very well prove to be more entertaining than simply moving static tiles because the final image is less obvious at the outset and thus may produce a greater degree of satisfaction/reward when it finally clicks into place. As has been mentioned, however, assessment of the quality of the game requires user studies [12] and is not the focus of this paper. As exemplified by Figure 6, the general aesthetic qualities of the GCS transformation may be of independent interest [14].

**VII. Conclusions**

In this paper we have introduced a novel strategy for generalizing existing puzzles and games using three basic steps: (1) Mathematically express the operations of the game; (2) derive mathematical generalizations of those expressions; and (3) implement a new variant of the game using the generalized operations. We demonstrated this strategy by generalizing the physical shift operator used in sliding-tile puzzles to obtain a result that adds a radically different interactive experience to this familiar type of puzzle/game. The value of the proposed strategy is that it offers a relatively focused approach for developing generalizations of existing puzzles and games to exploit the computational and graphical power of smart phones and related devices.

**References**