Active Contour without Edges

Tony Chan and Luminita Vese
Basic idea in classical active contours

Curve evolution and deformation (internal forces):

\[ \text{Min } \text{Length}(C) + \text{Area(inside}(C)) \]

Boundary detection: stopping edge-function (external forces)

\[ g \geq 0, \quad g \downarrow, \quad \lim_{t \to \infty} g(t) = 0 \]

Example:

\[ g(|| \nabla u_0 ||) = \frac{1}{1 + || \nabla G_\sigma * u_0 ||^p} \]

Snake model (Kass, Witkin, Terzopoulos 88)

\[ \inf_C F(C) = \alpha \int_0^1 |C'(s)|^2 ds + \beta \int_0^1 |C''(s)| ds - \lambda \int_0^1 |\nabla u_0(C(s))|^2 ds \]

Geodesic model (Caselles, Kimmel, Sapiro 95)

\[ \inf_C F(C) = 2 \int_0^1 |C'(s)| g(|| \nabla I(C(s)) ||) ds \]
Basic idea in classical active contours

Curve evolution and deformation (internal forces):

\[ \text{Min } \text{Length}(C) + \text{Area}(\text{inside}(C)) \]

Boundary detection: stopping edge-function (external forces)

\[ g \geq 0, \quad g \downarrow, \quad \lim_{t \to \infty} g(t) = 0 \]

Example:

\[ g(|\nabla u_0|) = \frac{1}{1 + |\nabla G_\sigma * u_0|^p} \]

\( G_\sigma * u_0 \) is a smoother version of \( u_0 \)

\[ g(|\nabla u_0|) \approx 0 \text{ only on the edges} \]

\[ g(|\nabla u_0|) \approx \text{constant} > 0 \text{ far from edges} \]
Classical Snakes and Active Contour Models

Models in variational formulation $s \rightarrow C(s)$ a parameterization of $C$

* **Snake** model (Kass, Witkin, Terzopoulos 88)

$$\inf_{\mathcal{C}} F(C) = \int_{0}^{1} |C'(s)|^2 ds + \lambda \int_{0}^{1} g(|\nabla I(C(s))|) ds$$

* **Geodesic** model (Caselles, Kimmel, Sapiro 95)

$$\inf_{\mathcal{C}} F(C) = 2 \int_{0}^{1} |C'(s)| g(|\nabla I(C(s))|) ds$$

Other related models:

* Caselles, Catte, Coll, Dibos 93
* Malladi, Sethian, Vemuri 93
* Caselles, Kimmel, Sapiro 95
* Kichenassamy, Kumar, Olver, Tannenbaum, Yezzi ‘95
\[ L_R := \int_0^1 g(\|\nabla I(C(q))\|)|C'(q)| \, dq. \quad (11) \]

Since \(|C'(q)| \, dq = ds\), we obtain

\[ L_R := \int_0^{L(C)} g(\|\nabla I(C(q))\|) \, ds. \quad (12) \]
**Some active contours with edge-function**

**Models in variational formulation**

* Snake model: Kass, Witkin, Terzopoulos ’88
* Geodesic model: Caselles, Kimmel, Sapiro ’95
* Kichenassamy, Kumar, Olver, Tannenbaum, Yezzi ‘95

**Models in level set formulation:**

* Geometric model: Caselles, Catte, Coll, Dibos ‘93
  Malladi, Sethian, Vemuri ‘93
* Geodesic model: Caselles, Kimmel, Sapiro ‘95
A fitting term “without edges”

\[
\int_{\text{inside}(C)} |u_0 - c_1|^2 \, dx\,dy + \int_{\text{outside}(C)} |u_0 - c_2|^2 \, dx\,dy
\]

where

- \( c_1 = \text{average}(u_0) \) inside \( C \)
- \( c_2 = \text{average}(u_0) \) outside \( C \)

Fit > 0

Minimize: (Fitting + Regularization)

Fitting not depending on gradient detects “contours without gradient”
An active contour model “without edges”

(C. + Vese 98)

Fitting + Regularization terms (length, area)

\[
\inf_{c_1, c_2, C} F(c_1, c_2, C) = \mu \cdot |C| + \nu \cdot \text{Area(inside}(C)) \\
+ \lambda \int_{\text{inside}(C)} |u_0 - c_1|^2 \, dx dy + \lambda \int_{\text{outside}(C)} |u_0 - c_2|^2 \, dx dy
\]

\(C = \text{boundary of an open and bounded domain}\)

\(|C| = \text{the length of the boundary-curve } C\)
**Advantages**

- detects objects without sharp edges
- detects cognitive contours
- robust to noise; no need for pre-smoothing

**Examples**
Relation with the Mumford-Shah segmentation model

* The Mumford-Shah functional `87
\[
\inf_{u,C} F^{MS} (u, C) = \mu \cdot |C| + \lambda \int_{\Omega} |u - u_0|^2 \, dxdy + \int_{\Omega - C} |\nabla u|^2 \, dxdy
\]
- \(u\) is an optimal approximation of the initial image
- \(C\) is the set of jumps or edges of \(u\)
- \(u\) is smooth outside the edges \(C\)

* The model is a particular case of the Mumford-Shah functional
\[
u = \begin{cases} 
c_1 = \text{average}(u_0) & \text{inside } C \\
c_2 = \text{average}(u_0) & \text{outside } C 
\end{cases}
\]
- \(u\) is the best approximation of \(u_0\) taking only two values \(c_1\) and \(c_2\), and with one edge \(C\)

- the snake \(C\) is the boundary between the sets \(\{u = c_1\}\) and \(\{u = c_2\}\)
- we do not need regularization in \(u\), because \(c_1, c_2\) are constant
**Level Set Representation (S. Osher - J. Sethian ‘87)**

\[ C = \{(x, y) | \varphi(x, y) = 0\} \]

Inside \( C \) \( \varphi < 0 \)

Outside \( C \) \( \varphi > 0 \)

\( \tilde{n} \) = boundary of an open domain

Normal \( \tilde{n} = \frac{\nabla \varphi}{|\nabla \varphi|} \)

Curvature \( K = \text{div} \left( \frac{\nabla \varphi}{|\nabla \varphi|} \right) \)

Example: *mean curvature motion*

* Allows automatic topology changes, cusps, merging and breaking.

• Originally developed for tracking fluid interfaces.
Variational Formulations and Level Sets
(Following Zhao, Chan, Merriman and Osher ’96)

\[ C = \{(x, y) \in \Omega : \varphi(x, y) = 0\} \]

The Heaviside function

\[ H(\varphi) = \begin{cases} 
1, & \text{if } \varphi \geq 0 \\
0, & \text{if } \varphi < 0 
\end{cases} \]

- Length \( |C| = \int_{\Omega} |\nabla H(\varphi)| \)
- \( Area(inside(C)) = \int_{\Omega} H(\varphi)dxdy \)

\[ u(x, y) = c_1 H(\varphi(x, y)) + c_2 (1 - H(\varphi(x, y))) \]

The level set formulation of the active contour model

\[ \inf_{c_1, c_2, \varphi} F(c_1, c_2, \varphi) \]

\[ F(c_1, c_2, \varphi) = \mu \int_{\Omega} |\nabla H(\varphi)| + \nu \int_{\Omega} H(\varphi)dxdy \]

\[ + \lambda \int_{\Omega} |u_0(x, y) - c_1|^2 H(\varphi)dxdy + \lambda \int_{\Omega} |u_0(x, y) - c_2|^2 (1 - H(\varphi))dxdy \]
Considering any $C^1$ approximation and regularization $H_\varepsilon$ of the Heaviside function as $\varepsilon \to 0$, and denoting by $\delta_\varepsilon = H'_\varepsilon$ (an approximation to the one-dimensional Dirac delta function $\delta_0$), we can formally write the associated Euler-Lagrange equations obtained by minimizing the above functionals with respect to $\phi$, respectively:

$$\delta_\varepsilon(\phi) \text{div}
\left(
\frac{\nabla \phi}{|\nabla \phi|}
\right) = 0, \quad \delta_\varepsilon(\phi) = 0.$$

In (Osher and Sethian, 1988), a rescaling is made replacing $\delta_\varepsilon(\phi)$ by $|\nabla \phi|$, and the gradient descent flows are considered, giving:

$$\frac{\partial \phi}{\partial t} = |\nabla \phi|, \quad \phi(x, y, 0) = \phi_0(x, y). \quad (3)$$

(motion with constant speed, minimizing the area), and

$$\frac{\partial \phi}{\partial t} = |\nabla \phi| \text{div}
\left(
\frac{\nabla \phi}{|\nabla \phi|}
\right), \quad \phi(x, y, 0) = \phi_0(x, y), \quad (4)$$

(motion by mean curvature, minimizing the length).
Existence of minimizers among characteristic functions of sets with finite perimeter (in any dim. \( N \))

\[
F(c_1(\phi), c_2(\phi), \phi) = \overline{F}(H(\phi)), \quad H(\phi) = \mathbb{1}_{\{\phi > 0\}}
\]

\[
\overline{F}(\mathbb{1}) = \mu \int_{\Omega} |\nabla \mathbb{1}| + \nu \int_{\Omega} \mathbb{1}(x) dx
\]

\[
+ \lambda \int_{\Omega} |u_0 - c_1(\mathbb{1})|^2 \mathbb{1}(x) dx + \lambda \int_{\Omega} |u_0 - c_2(\mathbb{1})|^2 (1 - \mathbb{1}(x)) dx
\]

**Theorem**: Let \( \Omega \subset \mathbb{R}^N \) be open, bounded with \( \partial \Omega \) Lipschitz. If \( u_0 \in L^\infty(\Omega) \), then the following minimization problem

\[
\inf_{\mathbb{1}} \overline{F}(\mathbb{1}), \mathbb{1} \in BV(\Omega), \mathbb{1}(x) \in \{0,1\} dx - a.e.,
\]

has a solution.

\( BV(\Omega) = \) space of functions of bounded variation

Proof: standard in calculus of variations: \( lsc \) of TV and compactness in BV
The Euler-Lagrange equations

\[ \inf_{c_1, c_2, \varphi} F(c_1, c_2, \varphi) \]

Using smooth approximations for the Heaviside and Delta functions

Equations for \( c_1 \) and \( c_2 \)

\[
\begin{align*}
c_1(\varphi) &= \frac{\int u_0 \cdot H(\varphi) dx dy}{\int H(\varphi) dx dy}, \quad c_2(\varphi) = \frac{\int u_0 (1 - H(\varphi)) dx dy}{\int (1 - H(\varphi)) dx dy}
\end{align*}
\]

Equation for \( \varphi = \varphi(t, x, y) \)

\[
\frac{\partial \varphi}{\partial t} = \delta_\varepsilon(\varphi) \left[ \mu \cdot \text{div} \left( \frac{\nabla \varphi}{|\nabla \varphi|} \right) - \nu - \lambda (u_0 - c_1)^2 + \lambda (u_0 - c_2)^2 \right]
\]

\( \varphi(0, x, y) = \varphi_0(x, y) \)
Two approximations of the Heaviside and Delta functions

\[ H_\varepsilon(x) = \begin{cases} 
1, & \text{if } x > \varepsilon \\
0, & \text{if } x < -\varepsilon \\
\frac{1}{2} \left[ 1 + \frac{x}{\varepsilon} + \frac{1}{\pi} \sin\left( \frac{\pi x}{\varepsilon} \right) \right], & \text{if } |x| \leq \varepsilon 
\end{cases} \]

- \( \varepsilon \) has small compact support \([-\varepsilon, \varepsilon]\)
- acts only locally, around the zero-level curve of \( \varphi \)
- only local minima are computed

\[ H_\varepsilon(x) = \frac{1}{2} \left( 1 + \frac{2}{\pi} \arctan\left( \frac{x}{\varepsilon} \right) \right) \]

- \( \varepsilon \) > 0 everywhere
- acts on all level curves of \( \varphi \)
- has the tendency to compute global minima
- detects interior contours automatically
\[
\frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = \delta_h(\phi_{i,j}^n) \left[ \frac{\mu}{h^2} \Delta^x - \right.
\]
\[
\cdot \left( \frac{\Delta^x \phi_{i,j}^{n+1}}{\sqrt{(\Delta^x \phi_{i,j}^n)^2/(h^2) + (\phi_{i,j+1}^n - \phi_{i,j-1}^n)^2/(2h)^2}} \right)
\]
\[
+ \frac{\mu}{h^2} \Delta^y
\]
\[
\cdot \left( \frac{\Delta^y \phi_{i,j}^{n+1}}{\sqrt{(\phi_{i+1,j}^n - \phi_{i-1,j}^n)^2/(2h)^2 + (\Delta^y \phi_{i,j}^n)^2/(h^2)}} \right)
\]
\[
- \nu - \lambda_1(u_{0,i,j} - c_1(\phi^n))^2 + \lambda_2(u_{0,i,j} - c_2(\phi^n))^2 \right].
\]
Finally, the principal steps of the algorithm are:

- Initialize $\phi^0$ by $\phi_0$, $n = 0$.
- Compute $c_1(\phi^n)$ and $c_2(\phi^n)$ by (6) and (7).
- Solve the PDE in $\phi$ from (9), to obtain $\phi^{n+1}$.
- Reinitialize $\phi$ locally to the signed distance function to the curve (this step is optional).
- Check whether the solution is stationary. If not, $n = n + 1$ and repeat.
\[ \begin{align*}
\psi_T &= \text{sign}(\phi(t))(1 - |\nabla \psi|) \\
\psi(0, \cdot) &= \phi(t, \cdot)
\end{align*} \]
Experimental Results

Evolution of $C$ Averages $(c_1, c_2)$

Advantages

- Automatically detects interior contours!
- Works very well for concave objects
- Robust w.r.t. noise
- Detects blurred contours
- The initial curve can be placed anywhere!
- Allows for automatical change of topology
Contours without gradient (cognitive contours)
A spiral from an art picture
(the initial curve is the border of the image)

Europe nightlights
The model detects clusters of points (cognitive contours)
For the results from Figs. 15 and 16, we replaced in our model $u_0$ by $\text{curvature}(u_0) = \text{div}(\nabla u_0/|\nabla u_0|)$, and by $\text{orientation}(u_0) = \tan^{-1}(u_{0,y}/u_{0,x})$ respectively (the angle of the normal to the level curves). Other discriminants may be considered.
A plane in a noisy environment

A galaxy
From Active Contours (2D) to Active Surfaces (3D)

MRI DATA FROM LONI-UCLA
Potential Application to Data Mining:

(Data from V. Kumar et al: Chameleon, IEEE Computer 2000)
Extension to Vector-Valued Images

Tony F. CHAN & Luminita VESE & B. Yezrielev SANDBERG

Applications

* Color images (RGB)
* Multi-spectral (PET, MRI, CT)

Vector-Valued image: \( \vec{u}_0 = (u_{0,1}, u_{0,2}, \ldots, u_{0,N}) \), \( N \) channels

\( \vec{c}_1, \vec{c}_2 \) are constant vectors (averages of each channel inside and outside the curve)

The model

\[
\inf_{\vec{c}_1, \vec{c}_2, C} F(\vec{c}_1, \vec{c}_2, C)
\]

\[
F(\vec{c}_1, \vec{c}_2, C) = \mu \cdot \text{Length}(C) + \nu \cdot \text{Area(inside}(C))
\]

\[
+ \lambda \sum_i \left( \int_{\text{inside}(C)} |u_{0,i} - c_{1,i}|^2 \, dx \, dy + \int_{\text{outside}(C)} |u_{0,i} - c_{2,i}|^2 \, dx \, dy \right)
\]

* The model can be solved using level sets, similar with the scalar case
Extension to Vector-Valued Images

Channel 1 with occlusion          Channel 2 with noise

Recovered object and averages
**Color Images**

Color (RGB) picture          Intensity (gray-level) picture

Recovered objects and contours in RGB mode
Recovered object contours combined in RGB mode

Recovered averages combined in RGB mode
Active Contour for Texture

45 transforms of image with Gabor function

\[
\begin{array}{c|c|c|c|c|c}
\theta & 0 & \pi/6 & \pi/4 & \pi/3 & \pi/2 \\
\hline
F & \sigma=0.075 & \sigma=0.05 & \sigma=0.025 & \\
120 & \\
150 & \\
180 & \\
\end{array}
\]

User Picked Gabor transforms yield final contour
Texture Segmentation with User picked Gabor transforms

Original Image

Gabor Transforms

θ = π/6  σ = .0075  F = 120
θ = π/3  σ = .0075  F = 180
θ = π/3  σ = .005  F = 150

Initial Contour  Final Contour  Final Contour on the original image
Unsupervised Texture Segmentation Synthetic

Automatically select small subset of Gabor frames for AC model
Plan

Introduction and Motivations:

Active contours “without edges”

Chan-Vese, *Active Contours without Edges*, SS ’99, IEEE IP
Chan-Sandberg-Vese, *Active Contours without Edges for Vector-Valued Images*, JVCI

Generalization to the Mumford-Shah model:

The piecewise-constant case

Plan

Introduction and Motivations:

Active contours “without edges”

Generalization to the Mumford-Shah model:

The piecewise-constant case

The piecewise-smooth case
Variational Segmentation Models

Mumford-Nitzberg-Shiota, Zhu-Lee-Yuille,
Chambolle-Dal Maso,
Chambolle-Bourdin, Cortesani-Toader,
Malladi-Sarti-Sethian, …

Works Related to Ours:
Samson-Feraud-Aubert-Zerubia, Sifakis-Garcia-Tziritas,
Paragios-Deriche, Yezzi-Tsai-Willsky
\[
\min_{u,S} \int_{\Omega-S} \left( \alpha |\nabla u|^2 + \beta (u-u_0)^2 \right) + \int_S dS
\]

\(S=\text{“edges”}\)
Algorithms for Minimizing Mumford-Shah energy

Difficult problem in practice: non-convex; lower-dim. unknown $C$

Solutions proposed:

* Region growing and merging: use MS energy in greedy algorithm (Dal Maso - Morel - Solimini, Koepfler-Lopez-Morel)

* Elliptic approximations: embed $C$ in 2D phase-field function (Ambrosio - Tortorelli, Chambolle, Braides, March)

* Statistical framework: (Zhu-Lee-Yuille)

Our work:

* provides an efficient level set formulation for the M-S model

* inherits the advantages of level sets
* Our AC model is a particular case of the Mumford-Shah functional

\[ u = \begin{cases} 
  c_1 = \text{average}(I) \text{ inside } C \\
  c_2 = \text{average}(I) \text{ outside } C 
\end{cases} \]

- \( u \) is the best approximation of \( I \) taking only two values \( c_1 \) and \( c_2 \), and with one edge \( C \)
- the contour \( C \) is the boundary between the sets \( \{u = c_1\} \) and \( \{u = c_2\} \)
- we do not need regularization in \( u \), because \( c_1, c_2 \) are constant
Relation C-Vese AC & Mumford-Shah segmentation models

* The Mumford-Shah functional 87

\[
\inf_{u,C} F^{MS}(u, C) = \mu \cdot \text{Length}(C) + \lambda \int_{\Omega} |u - I|^2 \, dx \, dy + \int_{\Omega - C} |\nabla u|^2 \, dx \, dy
\]

- \(u\) is an optimal approximation of \(I\)
- \(C\) is the set of jumps or edges of \(u\)
- \(u\) is smooth outside the edges \(C\)

* Our AC model is a particular case of the Mumford-Shah functional

\[
u = \begin{cases} 
  c_1 = \text{average}(I) \text{ inside } C \\
  c_2 = \text{average}(I) \text{ outside } C 
\end{cases}
\]

- \(u\) is the best approximation of \(I\) taking only two values \(c_1\) and \(c_2\), and with one edge \(C\)
- the snake \(C\) is the boundary between the sets \(\{u = c_1\}\) and \(\{u = c_2\}\)
- we do not need regularization in \(u\), because \(c_1, c_2\) are constant
Generalizing CV-AC to MS

• From 2 segments (1 level set function) to multiple segments and level set functions
• From p.w. constant to p.w. smooth approximations
Generalization to Multiple (>2) Segments

The Mumford-Shah piecewise-constant segmentation model

\[ F_{MS} (u, C) = \sum_i |u_0 - c_i|^2 \, dx \, dy + \nu |C| \]

Need > 1 level set function to represent > 2 segments, triple junctions, …

Classical methods for Multiphase Level Set Representation:

(Zhao-Chan-Merriman-Osher):
- associate one level set function to each phase, allows for triple junctions
- high computational cost
- needs additional constraints to prevent vacuum and overlap

New Multiphase Level Set Representation:
- using \( n \) level set functions, we can represent \( 2^n \) phases or segments
- reduced computational cost
- by definition of the partition, no vacuum and no overlap
- allows for triple junctions and other complex topologies
Multiphase level set representations and partitions allows for triple junctions, with no vacuum and no overlap of phases

\[ \Phi = (\varphi_1, \ldots, \varphi_n) \Rightarrow 2^n \text{ phases} \]

Curves:
\[ \{\varphi = 0\} \]

2-phase segmentation
1 level set function

Curves:
\[ \{\varphi_1 = 0\} \cup \{\varphi_2 = 0\} \]

4-phase segmentation
2 level set functions

\[ \begin{align*}
+ & \quad \varphi > 0 \\
- & \quad \varphi < 0
\end{align*} \]

\[ \begin{align*}
++ & \quad \varphi_1 > 0, \varphi_2 > 0 \\
+- & \quad \varphi_1 > 0, \varphi_2 < 0 \\
-+ & \quad \varphi_1 < 0, \varphi_2 > 0 \\
-- & \quad \varphi_1 < 0, \varphi_2 < 0
\end{align*} \]
Figure 2. Two curves given by $\phi_1 = 0$ and $\phi_2 = 0$, partition the domain into four regions: $\{\phi_1 > 0, \phi_2 > 0\}$, $\{\phi_1 > 0, \phi_2 < 0\}$, $\{\phi_1 < 0, \phi_2 > 0\}$, $\{\phi_1 < 0, \phi_2 < 0\}$. 
Figure 3. Three curves given by $\phi_1 = 0$, $\phi_2 = 0$, and $\phi_3 = 0$, partition the domain into eight regions: \{\(\phi_1 > 0, \phi_2 > 0, \phi_3 > 0\)\}, \{\(\phi_1 > 0, \phi_2 > 0, \phi_3 < 0\)\}, \{\(\phi_1 > 0, \phi_2 < 0, \phi_3 > 0\)\}, \{\(\phi_1 > 0, \phi_2 < 0, \phi_3 < 0\)\}, \{\(\phi_1 < 0, \phi_2 > 0, \phi_3 > 0\)\}, \{\(\phi_1 < 0, \phi_2 > 0, \phi_3 < 0\)\}, \{\(\phi_1 < 0, \phi_2 < 0, \phi_3 > 0\)\}, \{\(\phi_1 < 0, \phi_2 < 0, \phi_3 < 0\)\}. 
**Example: two level set functions and four phases**

<table>
<thead>
<tr>
<th>2 level set functions ((\varphi_1, \varphi_2)) \iff 4 phases or segments:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\varphi_1 &gt; 0, \varphi_2 &gt; 0, \quad \varphi_1 &gt; 0, \varphi_2 &lt; 0, \quad \varphi_1 &lt; 0, \varphi_2 &gt; 0, \quad \varphi_1 &lt; 0, \varphi_2 &lt; 0)</td>
</tr>
</tbody>
</table>

\[\Phi = (\varphi_1, \varphi_2) \quad \text{The level set functions} \]

\[c = (c_{11}, c_{10}, c_{01}, c_{00}) \quad \text{Constant vector} \]

\[u = c_{11} H(\varphi_1) H(\varphi_2) + c_{10} H(\varphi_1)(1 - H(\varphi_2)) + c_{01}(1 - H(\varphi_1))H(\varphi_2) + c_{00}(1 - H(\varphi_1))(1 - H(\varphi_2))\]

**Energy**

\[
\inf_{(c, \Phi)} F(c, \Phi) = \int_{\Omega} |u_0 - c_{11}|^2 H(\varphi_1) H(\varphi_2) dx\,dy + \int_{\Omega} |u_0 - c_{10}|^2 H(\varphi_1)(1 - H(\varphi_2)) dx\,dy
\]

\[
+ \int_{\Omega} |u_0 - c_{01}|^2 (1 - H(\varphi_1)) H(\varphi_2) dx\,dy + \int_{\Omega} |u_0 - c_{00}|^2 (1 - H(\varphi_1))(1 - H(\varphi_2)) dx\,dy
\]

\[
+ \nu \int_{\Omega} |\nabla H(\varphi_1)| + \nu \int_{\Omega} |\nabla H(\varphi_2)|
\]

Existence of minimizers

Theorem:

Existence of minimizers among characteristic functions of sets with finite perimeter

Similar with the previous case

Remarks:

Non-convex minimization problem

No uniqueness of minimizers
**The Euler-Lagrange equations**

Using smooth approximations for the Heaviside and Delta functions

The constants

\[ c_{11} = \text{mean}(u_0) \text{ in } \{\varphi > 0, \varphi_2 > 0\} \]

\[ c_{10} = \text{mean}(u_0) \text{ in } \{\varphi > 0, \varphi_2 < 0\} \]

\[ c_{01} = \text{mean}(u_0) \text{ in } \{\varphi < 0, \varphi_2 > 0\} \]

\[ c_{00} = \text{mean}(u_0) \text{ in } \{\varphi < 0, \varphi_2 < 0\} \]

The PDE's in \( \Phi = (\varphi_1, \varphi_2) \)

\[
\frac{\partial \varphi_1}{\partial t} = \delta_\varepsilon(\varphi_1) \left\{ \nabla \text{div} \left( \frac{\nabla \varphi_1}{|\nabla \varphi_1|} \right) - \left[ (|u_0 - c_{11}|^2 - |u_0 - c_{01}|^2) H(\varphi_2) + (|u_0 - c_{10}|^2 - |u_0 - c_{00}|^2) (1 - H(\varphi_2)) \right] \right\}
\]

\[
\frac{\partial \varphi_2}{\partial t} = \delta_\varepsilon(\varphi_2) \left\{ \nabla \text{div} \left( \frac{\nabla \varphi_2}{|\nabla \varphi_2|} \right) - \left[ (|u_0 - c_{11}|^2 - |u_0 - c_{01}|^2) H(\varphi_1) + (|u_0 - c_{10}|^2 - |u_0 - c_{00}|^2) (1 - H(\varphi_1)) \right] \right\}
\]

Similarly for \( \Phi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \)
Results with 2 level set functions (4 phases)

* two different initializations
* interior contours are automatically detected
* model robust to noise

Faster
Energy Decrease With Iterations for 3 initial conditions

(here, only with (a) and (c) it converge to the global minimum)

Non-convex problem: different (I.C.) may converge to local/global min.
Detection of contours without gradient - cognitive contours

**Triple junctions** can be detected and represented using only two level set functions, without vacuum and without overlap (new).

\[ \{ \varphi_1 = 0 \} \quad \{ \varphi_2 = 0 \} \]
An MRI brain image

Phase 11
mean(11)=45

Phase 10
mean(10)=159

Phase 01
mean(01)=9

Phase 00
mean(00)=103
Numerical results on an MRI brain picture

Multiphase segmentation model using two level set functions and four phases
A real outdoor picture

Phase 11
mean(11)=159

Phase 10
mean(10)=205

Phase 01
mean(01)=23

Phase 00
mean(00)=97
Three level set functions representing up to eight phases
Six phases are detected, together with the triple junctions.

Evolution of the 3 level sets
Segmented image

Evolution of the 3 individual level sets.
Original gene chip data
(Terry Speed)

Detected contours

Segmented images for three different scale parameters
Original gene data (G)         Detected contours         Segmented image

1 Segment(11)=180        2 Segment(10)=93      3 Segment(01)=56

4 Segment(00)=6

Gene Microarray
Expression Data
It is useful to compute the level set function not on the whole image domain but in a narrow band near to the contour. $\text{Abs}(\phi) < d$

Decreasing the computational complexity.
Narrow Band Algorithm
Narrow Band Algorithm

- Tag *alive* points in narrow band
- Build *land mines* to indicate near edge
- Initialize *far away* points outside the narrow band with large positive (negative) values if values are outside(inside) the front itself.
- Solve level set equation until *land mine* hit
- Rebuild and loop.
- Bandwidth can be set as 12
Plan

Introduction and Motivations:

Active contours “without edges”

Chan-Vese, *Active Contours without Edges*, SS ’99, and IEEE IP

Chan-Sandberg-Vese, *Active Contours without Edges for Vector-Valued Images*, JVCI

**Generalization to the Mumford-Shah model:**

The piecewise-constant case

The piecewise-smooth case

Plan

Introduction and Motivations:

Active contours “without edges”

Generalization to the Mumford-Shah model:

The piecewise-constant case

The piecewise-smooth case
Generalization to Piecewise-Smooth Approximations
by the Mumford-Shah segmentation model

The Mumford-Shah segmentation model ‘89

\[
\inf_{u,C} F_{MS}^{MS}(u, C)
\]

\[
F_{MS}^{MS}(u, C) = \mu^2 \int_\Omega |u - u_0|^2 \, dx\,dy + \int_{\Omega - C} |\nabla u|^2 \, dx\,dy + \nu |C|
\]

A level set algorithm for the M-S model

(assume first that \( C \) is the boundary of an open domain)

\[
C = \{(x, y) \in \Omega \mid \varphi(x, y) = 0\}, \quad u(x, y) = \begin{cases} 
  u^+(x, y), \text{ if } \varphi \geq 0 \\
  u^-(x, y), \text{ if } \varphi < 0 
\end{cases}
\]

\[
u(x, y) = u^+(x, y)H(\varphi) + u^-(x, y)(1 - H(\varphi))
\]
\[ \inf_{u^+, u^-, \varphi} F(u^+, u^-, \varphi) \]

\[ F(u^+, u^-, \varphi) = \mu^2 \int_{\varphi \geq 0} |u^+ - u_0|^2 \, dx \, dy + \mu^2 \int_{\varphi < 0} |u^- - u_0|^2 \, dx \, dy \]

\[ \nu |C| + \int_{\varphi > 0} |\nabla u^+|^2 \, dx \, dy + \int_{\varphi < 0} |\nabla u^-|^2 \, dx \, dy \]

**Using the Heaviside function:**

\[ F(u^+, u^-, \varphi) = \mu^2 \int_{\Omega} |u^+ - u_0|^2 H(\varphi) \, dx \, dy + \mu^2 \int_{\Omega} |u^- - u_0|^2 (1 - H(\varphi)) \, dx \, dy \]

\[ + \int_{\Omega} |\nabla u^+|^2 H(\varphi) \, dx \, dy + \int_{\Omega} |\nabla u^-|^2 (1 - H(\varphi)) \, dx \, dy + \nu \int_{\Omega} |\nabla H(\varphi)| \]

The model is a common framework for active contours, denoising, segmentation and edge detection.
**The Euler-Lagrange equations**

\[
\mu^2 (u^+ - u_0) = \Delta u^+ \quad \text{on } \varphi > 0, \quad \frac{\partial u^+}{\partial \vec{n}} = 0 \quad \text{on } \{\varphi = 0\}
\]

\[
\mu^2 (u^- - u_0) = \Delta u^- \quad \text{on } \varphi < 0, \quad \frac{\partial u^-}{\partial \vec{n}} = 0 \quad \text{on } \{\varphi = 0\}
\]

\[
\varphi_t =
\delta\varepsilon(\varphi) \left[ \nabla \cdot \text{div} \left( \frac{\nabla \varphi}{|\nabla \varphi|} \right) - \mu^2 |u^+ - u_0|^2 + \mu^2 |u^- - u_0|^2 - |\nabla u^+|^2 + |\nabla u^-|^2 \right]
\]

Curvature  Jump of energy terms in \( u \)

The model is a common framework for active contours, denoising, segmentation and edge detection.
Remark: We have to extend \( u^+ \) on \( \varphi < 0 \) and \( u^- \) on \( \varphi > 0 \) by \( C^1 \) functions

Several solutions (by diffusion in the normal direction):

Interpolation of functions (Caselles, Morel, Sbert)

Minimizing Lipschitz extensions (Jensen, Crandall, Cao)

Ghost fluid method (Fedkiw, Aslam)

Normal \( N = (n_x, n_y) = \left( \frac{\varphi_x}{|\nabla \varphi|}, \frac{\varphi_y}{|\nabla \varphi|} \right) \)

Examples:

\[
    u_t^- = n_x^2 u_{xx}^- + 2n_x n_y u_{xy}^- + n_y^2 u_{yy}^- \quad \text{on} \quad \varphi > 0, \quad \frac{\partial u^-}{\partial n} = 0 \quad \text{on} \quad \partial \Omega
\]

Or:

\[
    u_t^- + n_x u_x^- + n_y u_y^- = 0 \quad \text{in} \quad \varphi > 0, ...
\]
Combining p.w. smooth approx. with multiple LS’s

We consider (again) two level set functions with four phases

Remark: Based on Four Color Thm., 2 LS’s should suffice.

\[
\begin{align*}
    u &= u^{++} H(\phi_1)H(\phi_2) + u^{+-} H(\phi_1)(1 - H(\phi_2)) + \\
    &= u^{--} (1 - H(\phi_1))H(\phi_2) + u^{-'-} (1 - H(\phi_1))(1 - H(\phi_2))
\end{align*}
\]

\[
\Phi = (\phi_1, \phi_2)
\]

The energy and the Euler-Lagrange equation can be written and solved in a similar manner.
To handle triple junctions and more general cases

We consider (again) two level set functions with four phases

**Remark:** Based on the Four Color Thm., it should suffice.

\[
\begin{align*}
    u &= u^{++} H(\phi_1) H(\phi_2) + u^{+-} H(\phi_1)(1 - H(\phi_2)) + \\
    &\quad u^{-+} (1 - H(\phi_1)) H(\phi_2) + u^{--} (1 - H(\phi_1))(1 - H(\phi_2))
\end{align*}
\]

\[
\Phi = (\phi_1, \phi_2)
\]

The energy and the Euler-Lagrange equation can be written and solved in a similar manner.
Existence of minimizers in the space SBV

(Special functions of Bounded Variation)

$$SBV(\Omega) \subset BV(\Omega) \subset L^{N/N-1}(\Omega)$$  
Proof: standard in calculus of variations

$$u \subset BV(\Omega)$$  
$$Du = \nabla u(x)dx + Cu + Ju$$  
Compactness and l.s.c. results on $SBV$ by L. Ambrosio

$$u \subset SBV(\Omega)$$  
$$Du = \nabla u(x)dx + J_u$$
Signal denoising - segmentation - jump detection
(numerical results in 1D; note jumps well located and preserved)

Remark: Related works by Anne Gelb and Eitan Tadmor using spectral methods
**Piecewise-smooth approximations**

Active Contours+Denoising+Segmentation

M-S: The three objects are detected with the correct values!
Before (A-C): all objects were detected, but with the same mean
2 level sets (four phases) in the piecewise-smooth case

Initial Noisy Image

Initial curves

Evolutions to the steady state (denoised - segmented image)
**Future works**

- Consider other surface energies, depending on the normal or on the curvature (beyond image processing applications)
- The multiphases can be used for occlusion (P. Perona)
- Higher order regularizations, to detect jumps in the derivatives
- Study the convergence of the algorithm, when adding some regularizing/elliptic terms
- Use the method for related problems arising in fracture mechanics and optimal design
- Viscosity solutions for these geometric curve evolution PDEs
- ...
Conclusion

Features & Advantages of PDE Imaging Models

* Use PDE & Geometry concepts: gradients, diffusion, curvature, level sets

* Exploit sophisticated PDE and CFD (e.g. shock capturing) techniques

Restoration:
- sharper edges, less edge artifacts, often morphological

Segmentation (using level set techniques):
- scale adaptivity, geometry-based, controlled regularity of boundaries, segments can have complex topologies

Newer, less well developed/accepted.
OUTLINE

• Review of Variational Level Set Segmentation
• Fast Algorithm
A Fast Algorithm for Level Set Based Optimization

(Bing Song and Tony Chan, 2003)
Available at
www.math.ucla.edu/applied/cam/index.html
CAM 02-68
(Related work by Gibou & Fedkiw)
Outline

• Motivation
• General Algorithm
• Application to the Chan-Vese segmentation model
• Convergence of algorithm
• Examples
• Closely related work by Gibou and Fedkiw
Motivation

• Minimize level set based functional

\[
\min_{\Phi} F(H(\Phi))
\]

• Examples:
  – Image segmentation
  – Inverse problem
  – Shape analysis and optimization
  – Data Clustering
  – …
How to solve it?

• Usually use gradient descent flow

  \[ \Phi_t = -\nabla_\phi F(H(\Phi)) \]

• Drawback
  – Slow (CFL)
  – \( F \) need to be differentiable
An Insight

• Segmentation only needs sign of $\Phi$ but not its value

• Direct search making use of objective function does not require derivatives of functional
New Algorithm

1. Initialization. Partition the domain into $\Phi > 0$ and $\Phi < 0$

2. Advance. For each point $x$ in the domain, if the energy $F$ lower when we change $\Phi(x)$ to $-\Phi(x)$, then update this point.

3. Repeat step 2 until the energy $F$ remains unchanged.
An example: Chan-Vese model

\[ F(H(\Phi), c_1, c_2) = \mu \left( \int_\Omega |\nabla H(\phi)| + \lambda_1 \int_\Omega |u_0 - c_1|^2 H(\phi) \right) + \lambda_2 \int_\Omega |u_0 - c_2|^2 (1 - H(\phi)) \]
Core Step of Algorithm

Change in $F$ if $\Phi > 0$ is changed to $\Phi < 0$ at a pixel:

\[ \tilde{c}_1 = c_1 + \frac{c_1 - u}{m - 1} \]
\[ \tilde{c}_2 = c_2 - \frac{c_2 - u}{n - 1} \]
\[ \tilde{F}_1 = F_1 - (u - c_1)^2 \frac{m}{m - 1} \]
\[ \tilde{F}_2 = F_2 + (u - c_2)^2 \frac{n}{n + 1} \]

\[ \Delta F_{12} = (u - c_2)^2 \frac{n}{n + 1} - (u - c_1)^2 \frac{m}{m - 1} \text{ + change in length} \]
How to calculate length term?

\[ |\nabla H(\phi_{ij})| = \sqrt{(H(\phi_{i+1,j}) - H(\phi_{i,j}))^2 + (H(\phi_{i,j+1}) - H(\phi_{i+1,j}))^2} \]

Change in length term can be updated locally.

Only possible value for length term locally is 0, 1, \sqrt{2} depending the 3 points in the length term belong to the same region or not.
A 2-phase example

(a), (b), (c), (d) are four different initial condition. All of them converge in one sweep!
Example with Noise

Converged in 4 steps.
(Gradient Descent on Euler-Lagrange took $> 400$ steps.)
Convergence of the algorithm

Theorem: For a two phase image, the algorithm will converge in one sweep, independent of sweeping order.

Proof considers the following 2 cases:

Green part of inside of contour will flip sign. Only one of the 2 black parts can change sign.
Why is 1-step convergence possible?

• Problem is global: usually cannot have finite step convergence based on local updates only

• But, in our case, we can exactly calculate the global energy change via local update (can update global average locally)
Application to piecewise linear CV model
(Vese 2002)

\[
F(H(\Phi), c_1, c_2) = \mu\int_{\Omega} |\nabla H(\phi)| + \lambda_1 \int_{\Omega} |u_0 - a_0 - a_1 x - a_2 y|^2 \, H(\phi) \\
+ \lambda_2 \int_{\Omega} |u_0 - b_0 - b_1 x - b_2 y|^2 (1 - H(\phi))
\]

Original P.W. Constant
Converged in 4 steps

P.W. Linear
Converged in 6 steps
Application to multiphase CV model
(C.- Vese 2000)

- Using n level set to represent $2^n$ phases.

$$F_n(c, \Phi) = \sum_{1 \leq I \leq n = 2^m} \int_\Omega |u_0 - c_I|^2 \, \chi_I \, dx \, dy + \sum_{1 \leq I \leq m} \nu \int |\nabla H(\phi_I)|$$
Multiphase CV model (cont’d)

Converged in 1 step.
Gibou and Fedkiw’s method (02)

- First, discard the stiff length term in E-L:
  \[
  \frac{d\phi}{dt} = -\lambda_1 (u - c_1)^2 + \lambda_2 (u - c_2)^2 := V(x)
  \]

- Take large time step enough to change sign

- If \( \phi(x)V(x) < 0 \), then \( \phi(x) = -\text{sign}(\phi(x)) \)

- Followed by anisotropic diffusion to handle noise

- Difference (Ours wrt GF):
  - A unified framework to handle length term
  - F does not need to be differentiable
Conclusion

• A fast algorithm to solve the Chan-Vese segmentation models.
• 1-step convergence for 2 phase images (no reg.).
• Robust wrt noise: 3-4 step convergence.
• The gradient of functional is not needed.
• We think it can be applied to more general level set based optimization problems.
• In general, cannot expect finite step convergence (need global change from local update). Can get stuck in local minima.
That’s all.

Thank you for your patience!
Logic Operators on Multi-Channel Active Contour
(Chan, Sandberg 2001)

- An Example for two channel logical segmentation:

  \[ A_1 \cup A_2 \]  
  \[ A_1 \cap A_2 \]  
  \[ A_1 \cap \sim A_2 \]
Redefining the fitting term of CV Model

\[
\min F(C, c_+, c_-) = \mu \cdot \text{length}(C) + \int_\Omega f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) dx dy
\]

using the logical variables:

\[
x_i(x, y) = \begin{cases} 
0 & (x, y) \text{ good fit inside } C \\
1 & \text{otherwise}
\end{cases}
\]

\[
y_i(x, y) = \begin{cases} 
0 & (x, y) \text{ good fit outside } C \\
1 & \text{otherwise}
\end{cases}
\]
Logical variables for CV Model:

\[ x_i(x, y) = \frac{|u^i_0 - c^i_+|^2}{\max_{(x, y) \in u^i_0} |u^i_o - c^i_+|^2} \]

\( (x, y) \) inside \( C \)

\[ y_i(x, y) = \begin{cases} 0 & \text{outside } C \\ 0 & \text{inside } C \end{cases} \]

\[ x'_i = 1 - x_i \]

\[ y'_i = 1 - y_i \]
Truth Tables

Intersection of objects:
intersection of \( x_i \) inside \( C \), and union of \( y_i \) outside \( C \).

Union of objects: (analogous)

Truth Table using parameters:

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( A_1 \cup A_2 )</th>
<th>( A_1 \cap A_2 )</th>
<th>( A_1 \cap \sim A_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>inside ( C )</td>
<td>1</td>
<td>1</td>
<td>0</td>
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<td>1</td>
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<td>outside ( C )</td>
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</tr>
</tbody>
</table>

Fitting the truth table for function \( f \):

\[
f_{A_1 \cup A_2} (x, y) = \left[ (x_1 (x, y)x_2 (x, y))^{1/2} + (y_1 (x, y) + y_2 (x, y)) / 2 \right]
\]

\[
f_{A_1 \cap A_2} (x, y) = \left[ (x_1 (x, y) + x_2 (x, y)) / 2 + (y_1 (x, y)y_2 (x, y))^{1/2} \right]
\]

\[
f_{A_1 \cap \sim A_2} (x, y) = \left[ (x_1 (x, y) + (1 - x_2 (x, y))) / 2 + (y_1 (x, y)(1 - y_2 (x, y)))^{1/2} \right]
\]
Examples using logic operators

Given the 3 channels 4 possible logic operations are implemented.

Time evolution for original example
Example of an Application

$A_1$  $A_2$

Original image with contour overlapping

Contour only

Time evolution for finding the “tumor” in the first image that is not in the second.
A Fast Algorithm for Variational Level Set Segmentation

Tony F. Chan
Department of Mathematics, Univ. Calif. Los Angeles

SONAD 2003

McMaster University, Ontario, Canada

May 2, 2003

Reprints: www.math.ucla.edu/~imagers

Supported by ONR and NSF

(Joint work with Bing Song)
OUTLINE

• Review of Variational Level Set Segmentation
• Fast Algorithm
Narrow band

• Initialization $n=0$

• repeat
  – $n++$
  – Determination of the narrow band
  – Computing $c_1$ and $c_2$
  – Evolving the level-set function on the narrow band
  – Re-initialization

• until the solution is stationary, or $n>n_{\text{max}}$
What is an “active contour”?

- giving an image $u_0 : \Omega \rightarrow \mathcal{R}$
- evolve a curve $C$ to detect objects in $u_0$
- the curve has to stop on the boundaries of the objects

Initial Curve $\rightarrow$ Evolutions $\rightarrow$ Detected Objects